Power Approximations for Test Statistics with Dominant Components

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Abstract

We consider approximating the power functions of some tests for several hypothesis testing problems in time series. The test statistics under consideration are ratios of quadratic forms in normal variables and their power is related to the distributions of weighted sums of Chi-square random variables. Conventionally, the power functions are evaluated from these distributions at each alternative numerically by Pearson's moment approximation, Imhof's procedure, Edgeworth type expansion or Monte Carlo method. In this study, we propose analytic approximations to the power functions when part of the weighted sum of Chi-square random variables can be well approximated by a scaled Chi-square variable in distribution. In applications, the proposed analytic approximation may be obtained easily by evaluating the power only at a few alternative values. Several illustrative examples are presented that show excellent agreement with the true power functions.

Keywords: Hypothesis testing, locally best invariant test, moving average unit root, power approximation.

1 Introduction

A large number of test statistics can be expressed as ratios of quadratic forms in normal variables, namely $r = \mathbf{x}'F\mathbf{x}/\mathbf{x}'B\mathbf{x}$, where the vector $\mathbf{x}$ has a $N(0, \Sigma)$ distribution, F is a nonstochastic matrix, B is positive definite, and Σ is a nonsingular covariance matrix; for example, the uniformly most power invariant (UMPI) test (King (1980)), the locally best invariant (LBI) test (King(1980), Nabeya and Tanaka (1988), Leybourne and McCabe (1989), Saikkonen and Luukkonen (1993)) and the Lagrange multiplier (LM) test (Tanaka (1990) and Kwiatkowski et al. (1992)). Let $G_1 \geq \cdots \geq G_{m-p} > 0 > G_{m-p+1} \geq \cdots \geq G_m$ denote the non-zero characteristic roots of $(cB - F)\Sigma$ with rank $m(\geq 2)$, then the power of the test statistic r is given by

$$\begin{aligned} P(r < c) &= P(\mathbf{x}'(cB - F)\mathbf{x} > 0) \\ &= P(\sum_{i=1}^{m} G_i W_i > 0), \qquad (1.1) \end{aligned}$$

where c is a critical value of the test and W_i's are independent Chi-square random variables with one degree of freedom. From (1.1), the power of the test statistic can thus be obtained through the distribution of the weighted sum of Chi-square random variables. For ease of notation, we will denote $QW = \sum_{i=1}^{m} G_i W_i$ throughout. Note that the $G_i's$ typically depend on a parameter of interest (to be denoted by λ) where $\lambda = 0$ corresponds to the null hypothesis. Treating $P(QW > 0)$ as a function of λ (i.e. the power function), it is desired to derive an analytic approximation to this function.

Although in most cases the exact distribution of QW is unknown, it can be evaluated by numerical procedures such as those of Imhof (1961), Davies (1973) and Shively, Ansley and Kohn (1990). Imhof and Davies used the characteristic function inversion formula to evaluate the distribution of QW by numerical integration. Shively et al. suggested a modification of the Kalman filter to solve this problem more effectively. Since the distribution of QW under the alternative hypothesis varies with the alternative value λ, to calculate the power by Imhof, Davies or Shively et al. numerical procedure, we have to evaluate improper integrals at each alternative value of λ. There are other approximation methods such as Pearson's moment approximation, Edgeworth type expansion or Monte Carlo method. Lugannani and Rice (1980) derived a saddle-point approximation (indirect Edgeworth expansion) which is superior to the Edgeworth expansion for the distribution of sum of independent random variables. Lieberman (1994) applied their result to give a saddle-point approximation formula for weighted sum of Chi-squares variables. Although Lieberman's saddle-point approximation avoids numerical integration, nonlinear equations have to be solved for the saddle points at each alternative value. Similarly Pearson's moment approximation and Monte Carlo method require to evaluate the power function at each alternative value, which is computationally intensive. In this study, we propose a methodology which needs to calculate the power only at a few alternative values to obtain an accurate analytic approximation to the entire power function. One of our main assumptions is that the power $P(\sum_{i=1}^{m} G_i W_i > 0)$ can be approximated by $P(\sum_{i=1}^{m-p} G_i W_i + \bar{G}\sum_{i=m-p+1}^{m} W_i > 0)$, where $\bar{G} = \frac{1}{p}\sum_{i=m-p+1}^{m} G_i$ is the average of the negative weights. The proposed methodology provides an efficient and accurate approximation. Several examples, such as the LBI test and Shively's (1988) special most powerful test for constancy of regression coefficients, are presented. The results of our approximation show excellent agreement with the exact power. Furthermore, the derived analytic power approximation can also be applied to compute the instantaneous increasing rates of the power functions in order to compare the local power of different tests.

This paper is organized as follows. In section 2, we introduce several models and test statistics that motivate this study. In section 3, we derive some theorems and propose approximate analytic forms of the power functions. In section 4, we present numerical results of the proposed method for several examples.

2 Models and Tests

In this section, we introduce the problem of testing for constancy of regression coefficients and of testing for a moving average unit root.

(I) Constancy of Regression Coefficients.

