Value at Risk

What is Value at Risk (VaR)?

- a measure of financial risk
- amount a position could decline in a given period with a specified probability
- mainly for market risk, but idea applies to credit risk and operational risk too.

A formal definition:

- time period given: $\Delta t = \ell$
- change in value: $\Delta V(\ell)$
- CDF of the change $F_\ell(x)$
- given probability: p
- a long position:

$$p = Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR}).$$

- a short position:

$$p = Pr[\Delta V(\ell) \geq \text{VaR}]$$

$$= 1 - Pr[\Delta V(\ell) \leq \text{VaR}]$$
$$= 1 - F_\ell(\text{VaR}).$$

Quantile: x_p is the pth quantile of $F_\ell(x)$ if

$$p = F_\ell(x_p)$$

and $F_\ell(.)$ is continuous. In general,

$$x_p = \inf_x \{x | F_\ell(x) \geq p\}.$$

Factors affecting VaR:

1. the probability p.
2. the time horizon ℓ.
3. data frequency.
4. the CDF $F_\ell(x)$.
5. the mark-to-market value of the position.

Why use log returns? Simplicity

log returns $\approx$ percentage changes.

VaR = Value $\times$ (VaR of log returns).

Alternatively, use

VaR = Value × (exp[VaR of log returns] −1)

VaR for portfolios

Assume two positions (stocks)

Let VaR_i be the VaR of i-th position

Let ρ be the correlation of the two returns

$$\rho = \frac{\Sigma_{t=1}^{T}(r_{1t} - \bar{r}_1)(r_{2t} - \bar{r}_2)}{\sqrt{\Sigma_{t=1}^{T}(r_{1t} - \bar{r}_1)^2 \, \Sigma_{t=1}^{T}(r_{2t} - \bar{r}_2)^2}}.$$

$\text{VaR} = \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho \text{VaR}_1 \text{VaR}_2}.$

Alternatively, construct portfolio returns and apply the usual VaR.

Methods available

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations:

Daily log returns of IBM stock

- span: July 3, 62 to Dec. 31, 98.
- size: 9190 points
- see Figure 7.1 (page 257)

Position: long on \$10 million.

RiskMetrics

- Developed by J.P. Morgan
- r_t given F_{t-1}: $N(0, \sigma_t^2)$
- σ_t^2 follows the special IGARCH(1,1) model

$$\sigma_t^2 = \alpha \sigma_{t-1}^2 + (1-\alpha) r_{t-1}^2, \quad 1 > \alpha > 0.$$

- VaR $= 1.65\sigma_t$ if $p = 0.05$.
- k-horizon: VaR$[k] = \sqrt{k}$VaR

  The square root of time rule
- Pros: simplicity and transparence
- Cons: model is not adequate

Example: IBM data

Model:

$$\begin{aligned} r_t &= a_t, \quad a_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.9396\sigma_{t-1}^2 + (1-0.9396)a_{t-1}^2 \end{aligned}$$

Because $r_{9190} = -0.0128$ and $\hat{\sigma}_{9190}^2 = 0.0003472$,

$\hat{\sigma}_{9190}^2(1) = 0.000336$.

For $p = 0.05$, VaR of $r_t = -1.65 \times \sqrt{0.000336} = -0.03025$

$$\text{VaR} = \$10,000,000 \times 0.03025 = \$302,500.$$

For $p = 0.01$, VaR of $r_t = -2.3262 \times \sqrt{0.000336} = -0.04265$, and VaR = \$426,500.

Econometric models

- $r_t = \mu_t + a_t$ given F_{t-1}
- μ_t: a mean equation (Ch.2)
- σ_t^2: a volatility model (Ch. 3 or 4)
- Pros: sound theory
- Cons: a bit complicated.

IBM data:

Case 1: Gaussian

$$r_t = 0.00066 - 0.0247 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.00000389 + 0.0799 a_{t-1}^2 + 0.9073 \sigma_t^2.$$

From $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.00033455$, we have

$$\hat{r}_{9190}(1) = 0.00071 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003211.$$

If $p = 0.05$, then

$$0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.$$

VaR = \$10,000,000×0.02877 = \$287,700.

