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Solutions to Midterm

Problem A: (40 pts) Let r_t be the log return, including dividends, of an asset at time t and R_t be the corresponding simple return.

1. What is the simple return R_t if the log return is $r_t = 0.04204$?

   A: $R_t = e^{r_t} - 1 = 0.04294$.

2. What is the log return r_t if the corresponding simple return is -0.042?

   A: $r_t = \ln(R_t + 1) = -0.04291$.

3. What are the weak stationarity conditions of the series $\{r_t\}$?

   A: The first two moments are time-invariant. That is, $E(r_t) = \mu$ and $\text{Cov}(r_t, r_{t-k}) = \gamma_k$.

4. Suppose that the data $\{r_t | t = 1, \cdots, T\}$ are available. Define the sample mean and sample variance of the data.

   A: $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$ and $s^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2$.

5. Describe a test statistic that can be used to test the hypothesis $H_o : \mu = 0$ vs $H_a : \mu \neq 0$, where $\mu = E(r_t)$ is the expectation of r_t. Assume that the return series is weakly stationary.

   A: $t = \frac{\bar{r}}{\text{std}(\bar{r})}$.

6. Describe a method to test the null hypothesis $H_0 : \rho_3 = 0$ vs $H_a : \rho_3 \neq 0$, where ρ_3 is the lag-3 autocorrelation of r_t. What is the test statistic? What is the reference distribution?

   A: $t = \frac{\hat{\rho}_3}{\text{std}(\hat{\rho}_3)}$ or approximately $t = \sqrt{T}\hat{\rho}_3$. The test statistic is asymptotically $N(0, 1)$.

7. **For questions 7 to 12**, assume that the log return r_t follows the model

   $$r_t = 0.01 + 0.2 r_{t-1} + a_t,$$

   where $\{a_t\}$ is a Gaussian white noise series with mean zero and variance σ^2. What is the expected value of the return r_t?

   A: $E(r_t) = \frac{0.01}{1-0.2} = 0.0125$.

8. Describe a method to test for conditional heteroscedasticity in r_t, i.e. test for ARCH effects. You may use the first m lags of autocorrelation involved. What is the test statistic? What is the reference distribution?

   A: The test statistic is $Q(m) = T(T+2)\sum_{i=1}^{m}\frac{\hat{\rho}_i^2}{T-i}$, where $\hat{\rho}_i$ is the lag-i sample ACF of a_t. $Q(m)$ follows asymptotically a chi-squared distribution with $m - 1$ degrees of freedom.

9. If the variance of a_t is $\sigma^2 = 0.096$. What is the variance of r_t?

   A: $\text{Var}(r_t) = \frac{\sigma^2}{1-\phi_1^2} = \frac{0.096}{1-0.2^2} = 0.1$.

10. What are the lag-1 and lag-2 autocorrelations of r_t?

    A: $\rho_1 = 0.2$ and $\rho_2 = 0.04$.

11. Assume further that $r_{100} = 0.04$. Based on the above AR(1) model, what is the 1-step ahead forecast of r_{101} at the forecast origin $t = 100$?

    A: $r_{100}(1) = 0.01 + 0.2 r_{100} = 0.01{+}0.2(0.04) = 0.018$.

12. What is the variance of forecast error of Question 11, assuming $\sigma^2 = 0.096$?

    A: Same as variance of $a_t = 0.096$.

13. **For questions 13 to 15**, assume that the log return r_t follows the MA(3) model

$$r_t = 0.004 + a_t - 0.2 a_{t-3},$$

    where $\{a_t\}$ is a white noise series with mean zero and variance σ^2. What is the expected value of r_t?

    A: $E(r_t) = 0.004$.

14. What is the variance of r_t, assuming that $\sigma^2 = 0.2$?

    A: $\text{Var}(r_t) = \sigma^2(1 + \theta_3^2) = 0.2(1 + 0.2^2) = 0.208$.

15. Using the above MA(3) model, what is the 4-step ahead forecast of r_{104} at the forecast origin $t = 100$?

    A: $r_{100}(4) = 0.004$.