First, consider the time series regression model:

$$\begin{aligned} y_t &= \alpha_t + z_t\beta + \varepsilon_t \\ \alpha_t &= \alpha_{t-1} + a_t, \qquad t = 1, \cdots, n, \end{aligned} \tag{2.1}$$

where the $\varepsilon_t's$ are independent $N(0, \sigma^2)$, the $a_t's$ are independent $N(0, \lambda\sigma^2)$ $(\lambda \geq 0)$, ε_t and a_s are independent for all t and s, z_t is an independent regressor, α_t is a scalar variable and β is a scalar parameter. Without loss of generality, we assume $\sigma^2 = 1$ and α_0 to be unknown. The following two cases are considered,

$$\text{(i)} \quad z_t = 0, t = 1, 2, \cdots, n; \tag{2.2}$$

$$\text{(ii)} \quad z_t = t, t = 1, 2, \cdots, n. \tag{2.3}$$

Applications of Model (2.1) can be found in Jazwiniski (1970), Nicholls and Pagan (1985), Fama and Gibbons (1982), Brown, Kleidon and Marsh (1983) and Trzcinka

(1982). An important problem is to test whether α_t really exhibits variation, that is to test

$$H_0 : \lambda = 0 \ \ vs. \ \ H_1 : \lambda > 0. \tag{2.4}$$

A UMPI test does not exist for this problem. Thus Nabeya and Tanaka (1988) proposed a LBI test which rejects H_0 for large values of

$$R_n = \frac{\mathbf{y}' M V M \mathbf{y}}{\mathbf{y}' M \mathbf{y}}, \tag{2.5}$$

where $\mathbf{y} = (y_1, \cdots, y_n)^{'}$, $M = I_n - (\mathbf{1}, Z)((\mathbf{1}, Z)'(\mathbf{1}, Z))^{-1}(\mathbf{1}, Z)'$, I_n is the $n \times n$ identity matrix, $\mathbf{1} = (1, \cdots, 1)^{'}$, $Z = (z_1, \cdots, z_n)'$ and $V = (v_{i,j})$ with $v_{i.j} = min(i, j), 1 \leq i, j \leq n$. If $z_t = 0$ for $t = 1, 2, \cdots, n$, then $M = I_n - \mathbf{1}(\mathbf{1}^{'}\mathbf{1})^{-1}\mathbf{1}^{'}$.

On the other hand, Shively (1988) constructed an exact small-sample test for (2.4) which rejects H_0 for small values of

$$T(\lambda_1) = \frac{\mathbf{w}'\{P(I_n + \lambda_1 V)P'\}^{-1}\mathbf{w}}{\mathbf{w}'\mathbf{w}}, \tag{2.6}$$

where $\mathbf{w} = P\mathbf{y}$, P is an $m \times n$ matrix, where $m = n - 1$ for Model (2.2) and $m = n - 2$ for Model (2.3), such that $PP^{'} = I_m$ and $P^{'}P = M$ in (2.5) and λ_1 is chosen such that the power of $T(\lambda_1)$ at λ_1 is 0.5 at the level 0.05. Shively showed numerically that the test is approximately UMPI and has better power than the LBI test for Model (2.2) when $\lambda > 0.01$. For brevity, we will denote Shively's test by SMP.

The power of the LBI test at λ with critical value c is

$$\begin{aligned} P_\lambda(R_n > c) &= P_\lambda(\frac{\mathbf{w}'(PVP')\mathbf{w}}{\mathbf{w}'\mathbf{w}} > c) \\ &= P(\sum_{i=1}^{m} L_i(\lambda)\xi_i^2 > 0),, \end{aligned} \tag{2.7}$$

where $\mathbf{w} = P\mathbf{y} \sim N(0, I_m + \lambda PVP'), L_i(\lambda) = (1 + \lambda d_i)(d_i - c),\ i = 1, 2, \cdots, m$ and $L_1 > \cdots > L_{m-p} > 0 > L_{m-p+1} > \cdots > L_m$, $\xi_i's$ are i.i.d. $N(0,1)$ variables and $d_i's$ are the eigenvalues of PVP'. In Fig.1, we plot the eigenvalues $L_i's$ for Model (2.2) with $n = 31$. We observe the following three main characteristics of $L_i's$.

(a) The ratio $(-L_1/\bar{L})$ increases and becomes dominant as either n increases for fixed λ (see Table 1) or as λ increases for fixed n, where $\bar{L} = \frac{1}{p}\sum_{i=m-p+1}^{m} L_i$ the average of the negative weights.

(b) The number p of negative weights dominates the value of $(-L_1/\bar{L})$ and increases faster than $(-L_1/\bar{L})$ as n increases (see Table 1).

