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Solutions to Midterm

Problem A: (40 pts) Consider an asset which pays no dividends. Let P_t be its price at time t. Briefly answer the following questions:

1. What is the simple return R_t of holding the asset from time $t-1$ to time t?

   Answer: $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ or $R_t = \frac{P_t}{P_{t-1}} - 1$.

2. What is the continuously compounded (or log) return r_t of the asset from time $t-1$ to time t?

   Answer: $r_t = \ln(P_t) - \ln(P_{t-1})$.

3. What is the simple return of holding the asset from time $t-5$ to time t?

   Answer: $R_t(5) = \frac{P_t - P_{t-5}}{P_{t-5}}$ or $R_t(5) = (1+R_t)(1+R_{t-1})\cdots(1+R_{t-4}) - 1$.

4. Assume that t is measured in months, what is the annual log return of the asset from month $t-1$ to month $t+11$?

   Answer: $r_t(12) = \ln(P_{t+11}) - \ln(P_{t-1})$ or $r_t(12) = r_t + r_{t+1} + \cdots + r_{t+11}$.

5. Denote the log return by r_t. Assume that there are T observations available, namely $r_1, \cdots, r_T$. What is the lag-1 sample autocorrelation of r_t?

   Answer: $\hat{\rho}_1 = \frac{\sum_{t=2}^{T}(r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2}$, where $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$.

6. Describe a method to test the null hypothesis $H_0 : \rho_1 = 0$ vs $H_a : \rho_1 \neq 0$, where ρ_1 is the lag-1 autocorrelation of r_t. What is the test statistic? What is the reference distribution?

   Answer: The test statistic is the t-ratio $t = \frac{\hat{\rho}_1}{\sqrt{1/T}} = \sqrt{T}\hat{\rho}_1$. The reference distribution is $N(0,1)$, the standard normal distribution.

7. Assume that r_t has no serial correlations. Describe a method to test for conditional heteroscedasticity in r_t, i.e. test for ARCH effects. You may use the first m lags of autocorrelation involved. What is the test statistic? What is the reference distribution?

   Answer: Use the Ljung-Box statistics of the ACF of the squared return r_t^2. Let $Q(m) = T(T+2)\sum_{i=1}^{m}\frac{\hat{\rho}_i^2}{T-i}$, where $\hat{\rho}_i$ is the lag-i ACF of r_t^2. The reference distribution is chi-square with m degrees of freedom.

8. **For questions 8 to 12**, assume that the log return r_t follows the model

   $$r_t = 0.007 + 0.3 r_{t-1} + a_t,$$

   where $\{a_t\}$ is a white noise series with mean zero and variance σ^2. What is the expected value of the return r_t?

   Answer: $E(r_t) = \frac{0.007}{1-.3} = \frac{0.007}{0.7} = 0.01$.

9. If the variance of a_t is $\sigma^2 = 0.091$. What is the variance of r_t?

   Answer: $\text{var}(r_t) = \frac{\sigma^2}{1-\phi_1^2} = \frac{0.091}{1-0.09} = 0.1$.

10. What is the implication of the constant term 0.007 of the above AR(1) model on the log price $\ln(P_t)$?

    Answer: The constant term 0.007 implies that the expectation of r_t is non-zero. This in turn implies that $\ln(P_t)$ has a time-trend with slope related to 0.007.

11. Assume further that $r_{100} = -0.02$. Based on the above AR(1) model, what is the 1-step ahead forecast of r_{101} at the forecast origin $t = 100$?

    Answer: 1-step ahead forecast $r_{100}(1) = 0.007 + 0.3r_{100} = 0.007 - 0.006 = 0.001$.

12. What is the variance of forecast error of Question 11, assuming $\sigma^2 = 0.091$?

    Answer: Forecast error $= r_{101} - r_{100}(1) = a_{101}$. $\text{Var}(\text{error}) = \sigma^2 = 0.091$.

