# Financial Time Series Analysis

What is financial time series analysis?

Theory and practice of asset valuation over time.

Different from other T.S. analysis?

Not exactly, but with an added uncertainty.

For example, ever-changing business & economic environments and volatility is not directly observed.

## Objective of the course

- to provide some basic knowledge of financial time series
- to introduce some statistical tools & econometric models useful for analyzing these series.
- to gain empirical experience in analyzing financial T.S.
- to study methods for assessing market risk

## Examples of financial time series

1. Daily log returns of GE stock
2. Quarterly earnings of Johnson & Johnson

   Seasonal time series useful in

   - earning forecasts
   - pricing weather related derivatives (e.g. energy)
   - modeling intraday behavior of asset returns
3. US monthly interest rates

   Relations between the two series? Term structure of interest rates
4. Exchange rate between US Dollar vs Japanese Yen

   Fixed income, hedging

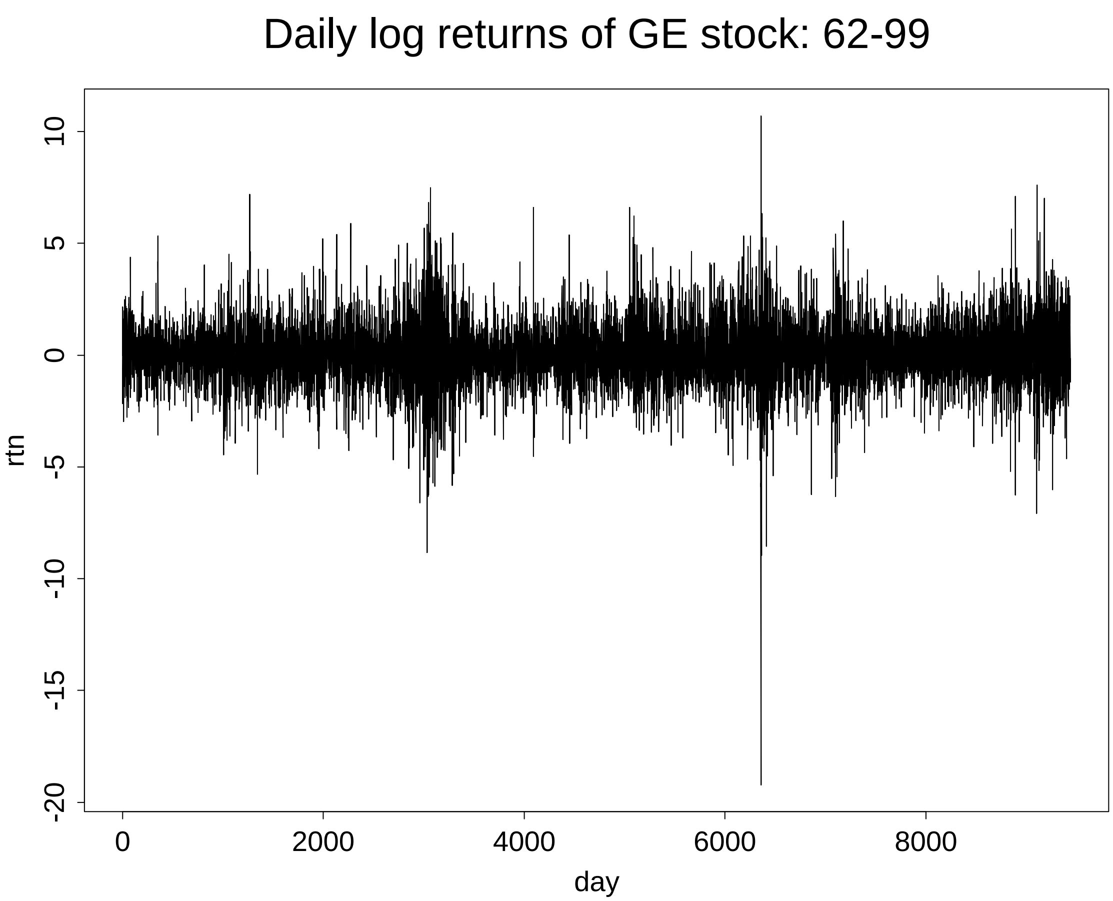


Figure 1: Dailey log returns of GE stock

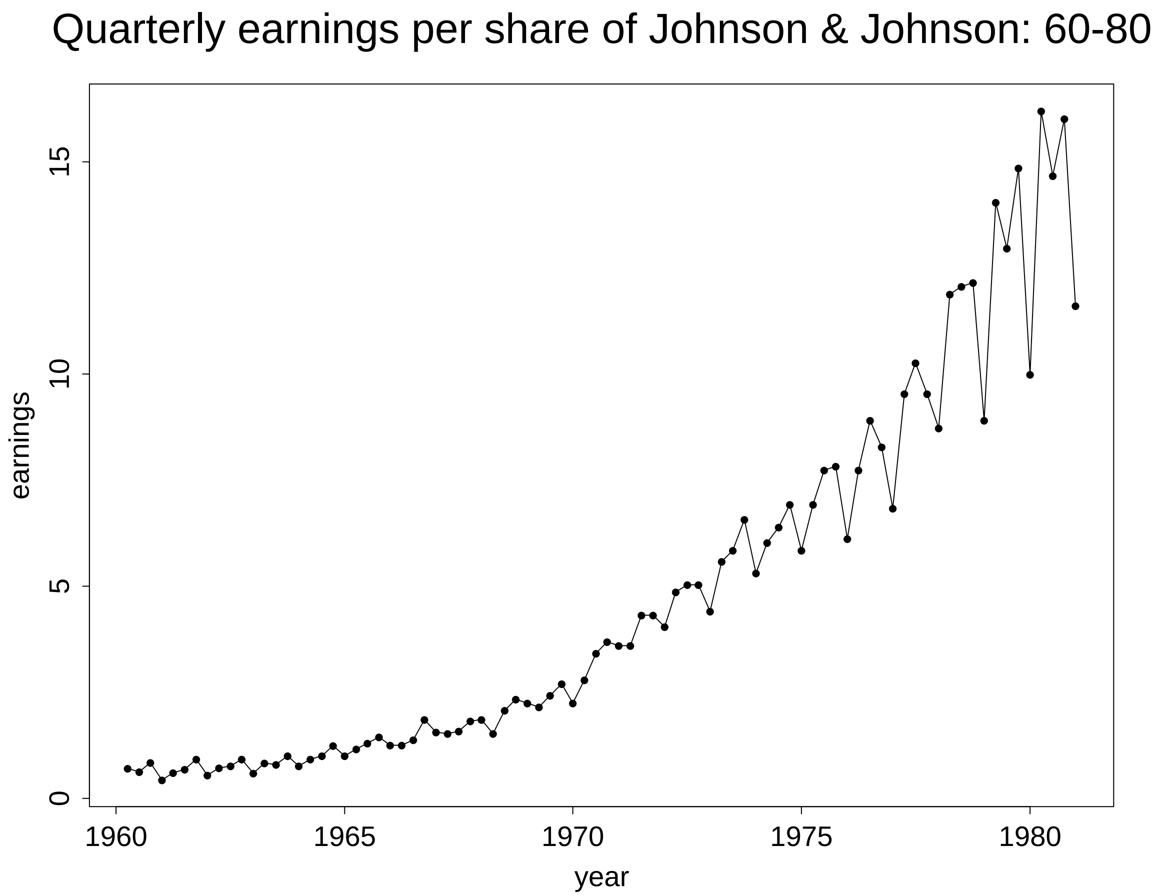


Figure 2: Quarterly earnings per share of Johnson and Johnson

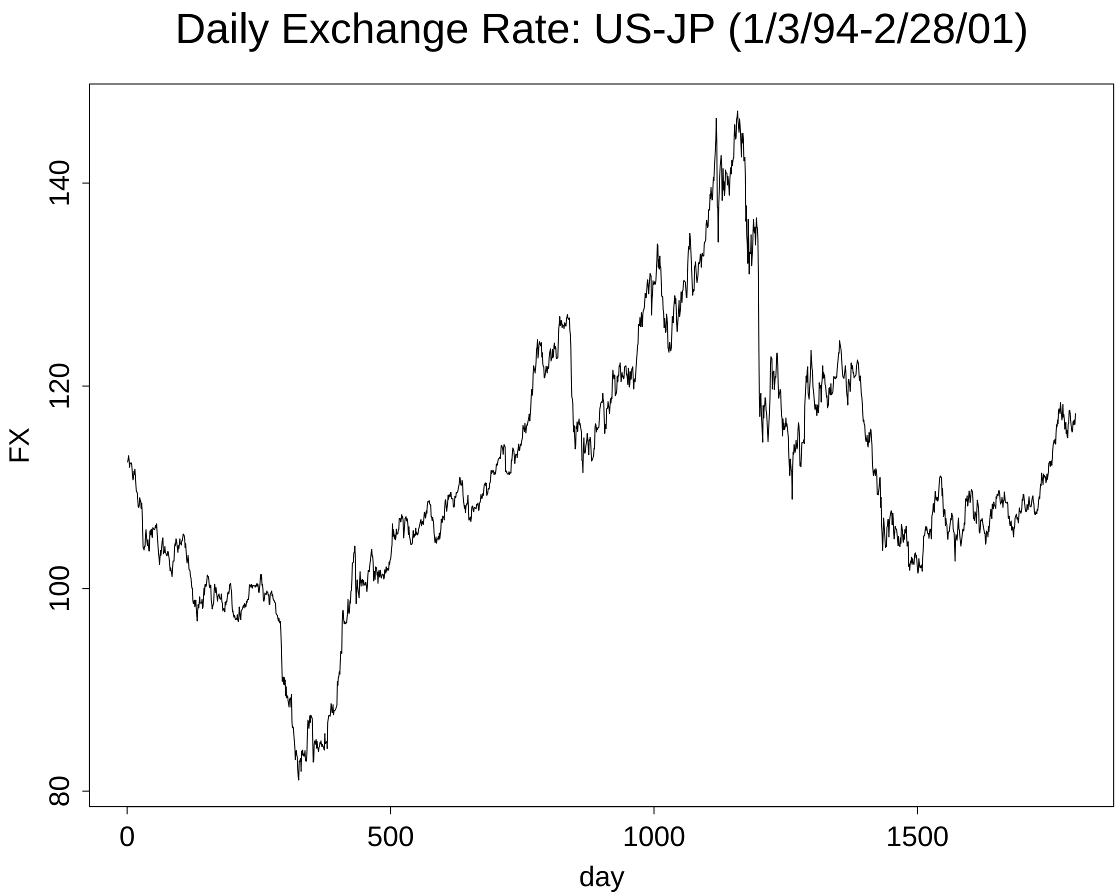


Figure 3: Daily Exchange Rate: Dollar vs Yen

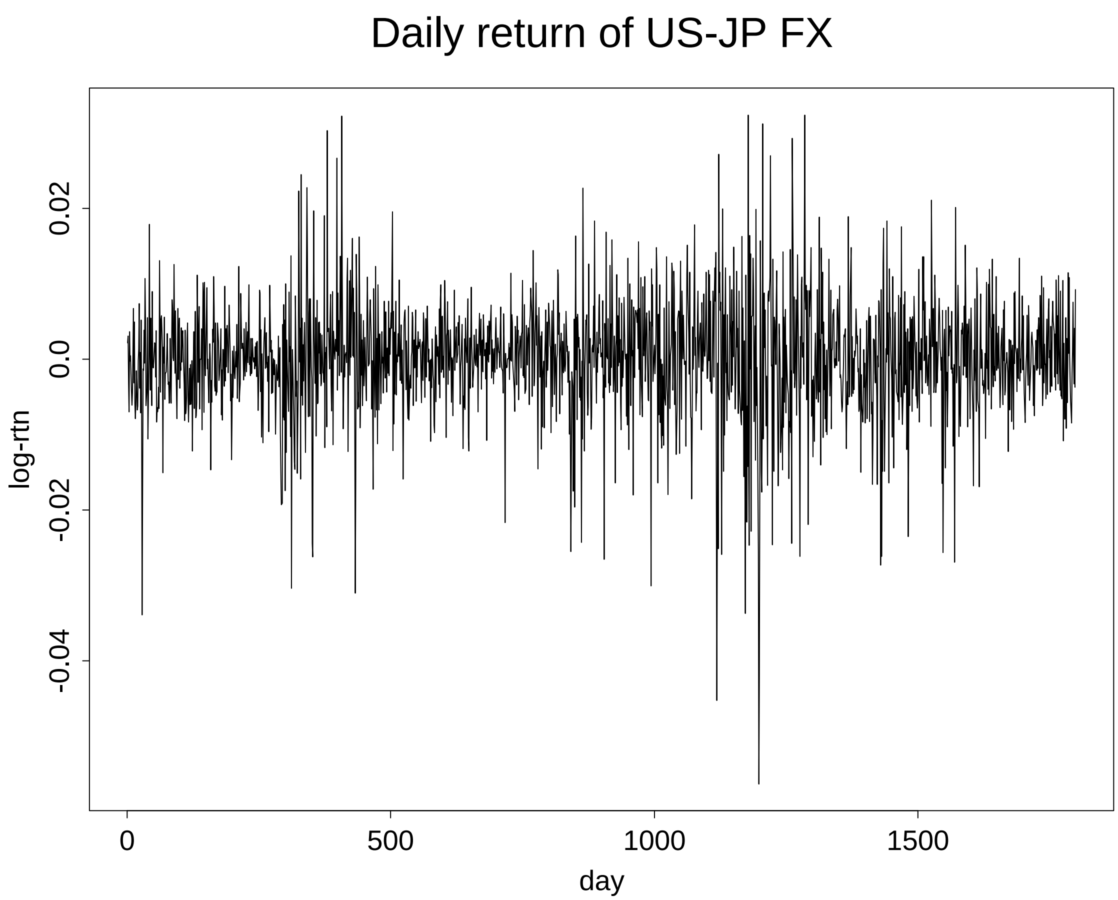


Figure 4: Daily log returns of FX (Dollar vs Yen)