(c) The negative $L_i's$ are nearly equal with properties such as

$$\frac{s_3}{s_2} \simeq -\bar{L} \quad \text{and} \quad \frac{s_1 s_2}{s_3} \simeq \frac{s_2^3}{s_3^2} \simeq p \quad \text{where} \quad s_k = \sum_{i=m-p+1}^{m} (-L_i)^k, k = 1, 2, 3.$$

Similarly, the negative eigenvalues of LBI test for Model (2.3) exhibit these three characteristics, see Table 2 for the values of $(-L_1/\bar{L})$ and p. The power of SMP test at λ with critical value c is

$$P_\lambda\left(\frac{\mathbf{w}'(I_m + \lambda_1 PVP')^{-1}\mathbf{w}}{\mathbf{w}'\mathbf{w}} < c\right) = P\left(\sum_{i=1}^{m} O_i(\lambda)\xi_i^2 > 0\right), \tag{2.8}$$

where $O_i(\lambda) = (1 + \lambda d_i)(c - \frac{1}{1+\lambda_1 d_i}),\ i = 1, 2, \cdots, m$, $\mathbf{w}$, $\xi_i's$ and d_i's are defined as in (2.7). The negative eigenvalues O_i's also possess characteristics (a)-(c) for Model (2.2) and Model (2.3), see Tables 1 and 2 for the values of p and $(-O_1/\bar{O})$.

(II) Moving Average Unit Root.

We can consider two models for the moving average unit root testing problem.

(i) Pure MA(1) model:

$$y_t = \varepsilon_t - \rho\varepsilon_{t-1}, \quad t = 1, 2, \cdots, n, \tag{2.9}$$

where $|\rho| \leq 1$, the $\varepsilon_t{}'s$ are independent $N(0, \sigma^2)$ $r.v.'s$. The problem is to test

$$H_0 : \rho = 1 \quad v.s. \quad H_1 : |\rho| < 1. \tag{2.10}$$

Tanaka (1990) suggested a score type test which rejects H_0 for large values of

$$S_n = \frac{1}{n}\frac{\mathbf{y}'\Sigma_1^{-2}\mathbf{y}}{\mathbf{y}'\Sigma_1^{-1}\mathbf{y}}, \tag{2.11}$$

where Σ_1 is the covariance matrix of $\mathbf{y} = (y_1, \cdots, y_n)'$ under H_0. Note that this test is unbiased and LBI. Without loss of generality, we assume $\sigma^2 = 1$. By $Cor.$1 of Tanaka (p.436), we have

$$P_\rho(S_n > c) = P(\sum_{i=1}^{n} Q_i(\rho)\xi_i^2 > 0) \ , \tag{2.12}$$

where $\{\xi_i\} \sim NID(0, 1)$ and $Q_i(\rho) = ((1-\rho)^2 + \rho\eta_i)(\frac{1}{n\eta_i^2} - \frac{c}{\eta_i})$ with $\eta_i = 4sin^2\frac{i\pi}{2(n+1)}$. The negative eigenvalues $Q_i's$ also possess the characteristcs (a)-(c), see Table 3.

(ii) ARIMA model:

$$\begin{aligned} y_1 &= \mu + e_1 \\ \Delta y_i &= e_i - \rho e_{i-1}, \\ \phi(B)e_i &= \beta(B)\varepsilon_i, \qquad i = 1, \cdots, n \ , \end{aligned}$$

where Δ is the difference operator, μ and ρ are fixed parameters, $\varepsilon_i \sim NID(0, \sigma^2)$, $\phi(B) = 1 + \alpha_1 B + \cdots + \alpha_p B^p$ and $\beta(B) = 1 + \beta_1 B + \cdots + \beta_q B^q$ are polynomials in the backward shift operator B such that the zeros of $\phi(B)$ and $\beta(B)$ lie outside the unit circle. We assume that $\phi(B)$ and $\beta(B)$ have no common zeros and either $\alpha_p \neq 0$ or $\beta_q \neq 0$. The covariance matrix of the error vector $\mathbf{e} = (e_1, \cdots, e_n)'$ is $\sigma^2\Sigma_2$, where Σ_2 is a known function of the parameter vector $\tau = (\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)'$, and let

$\Omega(\rho)$ denote the covariance matrix of $\mathbf{y} = (y_1, \cdots, y_n)'$. We wish to test $H_0 : \rho = 1$ against $H_1 : |\rho| < 1$. When the value of μ is unknown, Saikkonen and Luukkonen (1993) derived a LBIU test that rejects H_0 for large values of the test statistic,

$$U_n = \frac{\hat{u}'\Sigma_2^{-1} D\Sigma_2 D'\Sigma_2^{-1}\hat{u}}{n\hat{u}'\Sigma_2^{-1}\hat{u}} \ , \tag{2.13}$$

where $\hat{u} = (I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})\mathbf{y}$ is the generalized least square residuals obtained under H_0, and $D = (d_{ij})$ is a lower triangular matrix with 1s on all diagonals on and below the main diagonal, that is d_{ij} is 1 for $i \geq j$ and 0 elsewhere. Denote the eigenvalues of $(I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})$ $(\Sigma_2^{-1} D\Sigma_2 D'\Sigma_2^{-1} - cn\Sigma_2^{-1})$ $(I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})$ $\Omega(\rho)$ by $SL_i's$, where c is the critical value of U_n. The negative eigenvalues SL_i's also possess the characteristics (a)-(c), see Table 4.