16. Give a reason that the Durbin-Watson statistic is not sufficient in detecting serial correlations in the residuals of a linear regression model.

    A: DW statistic only checks the lag-1 residual serial correlation.

17. Describe a situation under which a seasonal time series model is useful in finance.

    A: Forecasting quarterly earning per share of a company.

18. Write down an EGARCH(1,1) model for $r_t = \sigma_t \epsilon_t$, where the shock ϵ_t is a standard Gaussian white noise series.

    A: $\ln(\sigma_t^2) = \alpha_0 + \frac{(1-\beta_1 B)}{(1-\alpha_1 B)} g(\epsilon_{t-1})$, where $g(\epsilon_t) = \theta\epsilon_t - \gamma(|\epsilon_t| - 0.7979)$.

19. Give two advantages of EGARCH models over the GARCH models.

    A: (1) Asymmetric responses between positive and negative previous returns, (2) use log volatility to ensure positiveness.

20. What is the main reason for employing a mean equation for r_t before considering its volatility model?

    A: To remove serial correlations of the return series.

Problem B. (20 pts) This problem is concerned with monthly log returns of GE stock from January 1926 to December 1999 for 888 observations. The returns are in percent and the necessary SCA output is attached. Answer the following questions:

1. Is the average monthly log return significantly different from zero? Use a 5% test to answer the question.

   A: Yes, t-ratio of the sample mean is 3.99, significant at the 5% level.

2. What is the excess kurtosis of the data? What is the skewness measure of the data?

   A: Excess kurtosis = 5.35 and skewness = −0.2517.

3. Are the monthly log returns predictable? State the null hypothesis and you may use the 5% significance level.

   A: Yes. You may use some lags of ACF to do the test. For example, if you use $Q(12)$, then $H_0 : \rho_1 = \cdots = \rho_{12} = 0$ and $Q(12) = 36.4$, which is highly significant.

4. An AR(4) model is estimated for the log return series. Write down the fitted model.

   A: $r_t = 1.004 + 0.0709 r_{t-1} - 0.124 r_{t-3} + 0.07 r_{t-4} + a_t$.

5. (4 pts) Is the fitted AR(4) model adequate? You may answer the question using 12 lags of the residual autocorrelations and the 5% significance level. Why is the chi-squared distribution used has $m - 3$ degrees of freedom?

   A: $Q(12) = 16.6$ with p-value 0.055. Cannot reject the null hypothesis that the residuals have no serial correlations. $Q(12)$ is asymptotically a chi-squared distribution with 9 degrees of freedom, because there are 3 AR coefficients in the fitted model.

6. (4 pts) Do the monthly log return show any ARCH effects? Why? You should provide the test statistic you use to draw the conclusion.

   A: Yes. You may select some $Q(m)$ statistic for the squared residuals. For instance, $Q(12) = 433$, which is highly significant.

Problem C. (20 pts) This problem is concerned with GARCH(1,1) models for the monthly log returns of GE stock from January 1926 to December 1999. The RATS output is attached. Answer the following questions:

1. Use a Gaussian AR(3)-GARCH(1,1) model for the data. What are the fitted mean and volatility equations?

   A: $r_t = 1.343 - 0.071 r_{t-3} + a_t$ and $\sigma_t^2 = 2.182 + 0.107 a_{t-1}^2 + 0.857 \sigma_{t-1}^2$.

2. (4 pts) Is the fitted model adequate? State the null hypotheses used for checking the mean and volatility equations. What are the test statistics? You may use the 5% significance level?

   A: For the mean equation, $H_0 = \rho_1 = \cdots = \rho_{10} = 0$, where ρ_i is the lag-i ACF of standardized residuals. The test statistic is $Q(10) = 5.57$ with p-value 0.85. Cannot reject the null hypothesis.

   For For the volatility equation, $H_0 = \rho_1 = \cdots = \rho_{10} = 0$, where ρ_i is the lag-i ACF of the squared standardized residuals. The test statistic is $Q(10) = 4.78$ with p-value 0.91. Cannot reject the null hypothesis.