13. **For questions 13 to 15**, assume that the log return r_t follows the MA(2) model

$$r_t = 0.005 + a_t - 0.2a_{t-1} + 0.1a_{t-2},$$

    where $\{a_t\}$ is a white noise series with mean zero and variance σ^2. What is the expected value of r_t?

    Answer: $E(r_t) = 0.005$.

14. What is the variance of r_t, assuming that $\sigma^2 = 0.2$?

    Answer: $\text{Var}(r_t) = [1 + (-0.2)^2 + 0.1^2]\sigma^2 = 1.05 \times 0.2 = 0.21$.

15. Using the above MA(2) model, what is the 3-step ahead forecast of r_{103} at the forecast origin $t = 100$?

    Answer: From $r_{103} = 0.005 + a_{103} - 0.2a_{102} + 0.1a_{101}$, $r_{100}(3) = 0.005$.

16. Give a weakness of the GARCH models in modeling volatility.

    Answer: Assume symmetric response to positive and negative shocks. Or provide no new insight into the cause of volatility or not flexible (or too restrictive).

17. Give a possible reason that daily returns of stock market indexes tend to have significant lag-1 autocorrelation.

    Answer: Non=synchronous trading.

18. If you deposit US$100 in a bank that pays interests quarterly with a fixed annual interest rate of 6%, what is the total amount you would have two years later?

    Answer: $\$100(1 + \frac{0.06}{4})^8 \approx \$112.65$.

19. Assume that the continuously compounded interest rate is 6%. What is the present value of a zero-coupon bond with a face value of US$100 that matures in two years from now?

    Answer: $\$100e^{-2\times 0.06} \approx \$88.69$.

20. Consider an AR(2) model

$$x_t = 0.87x_{t-1} - 0.27x_{t-2} + a_t$$

    where a_t is a white noise with mean zero and variance $\sigma^2 > 0$. Does the model imply existence of a business cycle?

    Answer: Rewrite the model as $x_t - 0.87x_{t-1} - (-0.27)x_{t-2} = a_t$. We have $\phi_1^2 + 4\phi_2 = (0.87)^2 + 4(-0.27) = -0.3231 < 0$. Yes, there exists a business cycle.

Problem B. (20 pts) This problem is concerned with monthly log returns of IBM from January 1926 to December 1997 with 864 data points. SCA output is attached. Answer the following questions:

1. What is the average monthly return of the series?

   Answer: $0.0119 = 1.19\%$.

2. Do the data support the heavy-tails hypothesis at the 5% level? Why?

   Answer: Yes, the excess kurtosis is 2.04 with standard error 0.166. The t-ratio 2.04/0.166 is significant at the 5% level.

3. Do the data show significant skewness at the 5% level? Why?

   Answer: The t-ratio of the skewness is $-0.2202/0.0832 = -2.65$, which is significant at the 5% level.

4. (4 pts) Based on the sample autocorrelations, do the data have significant serial correlations at the 5% level? Why? [You may choose the number of ACF in your test.]

   Answer: Use lag-1 ACF. The t-ratio is $\sqrt{T}\hat{\rho}_1 = 2.35$ which is significant at the 5% level.

5. Use the sample ACF or PACF to identify a mean equation for the series?

   Answer: AR(1) or MA(1) model with a constant term.

6. (4 pts) Do the data show significant conditional heteroscedasticity at the 5% level? Why? You may use the first 5 autocorrelations involved.

   Answer: $Q(5) = 85.6$, which is highly significant. Therefore, there are ARCH effects at the 5% level.

Problem C. (20 pts) This problem is concerned with an AR(1)-GARCH(1,1) model for the monthly log returns of IBM from January 1926 to December 1997. RATS output is attached. Answer the following questions:

1. What is the fitted mean equation?

   Answer: $r_t = 0.0118 + 0.1095 r_{t-1} + a_t, \quad a_t = \sigma_e \epsilon_t.$

2. What is the fitted volatility equation?

   Answer: $\sigma_t^2 = 0.00080 + 0.1547 a_{t-1}^2 + 0.6639 \sigma_{t-1}^2.$

3. (4 pts) Is the fitted model adequate? Why? You may use the 5% significance level?