3 Theorems

In section 2, we introduce several hypothesis testing problems such as $H_0 : \lambda = 0 \ \ vs. \ \ H_1 : \lambda > 0$ or $H_0 : \rho = 1 \ \ vs. \ \ H_1 : |\rho| < 1$ with test statistics that can be expressed as ratios of quadratic forms. To simplify notation, we denote the parameter of interest by λ through this section. The power of a test statistics at λ is given by $P(QW > 0)$, where $QW = \sum_{i=1}^{m} G_i(\lambda) W_i$, $W_i's$ are independent $\chi^2(1)$ random variables and $G_1(\lambda) \geq \cdots \geq G_{m-p}(\lambda) > 0 > G_{m-p+1}(\lambda) \geq \cdots \geq G_m(\lambda)$. For simplicity, we write G_i for $G_i(\lambda)$ for $i = 1, \cdots, m$ hereinafter. Recall from the last section that the negative weights $G_i's$ possess the characteristic (c). For the extreme case, when all the negative $G_i's$ equal $\bar{G} = \frac{1}{p}\sum_{i=m-p+1}^{m} G_i$, we have $\sum_{i=m-p+1}^{m} G_i W_i = \bar{G} W$, where $W = \sum_{i=m-p+1}^{m} W_i \sim \chi^2(p)$ so that $P(QW > 0) = P(\sum_{i=1}^{m-p} G_i W_i + \bar{G} W > 0)$. When the negative G_i's are nearly equal, it seems reasonable to approximate $P(QW > 0)$ by

$$P(\sum_{i=1}^{m-p} G_i W_i + \bar{G} W > 0). \tag{3.1}$$

The accuracy of the approximation (3.1) is subject to an empirical investigation in the following. We compute

$$\max_{\lambda \in A} |P(QW(\lambda) > 0) - P(G_1(\lambda)W_1 + \cdots + G_{m-p}(\lambda)W_{m-p} > -\bar{G}(\lambda)W)|, \tag{3.2}$$

by Imhof's procedure, where $A = (0, \lambda_{\max})$ and

$$\lambda_{\max} = \inf\{\lambda : \lim_{\lambda^* \to \infty} P(QW(\lambda^*) > 0) - P(QW(\lambda) > 0) < 0.01\}$$

for LBI and SMP tests for Model (2.2) and Model (2.3). For each model, the set A changes as the sample size n changes, e.g. in Table 1, $A = (0, 5)$ for $n = 31; A = (0, 1)$ for $n = 51; A = (0, 0.2)$ for $n = 101$ and $A = (0, 0.05)$ for $n = 201$ for the LBI test. The maximum discrepancy for the score test for Model (2.9) and LBIU test for Model (4.1) and Model (4.2) are also computed for $\lambda \in A$. The results are listed in Tables 1, 2, 3 and 4, respectively. As a benchmark, we regard the power computed by Imhof's procedure as the true values. The discrepancy between the computed values using Imhof's procedure and using another approximation method is regarded as the error of the latter method. Since all the maximum absolute errors of (3.1) are relatively small (compared with Monte Carlo method), it seems reasonable to regard (3.1) as an accurate approximation for these cases. We now turn to deriving an analytic approximation for the probability

$$P(G_1 W_1 + \cdots + G_{m-p} W_{m-p} > -GW),$$

where $G_1 \geq G_2 \geq \cdots > G_{m-p} > 0 > G$ and W is a $\chi^2(p)$ random variable independent of the $\chi^2(1)$ variables $W_i's$, $1 \leq i \leq m - p$.

Lemma 3.1 *Let $W_1 \sim \chi^2(1)$ and $W \sim \chi^2(p)$ be independent random variables, where p is a positive integer. Let $U_1 = G_1W_1 + GW$ with $G_1 > 0 > G$. Then*

(i) $P(U_1 > 0) = \frac{2}{B(\frac{p}{2},\frac{1}{2})} \int_0^{\theta_1} sin^{p-1}\theta d\theta$, *where* $B(\frac{p}{2},\frac{1}{2}) = \Gamma(\frac{p}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{p+1}{2})$ *and* $\theta_1 = tan^{-1}\sqrt{\frac{G_1}{-G}}$.

(ii) If p is even, then the p.d.f. of U_1 is, for $u_1 < 0$,

$$f_{U_1}(u_1) = \sum_{j=0}^{k} A_j \cdot (-u_1)^{k-j} e^{\frac{u_1}{-2G}},$$

where

$$A_j = \frac{\Gamma(j+\frac{1}{2})}{j!(k-j)!\sqrt{\pi}} \cdot \frac{cos\theta_1}{(-2G)^{k-j+1}} \cdot sin^{2j}\theta_1$$

and $k = \frac{p}{2} - 1$.

proof:

(i) Since $\frac{W/p}{W_1}$ has a $F_{p,1}$ distribution, the probability $P(U_1 > 0)$ is equal to

$$F_{p,1}(\frac{G_1}{-G \cdot p}) = \frac{1}{B(\frac{p}{2},\frac{1}{2})} \int_0^{\frac{G_1}{-G}} \frac{x^{\frac{p}{2}-1}}{(1+x)^{\frac{p+1}{2}}} dx.$$

The proof of (i) is completed by taking the transformation $x = tan^2\theta$.