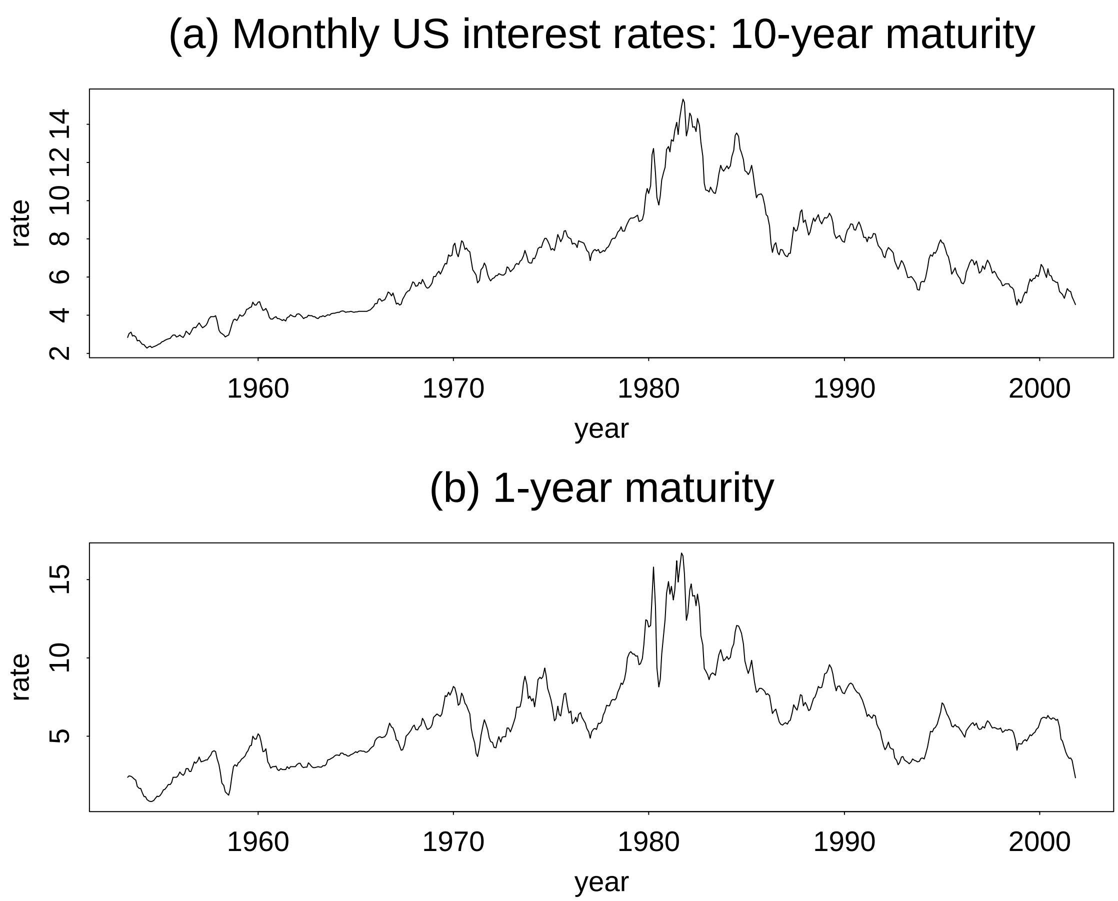


Figure 5: Monthly US interest rates

## Outline of the course

- Returns & their characteristics
- Simple linear time series models
- Univariate volatility modeling
- Nonlinearity in level and volatility
- Neural network
- High-frequency financial data and market micro-structure
- Continuous-time models and derivative pricing
- Value at Risk and extreme value theory
- Multivariate models: dynamic and cross dependence

## Asset Returns

Define $P_t$: price of an asset at time $t$.

One-period simple return: Gross return

$$1 + R_t = \frac{P_t}{P_{t-1}} \quad \text{or} \quad P_t = P_{t-1}(1 + R_t)$$

**Simple return**:

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Multiperiod simple return: Gross return

$$\begin{aligned} 1 + R_t(k) &= \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\ &= (1+R_t)(1+R_{t-1})\cdots(1+R_{t-k+1}). \end{aligned}$$

The $k$-period simple net return is $R_t(k) = \frac{P_t}{P_{t-k}} - 1$.

**Example**: Suppose the daily closing prices of a stock are

| Day | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Price | 37.84 | 38.49 | 37.12 | 37.60 | 36.30 |

1. What is the simple return from day 1 to day 2?

   Ans: $R_2 = \frac{38.49-37.84}{37.84} = 0.017.$

2. What is the simple return from day 1 to day 5?

   Ans: $R_5(4) = \frac{36.30-37.84}{37.84} = -0.041.$

3. Verify that $1 + R_5(4) = (1+R_2)(1+R_3)\cdots(1+R_5)$.

**Time interval** is important! Default is one year.

**Annualized (average) return**:

$$\text{Annualized}[R_t(k)] = \left[\prod_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k} - 1.$$

An approximation:

$$\text{Annualized}[R_t(k)] \approx \frac{1}{k}\sum_{j=0}^{k-1} R_{t-j}.$$

Continuously compounding: Illustration of the power of compounding (int. rate 10% per annum)

| Type | #(payment) | Int. | Net |
|---|---|---|---|
| Annual | 1 | 0.1 | $1.10000 |
| Semi-Annual | 2 | 0.05 | $1.10250 |
| Quarterly | 4 | 0.025 | $1.10381 |
| Monthly | 12 | 0.0083 | $1.10471 |
| Weekly | 52 | $\frac{0.1}{52}$ | $1.10506 |
| Daily | 365 | $\frac{0.1}{365}$ | $1.10516 |
| Continuously | $\infty$ | | $1.10517 |

$$A = C \exp[r \times n]$$

where $r$ is the interest rate per annum, $C$ is the initial capital, $n$ is the number of years, and exp is the exponential function.

**Present value**:

$$C = A \exp[-r \times n]$$

**Continuously compounded (or log) return**

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1},$$

where $p_t = \ln(P_t)$.

Multiperiod log return:

$$\begin{aligned} r_t(k) &= \ln[1 + R_t(k)] \\ &= \ln[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})] \\ &= \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

**Example** (continued). Use the previous daily prices.

1. What is the log return from day 1 to day 2?

A: $r_2 = \ln(38.49) - \ln(37.84) = 0.017$.

2. What is the log return from day 1 to day 5?

   A: $r_5(4) = \ln(36.3) - \ln(37.84) = -0.042$.

3. It is easy to verify $r_5(4) = r_2 + \cdots + r_5$.

Portfolio return: $N$ assets

$$R_{p,t} = \sum_{i=1}^{N} w_i R_{it}$$

**Example**: An investor holds stocks of IBM, Microsoft and Citi-Group. Assume that her capital allocation is 30%, 30% and 40%. Use the monthly simple returns in Table 1.2. What is the mean simple return of her stock portfolio?

