# Stochastic Diffusion & Option Pricing

**Stock Options**:

- A contract giving its holder the right, but not obligation, to trade shares of a common stock by a certain date for a specified price.
- Call option: to buy
- Put option: to sell
- Specified price: strike price $K$
- date: expiration $T$

**Factors** affecting the price of an option

- Current stock price: $P_t$
- time to expiration: $T - t$
- Risk-free interest rate: $r$ per annum
- Stock volatility: $\sigma$ annualized

**Payoff** for European options (exercised at $T$ only)

**Call option**:

$$V(P_T) = (P_T - K)_+ = \begin{cases} P_T - K & \text{if } P_T > K \\ 0 & \text{if } P_T \leq K \end{cases}$$

The holder only exercises her option if $P_T > K$ (buys the stock via exercising the option and sells the stock on the market).

**Put option**:

$$V(P_T) = (K - P_T)_+ = \begin{cases} K - P_T & \text{if } P_T < K \\ 0 & \text{if } P_T \geq K \end{cases}$$

The holder only exercises her option if $P_T < K$ (buys the stock from the market and sells it via option).

**Mathematical framework**

- Stock price follows a diffusion equation, i.e. a continuous-time continuous stochastic process
- In a complete market, use hedging to derive the price of an option (no arbitrage argument).

- In an incomplete market (e.g. existence of jumps), specify risk and a hedging strategy to minimize the risk.

**Stochastic processes**

- Wiener process (or Standard Brownian motion)
  - notation: $w_t$
  - initial value: $w_0 = 0$
  - small increments are independent and normal

$$\text{time poits:} \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$$

$$\{\Delta w_i = w_{t_i} - w_{t_{i-1}}\} \quad \text{are independent}$$

$$\Delta w_t = w_{t+\Delta t} - w_t \sim N(0, \Delta t).$$

  - property: $w_t \sim N(0, t)$
  - zero drift and rate of variance change is 1.
  - use the program "wiener" on GSBMBA to simulate some approximations of a Wiener process.

– To run “wiener”,

1. copy the program into your directory.
2. type “wiener” on GSBMBA.

   The program will ask for the number of data points and a seed. Type 5000 (say) and an arbitrary positive number. The output file is “fort.25”. You may plot the data using a software of your choice.

- Generalized Wiener process

$$dx_t = \mu dt + \sigma dw_t$$

Drift $\mu$ & rate of variance change $\sigma^2$.

- Ito’s process

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t$$

Drift and volatility are time-varying.

- Geometric Brownian motion

$$dP_t = \mu P_t dt + \sigma P_t dw_t$$

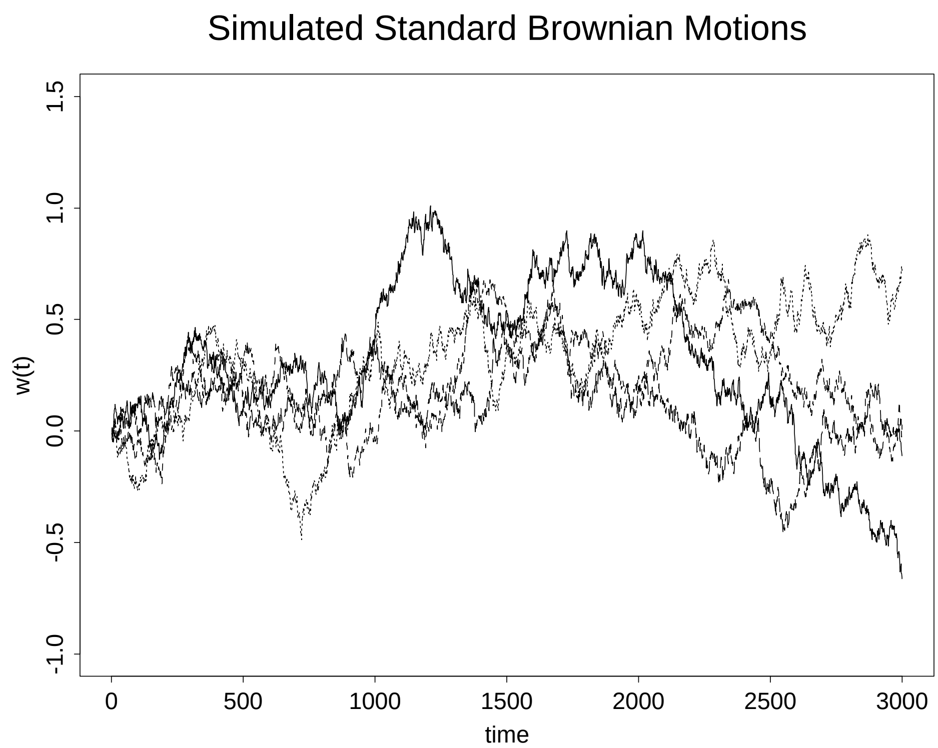


Figure 1: Dailey log returns of GE stock

**Illustration**: Four simulated standard Brownian motions. key feature: variability increases with time.

Assume that the price of a stock follows a geometric Brownian motion. What is the distribution of the log return? To answer this question, we need Ito's calculus.