(ii) The joint p.d.f. of W_1 and U_1 is

$$f(w_1, u_1) = \frac{1}{-G} \cdot f_{W_1}(w_1) \cdot f_W(\frac{G_1w_1 - u_1}{-G}),$$

where f_{W_1} and f_W are the p.d.f.'s of $\chi^2(1)$ and $\chi^2(p)$ random variables, respectively. Since $W_1 > \max\{0, \frac{U_1}{G_1}\}$, the p.d.f. of U_1 is, for $u_1 < 0$

$$\begin{aligned} f_{U_1}(u_1) &= \int_0^\infty f(w_1, u_1) dw_1 \\ &= \frac{e^{\frac{u_1}{-2G}}}{(-2G)^{\frac{p}{2}}\Gamma(\frac{p}{2})\sqrt{2\pi}} \int_0^\infty w_1^{-\frac{1}{2}} (G_1w_1 - u_1)^{\frac{p}{2}-1} e^{-\frac{w_1}{2}(\frac{G_1-G}{-G})} dw_1. \end{aligned}$$

By taking the binomial expansion of $(G_1 w_1 - u_1)^{\frac{p}{2}-1}$, *the integral equals*

$$\sum_{j=0}^{k} \binom{k}{j} (-u_1)^{k-j} G_1^j \Gamma(j+\frac{1}{2})(\frac{-2G}{G_1 - G})^{j+\frac{1}{2}} \quad .$$

The result of (ii) is obtained by noticing $\sin\theta_1 = \sqrt{\frac{G_1}{G_1-G}}$ *and* $\cos\theta_1 = \sqrt{\frac{-G}{G_1-G}}$.

□

Remark 3.1 *(i) If* $p = 1$, *then* $P(G_1 W_1 + GW > 0) = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{G_1}{-G}}$.

(ii) If $(-G_1(\lambda)/G(\lambda))$ *is an increasing function of* λ *for* $\lambda \in A$ *(cf. (3.2)) then so does* $\theta_1(\lambda) = \tan^{-1}\sqrt{\frac{G_1}{-G}}$ *and the integral* $\int_0^{\theta_1(\lambda)} sin^{p-1}\theta d\theta$ *can be decomposed as*

$$\int_0^{\theta_1(0)} sin^{p-1}\theta d\theta + \int_{\theta_1(0)}^{\theta_1(\lambda)} sin^{p-1}\theta d\theta.$$

In some applications, the function $sin^{p-1}(\theta)$ *can be well approximated by a linear function of* θ *(say,* $a_1 + b_1\theta$*) for* $\theta_1(0) \le \theta \le \theta_1(\lambda_{max})$ *where* λ_{max} *is defined as in (3.2). Based on this linear approximation, we can approximate* $P(U_1 > 0)$ *by*

$$F_{app}(\lambda) = P_0 + a_1[\theta_1(\lambda) - \theta_1(0)] + b_1[\theta_1(\lambda) - \theta_1(0)]^2, \tag{3.3}$$

where a_1 *and* b_1 *satisfy*

$$\begin{cases} a_1(\theta_1(\lambda_1) - \theta_1(0)) + b_1(\theta_1(\lambda_1) - \theta_1(0))^2 = P_{\lambda_1} - P_0 \\ a_1(\theta_1(\lambda_{max}) - \theta_1(0)) + b_1(\theta_1(\lambda_{max}) - \theta_1(0))^2 = P_{\lambda_{max}} - P_0 \ , \end{cases}$$

and λ_1 *is either given by SMP test or chosen to satisfy* $P_{\lambda_1} = \frac{1}{2}P_{\lambda_{max}}$ *for other tests. (Here* P_λ *denotes the power at* λ*.)*

(iii) Notice that, when $u_1 < 0$, $f_{U_1}(u_1)$ *can be viewed as a weighted sum of Gamma p.d.f..* □

Theorem 3.1 *Let* $U_1 = G_1W_1 + GW$ *and* $U_2 = G_1W_1 + G_2W_2 + GW$, *where* $W_1 \sim \chi^2(1), W_2 \sim \chi^2(1)$, *and* $W \sim \chi^2(p)$ *are independent,* $G_1 > G_2 > 0 > G$. *If* p *is a positive even integer, then*

$$P(U_2 > 0) = P(U_1 > 0) + cos\theta_1 sin^{p-1}\theta_1 \sum_{j=0}^{k} H_j K_j, \tag{3.4}$$

where

$$K_j = \frac{2}{B(\frac{p}{2} - j, \frac{1}{2})} \int_0^{\theta_2} (\frac{sin\theta}{sin\theta_1})^{p-2j-1} d\theta,$$

$\theta_i = tan^{-1}\sqrt{\frac{G_i}{-G}}, i = 1, 2,\ H_j = \frac{\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}}$ *and* $k = \frac{p}{2} - 1$.

proof: *Note that* $P(U_2 > 0) = P(U_1 > 0) + P_\lambda(U_2 > 0 \ \ and \ \ U_1 < 0)$. *The first term on the right hand side is given by Lemma 3.1 (i). The second term equals*

$$\int_0^\infty f_{W_2}(w_2) \int_{-G_2w_2}^0 f_{U_1}(u_1) du_1 dw_2$$

$$= \sum_{j=0}^{k} A_j \int_0^\infty f_{W_2}(w_2) \int_{-G_2w_2}^0 (-u_1)^{k-j} e^{\frac{u_1}{-2G}} du_1 dw_2 \ . \tag{3.5}$$