**Answer**: $E(R_t) = 0.3 \times 1.42 + 0.3 \times 4.26 + 0.4 \times 2.55$ $= 2.72$.

Dividend payment:

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \quad r_t = \ln(P_t + D_t) - \ln(P_{t-1}).$$

Excess return: (adjusting for risk)

$$Z_t = R_t - R_{0t}, \quad z_t = r_t - r_{0t}$$

where $r_{0t}$ denotes the log return of a reference asset (e.g. risk-free interest rate).

Relationship:

$$r_t = \ln(1 + R_t), \quad R_t = e^{r_t} - 1.$$

If the returns are in **percentage**, then

$$r_t = 100 \times \ln(1 + \frac{R_t}{100}), \quad R_t = [\exp(r_t/100) - 1] \times 100.$$

Temporal aggregation of the returns produces

$$\begin{aligned} 1 + R_t(k) &= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}), \\ r_t(k) &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

These two relations are important in practice, e.g. obtain annual returns from monthly returns.

**Example**: If the monthly log returns of an asset are

4.46%, −7.34% and 10.77%, then what is the corresponding quarterly log return?

**Answer**: $4.46 - 7.34 + 10.77 = 7.89\%$.

**Example**: If the monthly simple returns of an asset are 4.46%, −7.34% and 10.77%, then what is the corresponding quarterly simple return?

**Answer**: $R = (1 + 0.0446)(1 - 0.0734)(1 + 0.1077) - 1$
$= 1.0721 - 1 = 0.0721 = 7.21\%$

## Distributional properties of returns

Key: What is the distribution of
$\{r_{it}; i = 1, \cdots, N; t = 1, \cdots, T\}$?

**Some theoretical properies**:

Moments of a r.v. $X$: $\ell$-th moment

$$m'_\ell = E(X^\ell) = \int_{-\infty}^{\infty} x^\ell f(x) dx$$

First moment: mean or expectation of $X$.

$\ell$-th central moment

$$m_\ell = E[(X-\mu_x)^\ell] = \int_{-\infty}^{\infty} (x-\mu_x)^\ell f(x)dx,$$

2nd c.m.: Variance of $X$.

Skewness (symmetry) and kurtosis (fat-tails)

$$S(x) = E\left[\frac{(X-\mu_x)^3}{\sigma_x^3}\right], \quad K(x) = E\left[\frac{(X-\mu_x)^4}{\sigma_x^4}\right].$$

$K(x)-3$: *Excess kurtosis.*

Why are mean and variance of returns important?

They are concerned with long-term return and risk, respectively.

Why is symmetry of interest in financial study?

Symmetry has important implications in holding short or long financial positions and in risk management.

Why is kurtosis important?

Related to volatility forecasting, efficiency in estimation and tests, etc.

High kurtosis implies heavy (or long) tails in distribution.

**Estimation**:

Data:$\{x_1, \cdots, x_T\}$

- sample mean:

$$\hat{\mu}_x = \frac{1}{T}\sum_{t=1}^{T} x_t,$$

- sample variance:

$$\hat{\sigma}_x^2 = \frac{1}{T}\sum_{t=1}^{T} (x_t - \hat{\mu}_x)^2,$$

- sample skewness:

$$\hat{S}(x) = \frac{1}{T\hat{\sigma}_x^3}\sum_{t=1}^{T} (x_t - \hat{\mu}_x)^3,$$

- sample kurtosis:

$$\hat{K}(x) = \frac{1}{T\hat{\sigma}_x^4}\sum_{t=1}^{T} (x_t - \hat{\mu}_x)^4.$$

Under normality assumption,

$$\hat{S}(x) \sim N(0, \frac{6}{T}), \quad \hat{K}(x) - 3 \sim N(0, \frac{24}{T}).$$

Some simple tests for normality (for large $T$).

1. Test for symmetry:

$$S^* = \frac{\hat{S}(x)}{\sqrt{6/T}} \sim N(0,1)$$

   if normality holds.

2. Test for tail thickness:

$$K^* = \frac{\hat{K}(x) - 3}{\sqrt{24/T}} \sim N(0,1)$$

   if normality holds.

3. A joint test (Jarque-Bera test):

$$JB = (K^*)^2 + (S^*)^2 \sim \chi^2_2$$

   if normality holds, where $\chi^2_2$ denotes a chi-squared distribution with 2 degrees of freedom.

## Empirical properties of returns

Data sources:

- CRSP: Center for Research in Security Prices
- Various web sites.
- Data sets of the textbook:

  http://www.gsb.uchicago.edu/fac/ruey.tsay/teaching/fts/

See Figures and tables of Chapter 1 for summary, including comparison between empirical dist and normal dist

Empirical dist tends to be skewed with heavy tails. It also has higher peak.