If $p = 0.01$, then the quantile is

$$0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.$$

VaR = \$409,738.

Case 2: Student-t_5

$$r_t = 0.0003 - 0.0335 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.000003 + 0.0559 a_{t-1}^2 + 0.9350 \sigma_{t-1}^2.$$

From the data, $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma^2_{9190} = 0.000349$, we have

$$\hat{r}_{9190}(1) = 0.000367 \quad \text{and} \quad \hat{\sigma}^2_{9190}(1) = 0.0003386.$$

If $p = 0.05$, then the quantile is

$$0.000367 - 1.5608\sqrt{0.0003386} = -0.028354.$$

VaR $= \$10,000,000 \times 0.028352 = \$283{,}520$.

If $p = 0.01$, the quantile is

$$0.000367 - (3.3649/\sqrt{5/3})\sqrt{0.0003386} = -0.0475943.$$

VaR $= \$475{,}943$.

Discussion:

- Effects of heavy-tails seen with $p = 0.01$.
- Multiple step-ahead forecasts are needed.

Example 7.3 (continued). 15-day horizon.

$\hat{r}_{9190}[15] = 0.00998$ and $\sigma_t[15] = 0.0047948$.

If $p = 0.05$, the quantile is $0.00998 - 1.6449\sqrt{0.0047948} = -0.1039191$.

15-day VaR $= \$10{,}000{,}000 \times 0.1039191 = \$1{,}039{,}191$.

RiskMetrics: VaR $= \$287,700 \times \sqrt{15} = \$1{,}114{,}257$.

Empirical quantile

Sample of log returns: $\{r_t | t = 1, \cdots, n\}$.

Order statistics:

$$r_{(1)} \le r_{(2)} \le \cdots \le r_{(n)}$$

$r_{(i)}$ as the ith order statistic of the sample.

$r_{(1)}$ is the sample minimum

$r_{(n)}$ the sample maximum.

Idea: Use the empirical quantile to estimate the theoretical quantile of r_t.

For a given probability p, what is the empirical quantile?

If $np = \ell$ is an integer, then it is $r_{(\ell)}$.

If np is not an integer, find the two neighboring integers $\ell_1 < np < \ell_2$ and use interpolation.

The quantile is

$$\hat{x}_p = \frac{p_2 - p}{p_2 - p_1} r_{(\ell_1)} + \frac{p - p_1}{p_2 - p_1} r_{(\ell_2)}.$$

IBM data:

$n = 9190$. If $p = 0.05$, then $np = 459.5$.

5% quantile is $(r_{(459)} + r_{(460)})/2 = -0.021603$.

VaR = \$216,030.

If $p = 0.01$, then $np = 91.9$ and the 1% quantile is

$$\begin{aligned}
\hat{x}_{0.01} &= \frac{p_2 - 0.01}{p_2 - p_1} r_{(91)} + \frac{0.01 - p_1}{p_2 - p_1} r_{(92)} \\
&= \frac{.00001}{.00011}(-3.658) + \frac{0.0001}{0.00011}(-3.657) \\
&\approx -3.658.
\end{aligned}$$

VaR is $365,800.

Extreme value theory: Focus on the tail behavior of r_t.

Review of extreme value theory

A properly normalized $r_{(1)}$ assumes a special distribution:

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ \\ 1 - \exp[-\exp(x)] & \text{if } k = 0 \end{cases}$$

for $x < -1/k$ if $k < 0$ and for $x > -1/k$ if $k > 0$.

k: the *shape parameter*

$\alpha = -1/k$: tail index of the distribution.

Classification of distributions:

- Type I: $k = 0$, the Gumbel family. The CDF is

$$F_*(x) = 1 - \exp[-\exp(x)], \quad -\infty < x < \infty. \tag{1}$$

- Type II: $k < 0$, the Fréchet family. The CDF is

$$F_*(x) = \begin{cases} 1 - \exp[-(1+kx)^{1/k}] & \text{if } x < -1/k \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

- Type III: $k > 0$, the Weibull family. The CDF here is

$$F_*(x) = \begin{cases} 1 - \exp[-(1+kx)^{1/k}] & \text{if } x > -1/k \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function (pdf) of the normalized minimum is

$$f_*(x) = \begin{cases} (1+kx)^{1/k-1}\exp[-(1+kx)^{1/k}] & \text{if } k \neq 0 \\ \\ \exp[x - \exp(x)] & \text{if } k = 0 \end{cases}$$

where $-\infty < x < \infty$ for $k = 0$, $x < -1/k$ for $k < 0$ and $x > -1/k$ for $k > 0$.