The equality (3.5) follows from Lemma 3.1 (ii), where A_j *is defined in the same way. By letting* $y = \frac{u_1}{G}$, *the inner integral of (3.5) can be written as*

$$\int_0^{\frac{G_2}{-G}w_2} (-G)^{k-j+1} y^{k-j} e^{-\frac{y}{2}} dy = \Gamma(k-j+1)(-2G)^{k-j+1} \int_0^{\frac{G_2}{-G}w_2} f_Y(y) dy \tag{3.6}$$

where f_Y *is the p.d.f. of* $\chi^2(2(k-j+1))$. *Plugging (3.6) into (3.5), we have*

$$\int_0^\infty f_{W_2}(w_2) \int_0^{\frac{G_2}{-G}w_2} f_Y(y) dy dw_2 = P(G_2W_2 + GY > 0)$$

$$= \frac{2}{B(\frac{p}{2} - j, \frac{1}{2})} \int_0^{\theta_2} sin^{2(k-j)+1}\theta d\theta \ , \tag{3.7}$$

where $Y \sim \chi^2(2(k-j+1))$ *which is independent of* W_2, *and* $\theta_2 = tan^{-1}\sqrt{\frac{G_2}{-G}}$. *Finally, the result is obtained by plugging (3.7) into (3.5).* □

Remark 3.2 *(i) Similarly as in Remark 3.1 (ii), if we use linear functions (say $a_j^*+b_j^*\theta$) to approximate $(sin\theta)^{p-2j-1}$ for $j=0,1,\cdots,k$ and $\theta\in[\theta_2(0),\theta_2(\lambda_{max})]$, then (3.4) can be approximated by,*

$$F_{app}(\lambda)=P_0+a_1[\theta_1(\lambda)-\theta_1(0)]+b_1[\theta_1(\lambda)-\theta_1(0)]^2$$

$$+sin^{p-1}(\theta_1(\lambda))cos(\theta_1(\lambda))\{a_2[\theta_2(\lambda)-\theta_2(0)]+b_2[\theta_2(\lambda)-\theta_2(0)]^2\} \qquad (3.8)$$

where a_1,b_1,a_2 and b_2 satisfy

$$F_{app}(\lambda_i)=P_{\lambda_i},\quad i=1,2,3,max\ ,$$

where $(\lambda_1,P_{\lambda_1})$ and $(\lambda_{max},P_{\lambda_{max}})$ are chosen as suggested in Remark 3.1 (ii), and λ_2, λ_3 are chosen to satisfy $P_{\lambda_2}=\frac{1}{4}P_{\lambda_{max}}$ and $P_{\lambda_3}=\frac{3}{4}P_{\lambda_{max}}$.

(ii) Although, $F_{app}(\lambda)$ in (3.8) is derived for two positive $G_i's$, we found in some examples that (3.8) still provides good approximations when there are more than two positive G_i's.

We next extend the results to the case of more than two positive eigenvalues.

Lemma 3.2 *Let $W_i\sim\chi^2(1)$, $i=1,2,\cdots,m-p$, and $W\sim\chi^2(p)$ be independent, where p is a positive even integer, and $G_1\ge\cdots\ge G_{m-p}>0>G$. Let $U_{m-p}=\sum_{i=1}^{m-p}G_iW_i+GW$. Then the p.d.f. of U_{m-p} is, for $u_{m-p}<0$,*

$$f_{U_{m-p}}(u_{m-p})=\sum_{j_{m-p}=0}^{k}\sum_{j_{m-p-1}=0}^{k-j_{m-p}}\cdots\sum_{j_1=0}^{k-j_{m-p}-\cdots-j_2}A_{m-p}^*(\frac{1}{-G})(\frac{u_{m-p}}{-G})^{k-j_{m-p}-\cdots-j_1}e^{\frac{u_{m-p}}{-2G}},$$

where

$$A_{m-p}^*=K_{j_{m-p},\cdots,j_1}^*\prod_{t=1}^{m-p}[(\frac{-G}{G_t-G})^{\frac{1}{2}}\cdot(\frac{G_t}{G_t-G})^{j_t}],\quad\forall\ m-p\ge1\ ,$$

$j_i = 0$, $\forall\ i < 1$, $k = \frac{p}{2} - 1$ *and* $K^*_{j_{m-p},\cdots,j_1}$*'s are constants (not depending on the* G_i*'s).*

proof:

If $m - p = 1$*, the result follows by Lemma 3.1 (ii). If the result holds for* $m - p = q > 1$*, then the p.d.f. of* U_{q+1} *is, for* $u_{q+1} < 0$

$$\begin{aligned} f_{U_{q+1}}(u_{q+1}) &= \int_0^\infty f_{U_q}(u_{q+1} - G_{q+1}W_{q+1}) f_{W_{q+1}}(w_{q+1}) dw_{q+1} \\ &= \sum_{j_{q+1}=0}^{k} \sum_{j_q=0}^{k-j_{q+1}} \cdots \sum_{j_1=0}^{k-j_{q+1}-\cdots-j_2} A^*_{q+1}(\frac{1}{-G})(\frac{u_{q+1}}{-G})^{k-j_{q+1}-\cdots-j_1} e^{\frac{u_{q+1}}{-2G}}. \end{aligned}$$