   Answer: $Q(12) = 5.38$ for the standardized residuals and $Q(10) = 1.87$ for the squared standardized residuals. Both Q-statistics have high p-values. Thus, the model is adequate.

4. In the mean equation, test the hypothesis that the coefficient of r_{t-1} is zero, using the 5% significance level. What is your conclusion?

   Answer: t-ratio $= 2.69$ with p-value 0.0071, which is less than 0.05. Significant at the 5% level.

5. Based on the fitted volatility equation, what is the unconditional variance of the shock a_t?

   Answer: $\text{var}(a_t) = \frac{0.0008}{1-.1547-.6639} \approx .00441.$

6. (4 pts) Based on the fitted mean equation, what is the expected value of the monthly log return? What is the expected annualized log return?

   Answer: Expected value $= \frac{0.0118}{1-.1095} \approx 0.01331$. Annualized expectation $= 0.01331 \times 12 \approx 15.97\%$.

Problem D. (20 pts) This problem is concerned with an AR(1)-EGARCH(1,0) model for the monthly log returns of IBM from January 1926 to December 1997. The fitted model is in the from

$$r_t = \mu + \phi_1 r_{t-1} + a_t, \quad a_t = \sigma_t \epsilon_t,$$

$$\ln(\sigma_t^2) = \alpha_0 + \frac{1}{1 - \alpha_1 B} g(\epsilon_{t-1}), \quad g(\epsilon_{t-1}) = \theta \epsilon_{t-1} + \gamma[|\epsilon_{t-1}| - 0.7979],$$

where ϵ_t is an independent sequence of Standard Gaussian random variates. Note that the notation μ in the mean equation denotes a constant term, not the mean of r_t. RATS output is attached. Answer the following questions:

1. (4 pts) What is the fitted volatility equation?

   Answer: $\ln(\sigma_t^2) = -5.495 + \frac{1}{1-0.8465B} g(\epsilon_{t-1})$, where $g(\epsilon_{t-1}) = -0.0799 \epsilon_{t-1} + 0.2719[|\epsilon_{t-1}| - 0.7979]$.

2. Is the parameter θ significant at the 5% level? Why?

   Answer: t-ratio $= -2.948$ with p-value 0.0032. Yes, it is significant at the 5% level.

3. Is the parameter ϕ_1 significant at the 5% level? Why?

   Answer: t-ratio $= 2.31$ with p-value 0.0209. Yes, it is significant at the 5% level.

4. The volatility equation can also be written as

$$\ln(\sigma_t^2) = (1-\alpha_1)\alpha_0 + \alpha_1 \ln(\sigma_{t-1}^2) + g(\epsilon_{t-1}).$$

   Assume that $\epsilon_{t-1} \geq 0$. What is the resulting volatility equation?

   Amswer: $\ln(\sigma_t^2) = -1.0602 + 0.8465 \ln(\sigma_{t-1}^2) + 0.192\epsilon_{t-1}$.

5. Assume that $\epsilon_{t-1} < 0$. What is the resulting volatility equation?

   Answer: $\ln(\sigma_t^2) = -1.0602 + 0.8465 \ln(\sigma_{t-1}^2) - 0.3518\epsilon_{t-1}$.

6. (4 pts) Let $\sigma_t^2(+)$ denotes the fitted volatility when $\epsilon_{t-1} > 0$ and $\sigma_t^2(-)$ denotes the fitted volatility when $\epsilon_{t-1} < 0$. Assume that the magnitude of ϵ_{t-1} is 1. What is $\frac{\sigma_t^2(-)}{\sigma_t^2(+)}$?

   Answer: $\frac{\sigma_t^2(-)}{\sigma_t^2(+)} = e^{.3518-.192} \approx 1.173$. That is, when the magnitude of ϵ_{t-1} is 1, the negative shock has about 17.3% more impact on the volatility at time t.