How to use the EVT distribution?

If we know the three parameters, we can compute the quantile!

Empirical estimation

Divide the sample into non-overlapping subsamples.

Suppose there are T data points, we devide the data as

$$\{r_1, \cdots, r_n | r_{n+1}, \cdots, r_{2n} | r_{2n+1}, \cdots, r_{3n} | \cdots | r_{(g-1)n+1}, \cdots, r_{ng}\},$$

n: size of subgroup

Idea: find the minimum of each subgroup. These minima are the data used to estimate the three parameters.

Several estimation methods available. We use maximum likelihood estimates.

IBM data:

n	g	Scale α_n	Location β_n	Shape Par. k_n
(a) Minimal returns				
21	437	0.823(0.035)	−1.902(0.044)	−0.197(0.036)
63	145	0.945(0.077)	−2.583(0.090)	−0.335(0.076)
126	72	1.147(0.131)	−3.141(0.153)	−0.330(0.101)
252	36	1.542(0.242)	−3.761(0.285)	−0.322(0.127)
(b) Maximal returns				
21	437	0.931(0.039)	2.184(0.050)	−0.168(0.036)
63	145	1.157(0.087)	3.012(0.108)	−0.217(0.066)
126	72	1.292(0.158)	3.471(0.181)	−0.349(0.130)
252	36	1.624(0.271)	4.475(0.325)	−0.264(0.186)

EVT to VaR: Use a two-step procedure, because of the division into subgroups.

VaR for r_t:

$$\text{VaR} = \begin{cases} \beta_n - \frac{\alpha_n}{k_n}\left\{1 - [-n\ln(1-p)]^{k_n}\right\} & \text{if } k_n \neq 0 \\ \\ \beta_n + \alpha_n \ln[-n\ln(1-p)] & \text{if } k_n = 0. \end{cases}$$

For IBM data, if $n = 63$ (quarterly minima), then $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$. If $p = 0.01$, the VaR is

$$\begin{aligned} \text{VaR} &= -2.583 - \frac{0.945}{-0.335}\left\{1 - [-63\ln(1-0.01)]^{-0.335}\right\} \\ &= -3.04969 \end{aligned}$$

VaR is \$304,969.

If $p = 0.05$, then VaR is \$166,641.

For $n = 21$, the results are:

VaR = \$340,013 for $p = 0.01$;

VaR = \$184,127 for $p = 0.05$.

Discussion:

- Results depend on the choice of n
- VaR seems low, but it might be due to the choice of p.

  If $p = 0.001$, then

  VaR = \$546,641 for the Gaussian AR(2)-GARCH(1,1) model

VaR = \$666,590 for the extreme value theory with $n = 21$.

Summary of IBM data:

Position = \$10 millions.

If $p = 0.05$, then

1. \$302,500 for the RiskMetrics,
2. \$287,200 for an AR(2)-GARCH(1,1) model,
3. \$283,520 for an AR(2)-GARCH(1,1) with t_5
4. \$216,030 using the empirical quantile, and
5. \$184,127 for EVT with $n = 21$.

$p = 0.01$, then

1. \$426,500 for the RiskMetrics,
2. \$409,738 for an AR(2)-GARCH(1,1) model,
3. \$475,943 for an AR(2)-GARCH(1,1) model with t_5
4. \$365,800 for empirical quantile, and
5. \$340,013 for EVT with $n = 21$.

If $p = 0.001$, then

1. \$566,443 for the RiskMetrics,
2. \$546,641 for an AR(2)-GARCH(1,1) model,
3. \$836,341 for an AR(2)-GARCH(1,1) model with t_5
4. \$780,712 for quantile, and
5. \$666,590 for EVT with $n = 21$.

Multi-period VaR with EVT

$$\text{VaR}(\ell) = \ell^{1/\alpha}\text{VaR} = \ell^{-k}\text{VaR}$$

where α is the tail index and k is the shape parameter.