Review of differentiation

$G(x)$: a differentiable function of $x$.

What is $dG(x)$?

Taylor expansion:

$$\Delta G \equiv G(x+\Delta x) - G(x) = \frac{\partial G}{\partial x}\Delta x$$
$$+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 G}{\partial x^3}(\Delta x)^3 + \cdots.$$

Letting $\Delta x \to 0$, we have

$$dG = \frac{\partial G}{\partial x}dx.$$

How about $G(x, y)$?

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y$$
$$+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(\Delta y)^2 + \cdots.$$

Taking limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy.$$

Stochastic differentiation

Now, consider $G(x_t, t)$ with $x_t$ an Ito's process.

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$$

$$+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2 + \cdots.$$

A discretized version of the Ito's process is

$$\Delta x = \mu_* \Delta t + \sigma_* \epsilon \sqrt{\Delta t},$$

where $\mu_* = \mu(x_t, t)$ and $\sigma_* = \sigma(x_t, t)$. Therefore,

$$(\Delta x)^2 = \mu_*^2 (\Delta t)^2 + \sigma_*^2 \epsilon^2 \Delta t + 2\mu_* \sigma_* \epsilon (\Delta t)^{3/2}$$

$$= \sigma_*^2 \epsilon^2 \Delta t + H(\Delta t).$$

Thus, $(\Delta x)^2$ contains a term of order $\Delta t$.

$$\begin{aligned} E(\sigma_*^2 \epsilon^2 \Delta t) &= \sigma_*^2 \Delta t, \\ \mathrm{Var}(\sigma_*^2 \epsilon^2 \Delta t) &= E[\sigma_*^4 \epsilon^4 (\Delta t)^2] - [E(\sigma_*^2 \epsilon^2 \Delta t)]^2 = 2\sigma_*^4 (\Delta t)^2, \end{aligned}$$

where we use $E(\epsilon^4) = 3$. These two properties show that

$$\sigma_*^2 \epsilon^2 \Delta t \to \sigma_*^2 \Delta t \quad \text{as} \quad \Delta t \to 0.$$

Consequently,

$$(\Delta x)^2 \to \sigma_*^2 dt \quad \text{as} \quad \Delta t \to 0.$$

Using this result, we have

$$\begin{aligned} dG &= \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma_*^2 dt \\ &= \left(\frac{\partial G}{\partial x}\mu_* + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma_*^2\right) dt + \frac{\partial G}{\partial x}\sigma_* dw_t. \end{aligned}$$

This is the well-known Ito's lemma.

**Example**. Let $G(w_t, t) = w_t^2$. What is $dG(w_t, t)$?

Answer: Here $\mu_* = 0$ and $\sigma_* = 1$.

$$\frac{\partial G}{\partial w_t} = 2w_t, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial w_t^2} = 2.$$

Therefore,

$$dw_t^2 = (2w_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1)dt + 2w_t dw_t = dt + 2w_t dw_t.$$

If $P_t$ follows a geometric Brownian motion, what is the model for $\ln(P_t)$?

Answer: Let $G(P_t, t) = \ln(P_t)$. we have

$$\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{1}{2}\frac{\partial^2 G}{\partial P_t^2} = \frac{1}{2}\frac{-1}{P_t^2}.$$

Consequently, via Ito's lemma, we obtain

$$d\ln(P_t) = \left(\frac{1}{P_t}\mu P_t + \frac{1}{2}\frac{-1}{P_t^2}\sigma^2 P_t^2\right)dt + \frac{1}{P_t}\sigma P_t dw_t$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dw_t.$$

Thus, $\ln(P_t)$ follows a generalized Wiener Process with drift rate $\mu - \sigma^2/2$ and variance rate $\sigma^2$.

The log return from $t$ to $T$ is normal with mean $(\mu - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$.

Estimation of $\mu$ and $\sigma$

Assume that $n$ log returns are available, say $\{r_t | t = 1, \cdots, n\}$.

**Statistical theory**:

Estimate the mean and variance by the sample mean and variance.

$$\bar{r} = \frac{\Sigma_{t=1}^n r_t}{n},$$

$$s_r^2 = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2.$$

**Remember the length of time intervals!**

Let $\Delta$ be the length of time intervals measured in years.