The result is obtained by induction. $\Box$

Theorem 3.2 *Adopting the notations of Lemma 3.2, we have*

$P(U_{m-p} > 0) = P(U_{m-p-1} > 0) + P(U_{m-p} > 0$ *and* $U_{m-p-1} < 0)$*, where*

$P(U_{m-p} > 0$ *and* $U_{m-p-1} < 0)$

$= \sum_{j_{m-p-1}=0}^{k} \sum_{j_{m-p-2}=0}^{k-j_{m-p-1}} \cdots \sum_{j_1=0}^{k-j_{m-p-1}-\cdots-j_2} A^*_{m-p-1} \int_0^{\theta_{m-p}} (sin\theta)^{2(k+1-j_{m-p-1}-\cdots-j_1)-1} d\theta,$

and $\theta_{m-p} = tan^{-1}\sqrt{-G_{m-p}/G}$*.*

proof: *Since* $U_{m-p-1} \leq U_{m-p}$*, we have* $P(U_{m-p} > 0) = P(U_{m-p-1} > 0) + P(U_{m-p} > 0$ and $U_{m-p-1} < 0)$*. Furthermore, by Lemma 3.2, notice that when* $u_{m-p-1} < 0$*,* $f_{U_{m-p-1}}(u_{m-p-1})$ *can be expressed as a weighted sum of Gamma p.d.f.. Thus the result can be obtained by the same arguments as in the proof of Theorem 3.1.* $\Box$

Remark 3.3 *In the above theorems, in order to have the binomial expansion, we assume that p (the degree of freedom of W) is even. However, in the next section our simulation results show that (3.3) and (3.8) still provide good approximations when p is odd.*

4 Numerical Results

In this section, we present simulation results for several examples. By Remark 3.1(ii) and Remark 3.2 (i), we adopt the approximations $F_{app}(\lambda)$ (3.3) for $p = m - 1$ and $F_{app}(\lambda)$ (3.8) for $0 < p < m - 1$. In all cases, the critical value c of a test was chosen to correspond to the 5% level. We generated random variables and computed integrals by FORTRAN IMSL library on UNIX workstation.

Example 1. Constancy of Regression coefficients.

In Fig.2, the dots denote the power of LBI test computed by Imhof's method with sample size $n = 31$, and the line is the power computed by the proposed $F_{app}(\lambda)$ function. The maximum absolute errors of SMP and LBI tests with $n = 11, 31, 51, 101,$ and 201 are given in Table 1 for Model (2.2) and Table 2 for Model (2.3), in which the results of Monte Carlo method (the number of replication is 10,000 for each λ) and the method of (3.1) are also given. Obviously, (3.1) and $F_{app}(\lambda)$ give more accurate approximations than Monte Carlo method. Although as was expected, the precision of $F_{app}(\lambda)$ decreases as the number of positive eigenvalues increases, yet even in the worse case (when $n = 201$) its precision still kept to the second decimal point. Furthermore, the coefficients a_1, b_1, a_2, b_2 of $F_{app}(\lambda)$ of each model are given in Table 5 and Table 6.

We can also use $F_{App}(\lambda)$ to approximate the local power (in a neighbor of $\lambda = 0$) of SMP and LBI tests. When $m - p = 1$, the derivatives of $F_{app}(\lambda)$ of SMP and LBI tests at $\lambda = 0$ are

$$\frac{dF_{app}(\lambda)}{d\lambda}\bigg|_{\lambda=0} = a_1 \frac{d\theta_1(\lambda)}{d\lambda}\bigg|_{\lambda=0} + a_2 \frac{d\theta_2(\lambda)}{d\lambda}\bigg|_{\lambda=0} \cdot sin^{p-1}(\theta_1(0))cos(\theta_1(0)),$$

where $\theta_i(\lambda) = tan^{-1}\sqrt{\frac{O_i}{-O}}$, $i = 1, 2$, for SMP test and $\theta_i(\lambda) = tan^{-1}\sqrt{\frac{L_i}{-L}}$, $i = 1, 2$, for

LBI test, $L_i's$ and $O_i's$ are defined as in (2.7) and (2.8), respectively. When $n = 31$, the derivatives are 5.43 and 6.76 of SMP test and LBI test, respectively. This result not only confirms the fact that the power of LBI test is superior to SMP test in a small neighborhood of the null hypothesis, but also gives the instantaneous increasing rates of the power functions.

Example 2. Moving Average Unit Root.

The maximum absolute errors of the score type test with $n = 25, 50, 100$ are given in Table 3 and the coefficients of $F_{app}(\lambda)$ are in Table 7. Figure 3 plots the (approximate) power function by $F_{app}(\lambda)$ and by Imhof's method for $n = 100$. The maximum absolute errors of $LBIU$ tests of the ARIMA models for the following two cases,

$$e_i + 0.8e_{i-1} = \varepsilon_i, \tag{4.1}$$

and

$$e_i = \varepsilon_i + 0.8\varepsilon_{i-1}, \tag{4.2}$$

are given in Table 4 for $n = 100, \mu = 0$ and $\sigma^2 = 1$. Note that in this case, the maximum absolute errors are computed for ρ in $(-1, 1)$. We have similar conclusions as in Example 1.