## Normal and lognormal dists

$Y$ is lognormal if $X = \ln(Y)$ is normal.

If $X \sim N(\mu, \sigma^2)$, then $Y = \exp(X)$ is lognormal with mean and variance

$$E(Y) = \exp(\mu + \frac{\sigma^2}{2}), \quad V(Y) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

Conversely, if $Y$ is lognormal with mean $\mu_y$ and variance $\sigma_y^2$, then $X = \ln(Y)$ is normal with mean and variance

$$E(X) = \ln\left[\frac{\mu_y}{\sqrt{1+\frac{\sigma_y^2}{\mu_y^2}}}\right], \quad V(X) = \ln\left[1+\frac{\sigma_y^2}{\mu_y^2}\right].$$

**Application**: If the log return of an asset is normally distributed with mean 0.0119 and standard deviation 0.0663, then what is the mean and standard deviation of its simple return?

**Answer**: Solve this problem in two steps.

**Step 1**: Based on the prior results, the mean and variance of $Y_t = \exp(r_t)$ are

$$E(Y) = \exp\left[0.0119 + \frac{0.0663^2}{2}\right] = 1.014$$

$$V(Y) = \exp(2\times 0.0119 + 0.0663^2)[\exp(0.0663^2) - 1] = 0.0045$$

**Step 2**: Simple return is $R_t = \exp(r_t) - 1 = Y_t - 1$. Therefore,

$$E(R) = E(Y) - 1 = 0.014$$

$V(R) = V(Y) = 0.0045, \quad \text{standard dev} = \sqrt{V(R)} = 0.067$

**Remark**: See the monthly IBM stock returns in Table 1.2.

## Processes considered

- return series (e.g., ch. 1, 2, 5)
- volatility processes (e.g., ch. 3, 4, 9)
- continuous-time processes (ch. 6)
- extreme events (ch. 7)
- multivariate series (ch. 8, 9)

## Likelihood function

Finally, it pays to study the likelihood function of returns $\{r_1, \cdots, r_T\}$ discussed in Chapter 1.

**Basic concept**:

Joint dist = Conditional dist × Marginal dist, i.e.

$$f(x, y) = f(x|y)f(y)$$

For two consecutive returns $r_1$ and $r_2$, we have

$$f(r_2, r_1) = f(r_2|r_1)f(r_1).$$

For three returns $r_1$, $r_2$ and $r_3$, by repeated application,

$$\begin{aligned} f(r_3, r_2, r_1) &= f(r_3|r_2, r_1)f(r_2, r_1) \\ &= f(r_3|r_2, r_1)f(r_2|r_1)f(r_1). \end{aligned}$$

In general, we have

$$\begin{aligned} & f(r_T, r_{T-1}, \cdots, r_2, r_1) \\ = \; & f(r_T|r_{T-1}, \cdots, r_1)f(r_{T-1}, \cdots, r_1) \\ = \; & f(r_T|r_{T-1}, \cdots, r_1)f(r_{T-1}|r_{t-2}, \cdots, r_1)f(r_{T-2}, \cdots, r_1) \\ = \; & \vdots \\ = \; & \left[\prod_{t=2}^{T} f(r_t|r_{t-1}, \cdots, r_1)\right] f(r_1), \end{aligned}$$

where $\Pi_{t=2}^{T}$ denotes product.

If $r_t|r_{t-1}, \cdots, r_1$ is normal with mean $\mu_t$ and variance $\sigma_t^2$, then likelihood function becomes

$$f(r_T, r_{T-1}, \cdots, r_1) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[\frac{-(r_t - \mu_t)^2}{2\sigma_t^2}\right] f(r_1).$$

For simplicity, if $f(r_1)$ is ignored, then the likelihood function becomes

$$f(r_T, r_{T-1}, \cdots, r_1) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[\frac{-(r_t - \mu_t)^2}{2\sigma_t^2}\right].$$

This is the *conditional* likelihood function of the returns under normality.

Other dists, e.g. Student-$t$, can be used to handle heavy tails.

## Model specification

- $\mu_t$: discussed in chapter 2
- $\sigma_t^2$: Chapetrs 3 and 4.