Then, the distribution of $r_t$ is

$$r_t \sim N[(\mu - \sigma^2/2)\Delta, \sigma^2\Delta].$$

We obtain the estimates

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}}.$$

$$\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}.$$

**Example**. Daily log returns of IBM stock in 1998.

The data show $\bar{r} = 0.002276$ and $s_r = 0.01915$.

Since $\Delta = 1/252$ year, we obtain that

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198.$$

Thus, the estimated expected return was 61.98% and the standard deviation was 30.4% per annum for IBM stock in 1998.

**Example**. Daily log returns of Cisco stock in 1999.

Data show $\bar{r} = 0.00332$ and $s_r = 0.026303$,

Also, $Q(12) = 10.8$. Therefore, we have

$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.026303}{\sqrt{1.0/252.0}} = 0.418, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.924.$

Expected return was 92.4% per annum

Estimated s.d. was 41.8% per annum.

**Example**. Daily log returns of Cisco stock in 2001.

Data show $\bar{r} = -0.00301$ and $s_r = 0.05192$.

Therefore, $\hat{\sigma} = 0.818$ $\hat{\mu} = -0.412$.

Time-varying nature of mean and volatility is clearly shown.

## Distributions of stock prices

If the price follows

$$dP_t = \mu P_t dt + \sigma P_t dw_t,$$

then,

$$\ln(P_T) - \ln(P_t) \sim N\left[(\mu - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right].$$

Consequently, given $P_t$,

$$\ln(P_T) \sim N\left[\ln(P_t) + (\mu - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right],$$

and we obtain (log-normal dist; ch. 1)

$$\begin{aligned} E(P_T) &= P_t \exp[\mu(T-t)], \\ \text{Var}(P_T) &= P_t^2 \exp[2\mu(T-t)]\{\exp[\sigma^2(T-t)] - 1\}. \end{aligned}$$

The result can be used to make inference about $P_T$.

Simulation is often used to study the behavior of $P_T$.

**Black-Scholes equation**

- Price of stock: $P_t$ is a Geo. B. Motion
- price of derivative: $G_t = G(P_t, t)$ contingent the stock
- Risk neutral world: expected returns are given by the risk-free interest rate (no arbitrage)

From Ito's lemma:

$$dG_t = \left(\frac{\partial G_t}{\partial P_t}\mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2\right) dt + \frac{\partial G_t}{\partial P_t}\sigma P_t dw_t.$$

A discretized version of the set-up:

$$\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta w_t,$$

$$\Delta G_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G_t}{\partial P_t} \sigma P_t \Delta w_t,$$

Consider the **Portfolio**:

- short on derivative
- long $\frac{\partial G_t}{\partial P_t}$ shares of the stock.

Value of the portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t.$$

The change in value is

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t.$$

by substitution, we have

$$\Delta V_t = \left( -\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t.$$

**No stochastic** component involved.

The protfolio must be riskless during a small time interval.

$$\Delta V_t = r V_t \Delta t$$

where $r$ is the risk-free interest rate. We then have

$$\left(\frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2\right)\Delta t = r\left(G_t - \frac{\partial G_t}{\partial P_t}P_t\right)\Delta t.$$

and

$$\frac{\partial G_t}{\partial t} + rP_t\frac{\partial G_t}{\partial P_t} + \frac{1}{2}\sigma^2 P_t^2\frac{\partial^2 G_t}{\partial P_t^2} = rG_t,$$

the Black-Scholes differential equ. for derivative pricing.

**Example**. A forward contract on a stock (no dividend). Here

$$G_t = P_t - K\exp[-r(T-t)]$$

where $K$ is the delivery price. We have

$$\frac{\partial G_t}{\partial t} = -rK\exp[-r(T-t)], \quad \frac{\partial G_t}{\partial P_t} = 1, \quad \frac{\partial^2 G_t}{\partial P_t^2} = 0.$$

Substituting these quantities into LHS yields

$$-rK\exp[-r(T-t)] + rP_t = r\{P_t - K\exp[-r(T-t)]\},$$

which equals RHS.

## Black-Scholes formulas

A European call option: expected payoff

$$E_*[\max(P_T - K, 0)]$$

Price of the call: (current value)

$$c_t = \exp[-r(T-t)]E_*[\max(P_T - K, 0)].$$

In a risk-neutral world, $\mu = r$ so that

$$\ln(P_T) \sim N\left[\ln(P_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right].$$

Let $g(P_T)$ be the pdf of $P_T$. Then,

$$c_t = \exp[-r(T-t)]\int_K^\infty (P_T - K)g(P_T)dP_T.$$

After some algebra (appendix)

$$c_t = P_t\Phi(h_+) - K\exp[-r(T-t)]\Phi(h_-)$$

where $\Phi(x)$ is the CDF of $N(0,1)$,

$$h_+ = \frac{\ln(P_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$h_- = \frac{\ln(P_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = h_+ - \sigma\sqrt{T-t}.$$

See Chapter 6 for some interpretations of the formula.