5 Appendix

Table 1: The maximum absolute errors of the power of SMP and LBI tests by three different approximations under Model (2.2).

SMP test	n=11	n=31	n=51	n=101	n=201
p	8	25	44	91	187
$-O_1/\bar{O}$ at $\lambda = 0$	1.11	3.51	6.80	14.49	27.02
Monte-Carlo	0.0132	0.0196	0.0057	—	—
(3.1)	0.0044	0.0016	0.0006	0.0002	0.0001
$F_{app}(\lambda)$(3.3)	0.0017	0.0024	0.0050	0.0089	0.0106
LBI test	n=11	n=31	n=51	n=101	n=201
p	9	28	47	96	194
$-L_1/\bar{L}$ at $\lambda = 0$	1.59	6.41	10.75	23.26	45.45
Monte-Carlo	0.0084	0.0145	0.0001	—	—
(3.1)	0.0003	0.0080	0.0001	7.38×10^{-5}	2.47×10^{-5}
$F_{app}(\lambda)$(3.3)	0.0056	0.0027	0.0023	0.0056	0.0066

Table 2: The maximum absolute errors of the power of SMP and LBI tests by three different approximations under Model (2.3).

SMP test	n=11	n=31	n=51	n=101	n=201
p	7	23	41	87	180
$-O_1/\bar{O}$ at $\lambda = 0$	0.86	2.80	4.98	10.42	18.87
Monte-Carlo	0.0200	0.0204	0.0132	—	—
(3.1)	0.0068	0.0022	0.0013	0.0005	0.0002
$F_{app}(\lambda)$(3.8)	0.0007	0.0053	0.0139	0.0127	0.0125
LBI test	n=11	n=31	n=51	n=101	n=201
p	8	26	45	92	189
$-L_1/\bar{L}$ at $\lambda = 0$	1.31	5.08	8.85	18.18	37.04
Monte-Carlo	0.0210	0.0200	0.0120	—	—
(3.1)	0.0040	0.0008	0.0005	0.0002	6.92×10^{-5}
$F_{app}(\lambda)$(3.8)	0.0124	0.0010	0.0017	0.0050	0.0064

Table 3: The maximum absolute errors of the power of score-type test by three different approximations under Model (2.9).

Score-type test	n=25	n=50	n=100
p	23	47	96
$-Q_1/\bar{Q}$ at $\lambda = 0$	5.20	10.87	22.37
Monte-Carlo	0.0100	0.0081	0.0103
(3.1)	0.0025	0.0001	0.0001
$F_{app}(\lambda)$(3.8)	0.0048	0.0060	0.0138

Table 4: The maximum absolute errors of the power of LBIU tests by different approximations under Model (4.1) and Model (4.2) for n=100, $\mu = 0$ and $\sigma^2 = 1$.

LBIU test	Model (4.1)	Model (4.2)
p	95	95
$-SL_1/SL$ at $\rho = 1$	21.72	22.60
(3.1)	0.019	0.0196
$F_{app}(\lambda)(3.8)$	0.0043	0.0042

Table 5: The coefficients of $F_{app}(\lambda)(3.8)$ for Model (2.2).

SMP test	a_1	b_1	a_2	b_2
n=31, p=25	0.27	3.37	-0.06	15.80
n=51, p=44	0.36	6.10	0.26	31.72
n=101, p=91	0.9	10.1	-5.7	138.8
n=201, p=187	1.1	18.5	-11.8	406
LBI test	a_1	b_1	a_2	b_2
n=31, p=28	0.428	6.183	-1.009	1.626
n=51, p=47	0.593	9.735	-2.089	7.704
n=101, p=96	1.05	18.79	-5.16	35.06
n=201, p=194	1.4	36.6	-12.2	111.2

Table 6: The coefficients of $F_{app}(\lambda)$(3.8) for Model (2.3).

SMP test	a_1	b_1	a_2	b_2
n=31, p=23	7.08	-10.21	-21.47	36.08
n=51, p=41	11.60	-23.21	-45.09	94.19
n=101, p=87	15.3	-42.8	-83.7	257.6
n=201, p=180	18.7	-68.3	-143.9	650.5
LBI test	a_1	b_1	a_2	b_2
n=31, p=26	0.54	4.53	-1.64	10.81
n=51, p=45	0.75	7.20	-2.87	29.04
n=101, p=92	1.3	11.8	-6.8	109.8
n=201, p=189	1.8	23.6	-11.7	312.9

Table 7: The coefficients of $F_{app}(\lambda)$(3.8) for Model (2.9).

Score-type test	a_1	b_1	a_2	b_2
n=25 , p=23	0.395	5.121	-0.792	0.840
n=50 , p=47	0.66	9.27	-2.13	10.45
n=100, p=96	1.09	17.29	-5.76	45.15

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