3. What is the unconditional variance of the shocks as implied by the fitted volatility model?

   A: $\frac{2.182}{1-0.107-0.857} = 54.95$.

4. (4 pts) Compute 1-step ahead mean and volatility forecasts of the model at the forecast origin $t = 888$.

   A: $r_{888}(1) = 1.343 - 0.071 r_{886} = 1.343 - 0.071(13.353) = 0.395$

   $\sigma^2_{888}(1) = 2.182 + 0.107(16.66)^2 + 0.857(52.85) = 77.13$.

5. If a Student-t distribution with 5 degrees of freedom is used, what are the fitted mean and volatility equations?

   A: $r_t = 1.216 + a_t$ and $\sigma_t^2 = 3.844 + 0.1166 a_{t-1}^2 + 0.832\sigma_{t-1}^2$.

6. What is the unconditional variance of the stocks as implied by the fitted volatility model in question 5?

   A: $\frac{3.844}{1-0.1166-0.832} = 74.79$.

Problem D. (20 pts) This problem is concerned with an EGARCH(1,0)-M model for the monthly log returns of GE stock from January 1926 to December 1999. The fitted model is in the from

$$r_t = g_0\sigma_t^2 + a_t, \quad a_t = \sigma_t\epsilon_t,$$

$$\ln(\sigma_t^2) = \alpha_0 + \frac{1}{1-\alpha_1 B} g(\epsilon_{t-1}), \quad g(\epsilon_{t-1}) = \theta\epsilon_{t-1} + \gamma[|\epsilon_{t-1}| - 0.7979],$$

where ϵ_t is an independent sequence of Standard Gaussian random variates. The RATS output is attached. Answer the following questions:

1. (4 pts) What is the fitted volatility equation?

   A: $\ln(\sigma_t^2) = 3.984 + \frac{1}{1-0.892B} g(\epsilon_{t-1})$, where $g(\epsilon_t) = -0.0884\epsilon_t + 0.2447(|\epsilon_t| - 0.7979)$.

2. Is the model adequate at the 5% significance level? Why?

   A: Yes. Use the $Q(m)$ statistics of the standardized residuals and the squared standardized residuals. $Q(10) = 12.82$ with p-value 0.23 for the mean equation and $Q(10) = 2.54$ with p-value 0.99 for the volatility equation.

3. The volatility equation can also be written as

$$\ln(\sigma_t^2) = (1 - \alpha_1)\alpha_0 + \alpha_1 \ln(\sigma_{t-1}^2) + g(\epsilon_{t-1}).$$

Assume that $\epsilon_{t-1} \geq 0$. What is the resulting volatility equation?

A: $\ln(\sigma_t^2) = 0.43 + 0.892 \ln(\sigma_{t-1}^2) + (-0.0884\epsilon_{t-1} + 0.2447\epsilon_{t-1} - 0.1952) = 0.235 + 0.892 \ln(\sigma_{t-1}^2) + 0.1563\epsilon_{t-1}$.

4. Assume that $\epsilon_{t-1} < 0$. What is the resulting volatility equation?

A: $\ln(\sigma_t^2) = 0.235 + 0.892 \ln(\sigma_{t-1}^2) - 0.331\epsilon_{t-1}$.

5. Does the fitted model support the idea of risk premium at the 5% significance level? Why?

A: Yes. $r_t = 0.022\sigma_t^2 + a_t$, where the coefficient is significant at the 5% level.

6. (4 pts) Compute the 1-step ahead forecasts of the log return and its volatility at the forecast origin $t = 888$.

A: $\ln[\sigma_{888}^2(1)] = 0.235 + 0.892(3.955) + 0.1563\frac{16.45}{\sqrt{\exp(3.955)}} = 4.119$. Therefore, $\sigma_{888}^2(1) = 61.48$. (Note that $\epsilon_t = a_t/\sigma_t$.)

$r_{888}(1) = 0.022(61.48) = 1.353$.