For put option:

$$p_t = K\exp[-r(T-t)]\Phi(-h_-) - P_t\Phi(-h_+).$$

Alternatively, use the put-call parity:

$$p_t - c_t = K\exp[-r(T-t)] - P_t.$$

**Example**. $P_t = \$80$. $\sigma = 20\%$ per annum. $r = 8\%$ per annum.

What is the price of a European call option with a strike price of \$90 that will expire in 3 months?

From the assumptions, we have $P_t = 80$, $K = 90$, $T - t = 0.25$, $\sigma = 0.2$ and $r = 0.08$. Therefore,

$$\begin{aligned} h_+ &= \frac{\ln(80/90) + (0.08 + 0.04/2) \times 0.25}{.2\sqrt{0.25}} = -0.9278 \\ h_- &= h_+ - .2\sqrt{.25} = -1.0278. \end{aligned}$$

It can be found

$$\Phi(-.9278) = 0.1767, \quad \Phi(-1.0278) = 0.1520.$$

Therefore,

$$c_t = \$80\Phi(-0.9278) - \$90\Phi(-1.0278)\exp(-0.02) = \$0.73.$$

The stock price has to rise by \$10.73 for the purchaser of the call option to break even.

If $K = \$85$, then

$$c_t = \$85\Phi(-0.356246) - \$85\exp(-0.02)\Phi(-0.456246) = \$1.86.$$

## A note on computer program

- Copy the program "black" on my directory.
- Check the web:

  http://home.online.no/~espehaug/SayBlackScholes.html

  which contains the formula for many computer languages.

## Stochastic integral

The formula

$$\int_0^t dx_s = x_t - x_0$$

continues hold. In particular,

$$\int_0^t dw_s = w_t - w_0 = w_t.$$

From

$$dw_t^2 = dt + 2w_t dw_t$$

we have

$$w_t^2 = t + 2\int_0^t w_s dw_s.$$

Therefore,

$$\int_0^t w_s dw_s = \frac{1}{2}(w_t^2 - t).$$

Different from $\int_0^t y dy = (y_t^2 - y_0^2)/2$.

Assume $x_t$ is a Geo. Brownian motion,

$$dx_t = \mu x_t dt + \sigma x_t dw_t.$$

Apply Ito's lemma to $G(x_t, t) = \ln(x_t)$, we obtain

$$d\ln(x_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dw_t.$$

Taking integration, we have

$$\int_0^t d\ln(x_s) = \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t ds + \sigma \int_0^t dw_s.$$

Consequently,

$$\ln(x_t) = \ln(x_0) + (\mu - \sigma^2/2)t + \sigma w_t,$$

and

$$x_t = x_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t].$$

Change $x_t$ to $P_t$. The price is

$$P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t].$$

**Jump diffusion**

Weaknesses of diffusion models:

- no volatility smile
- fail to capture effects of rare events (tails)

Modification: jump diffusion and stochastic volatility

Jumps are governed by a probaility law:

Poisson process: $X_t$ is a Poisson process if

$$Pr(X_t = m) = \frac{\lambda^m t^m}{m!} \exp(-\lambda t), \quad \lambda > 0.$$

Use a special jump diffusion model by Kou (2000).

$$\frac{dP_t}{P_t} = \mu dt + \sigma dw_t + d\left(\sum_{i=1}^{n_t}(J_i - 1)\right),$$

- $w_t$: a Wiener process,
- $n_t$: a Poisson process with rate $\lambda$,
- $\{J_i\}$: iid such that $X = \ln(J)$ has a double exp. dist. with pdf

$$f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \quad 0 < \eta < 1.$$

- the above three processes are independent.

$n_t =$ the number of jumps in $[0, t]$ and Poisson$(\lambda t)$. At the $i$th jump, the proportion of price jump is $J_i - 1$.

For pdf of double exp. dist., see Figure 6.8 of the text.

Stock price under the jump diffusion model:

$$P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t] \prod_{i=1}^{n_t} J_i.$$

This result can be used to obtain the distribution for the return series.

Price of an option: Analytical results available, but complicated.

Use the program "kou" on GSBMBA.

**Example** $P_t = \$80$. $K = \$85$. $r = 0.08$ and $T - t = 0.25$.

Jump: $\lambda = 10$, $\kappa = -0.02$ and $\eta = 0.02$.

We obtain $c_t = \$2.25$, which is higher than \$1.86 of Example 6.6.

$p_t = \$5.57$, which is also higher.