For IBM data with $p = 0.05$,

$$\text{VaR}(30) = (30)^{0.335}\text{VaR} = 3.125 \times \$184,127 = \$575,397.$$

New approach to VaR

Based on Exceedances over a high threshold

Idea: frequency of big returns and their magnitudes are important.

Statistical theory:

Two-dimensional Poisson process

Two possible cases:

Homogeneous: parameters are fixed over time

Non-homogeneous case: parameters are time-varying, according to some explanatory variables.

IBM data: homogeneous model

Thr.	Exc.	Shape Par. k	Log(Scale) $\ln(\alpha)$	Location β
(a) Original log returns				
3.0%	175	$-0.30697(0.09015)$	$0.30699(0.12380)$	$4.69204(0.19058)$
2.5%	310	$-0.26418(0.06501)$	$0.31529(0.11277)$	$4.74062(0.18041)$
2.0%	554	$-0.18751(0.04394)$	$0.27655(0.09867)$	$4.81003(0.17209)$
(b) Removing the sample mean				
3.0%	184	$-0.30516(0.08824)$	$0.30807(0.12395)$	$4.73804(0.19151)$
2.5%	334	$-0.28179(0.06737)$	$0.31968(0.12065)$	$4.76808(0.18533)$
2.0%	590	$-0.19260(0.04357)$	$0.27917(0.09913)$	$4.84859(0.17255)$

VaR calculation:

$$\text{VaR} = \begin{cases} \beta + \frac{\alpha}{k}\left\{1 - [-T\ln(1-p)]^k\right\} & \text{if } k \neq 0 \\ \\ \beta + \alpha \ln[-T\ln(1-p)] & \text{if } k = 0 \end{cases}$$

where $T = 252$, the number trading days in a year.

IBM data: VaR of 5% & 1%

- Case I: original returns

1. $\eta = 3.0\%$: \$228,239 & \$359.303.

2. $\eta = 2.5\%$: \$219,106 & \$361,119.

3. $\eta = 2.0\%$: \$212,981 & \$368.552.

- Case II: remove sample mean

1. $\eta = 3.0\%$: \$232,094 & \$363,697.

2. $\eta = 2.5\%$: \$225,782 & \$364,254.

3. $\eta = 2.0\%$: \$217,740 & \$372,372.

Non-homogeneous case:

$$\begin{aligned} k_t &= \gamma_0 + \gamma_1 x_{1t} + \cdots + \gamma_v x_{vt} \equiv \gamma_0 + \boldsymbol{\gamma}' \boldsymbol{x}_t \\ \ln(\alpha_t) &= \delta_0 + \delta_1 x_{1t} + \cdots + \delta_v x_{vt} \equiv \delta_0 + \boldsymbol{\delta}' \boldsymbol{x}_t \\ \beta_t &= \theta_0 + \theta_1 x_{1t} + \cdots + \theta_v x_{vt} \equiv \theta_0 + \boldsymbol{\theta}' \boldsymbol{x}_t. \end{aligned}$$

For IBM data, explanatory variables include past volatilities, etc.

See Chapter 7 for more details and estimation results.

Illustration:

For December 31, 1998, we have $x_{3,9190} = 0$, $x_{4,9190} = 0.9737$ and $x_{5,9190} = 1.9766$. The parameters become

$$k_{9190} = -0.01195, \quad \ln(\alpha_{9190}) = 0.19331, \quad \beta_{9190} = 6.105.$$

If $p = 0.05$, then quantile = 3.03756% and

$$\text{VaR} = \$10,000,000 \times 0.0303756 = \$303,756.$$

If $p = 0.01$, then Var is \$497,425.

For December 30, 1998, we have $x_{3,9189} = 1$, $x_{4,9189} = 0.9737$ and $x_{5,9189} = 1.8757$ and

$$k_{9189} = -0.2500, \quad \ln(\alpha_{9189}) = 0.52385, \quad \beta_{9189} = 5.8834.$$

The 5% VaR becomes

$$\text{VaR} = \$10,000,000 \times 0.0269139 = \$269,139.$$

If p 0.01, then VaR becomes \$448,323.