- **Probability**, also theory of probability, branch of mathematics that deals with measuring or determining quantitatively the likelihood that an event or experiment will have a particular outcome.
- Probability is based on the study of permutations and combinations and is the necessary foundation for statistics.
- The foundation of probability is usually ascribed to the 17th-century French mathematicians Blaise Pascal and Pierre de Fermat.
- It is applied in such diverse areas as genetics, quantum mechanics, and insurance.

Preface

- This book is intended as an elementary introduction to the mathematical theory of probability for students in mathematics, engineering, social science, and management science.
- It attempts to present not only the mathematics of probability theory, but also, through numerous examples, the many diverse possible applications of this subject.

1.1 Introduction

A typical problem of interest involving probability:

- A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order.
- The resulting system will then be able to receive all incoming signals (functional) as long as no two consecutive antennas are defective.
- If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional?
- For instance: $n = 4$ and $m = 2$

0	1	1	0	o
0	1	0	1	o
1	0	1	0	o
0	0	1	1	x
1	0	0	1	x
1	1	0	0	x

1: function;   0: defect

- The probability of function is $3/6 = 1/2$.

Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur.

The mathematical theory of counting is formally known as *combinatorial analysis*.

- **Permutations and Combinations**, in mathematics, certain arrangements of objects or elements.
- In the case of combinations, no attention is paid to the order of arrangement.
- In permutations, however, different orderings are counted as distinct, and repetitions of the elements selected may or may not be allowed.

1.2 The basic principle of counting

The basic principle of counting Suppose that two experiments are to be performed. Then if experiment 1 can result in any one if m possible outcomes and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Example 1.2a. A small community consists of 10 women, each of whom has 3 children. If one women and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

- There are $10 \times 3 = 30$ possible choices.

The generalized basic principle of counting If r experiments that are to be performed are such that the first one many result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and so on, then there is a total of $n_1, n_2, \ldots, n_r$ possible outcomes of the r experiments.

Example 1.2b. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

- $3 \times 4 \times 5 \times 2 = 120$ possible choices.

Example 1.2c. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

- $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000$

Example 1.2d. How many functions defined on n points are possible if each functional value is either 0 or 1?

- $f(i) = 0, 1 \qquad i = 1, 2 \ldots, n$
- There are 2^n possible functions.

Example 1.2e. In Example 2c, how many license plates would be possible if repetition among letters or numbers were prohibited?

- $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000$ possible license plates.

1.3 Permutations

There are $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$ different permutations of the n objects.

Example 1.3a. How many different batting orders are possible for a baseball team consisting of 9 players?

- $9! = 362,880$ possible batting orders.

Example 1.3b. A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?

(b) If the men are ranked just among themselves and the women among themselves, how many different rankings are possible?

(a) $10! = 3,628,800$

(b) $(6!)(4!) = (720)(24) = 17,280$ possible rankings.

Example 1.3c. Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

- $4!4!3!2! = 6912$

Certain of objects are indistinguishable from each other:

Example 1.3d. How many different letter arrangements can be formed using the letters $PEPPER$?

- Consider $P_1E_1P_2P_3E_2R$.
- There are $6!/3!2! = 60$ possible letter arrangements of the letters $PEPPER$.

There are
$$\frac{n!}{n_1!n_2!\cdots n_r!}$$
different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike.

Example 1.3e. A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

- $\frac{10!}{4!3!2!1!} = 12,600$ different outcomes.

Example 1.3f. How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

- $\frac{9!}{4!3!2!} = 1260$ different signals.

1.4 Combinations

Notation and terminology We define $\binom{n}{r}$, for $r \le n$, by
$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$
and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.

Example 1.4a. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

- $\binom{20}{3} = \frac{20\cdot19\cdot18}{3\cdot2\cdot1} = 1140$ possible committees.

Example 1.4b. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

- $\binom{5}{2}\binom{7}{3} = (\frac{5\cdot4}{2\cdot1})\frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 350$ possible committees.
- If 2 of the men refuse to serve on the committee together, then there are $(\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2})\binom{5}{2} = 30\binom{5}{2} = 300$ possible committees.

Example 1.4c. Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defective and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defective are consecutive?

- There are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \le r \le n \tag{4.1}$$

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4.2}$$

Proof of the Binomial Theorem by Induction:

- When $n = 1$, $x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0 = y + x$
- Assume Eq. (4.2) for $n - 1$.

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1}\\ &= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}\end{aligned}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

- Letting $i = k + 1$ in the first sum and $i = k$ in the second sum,

$$\begin{aligned}
(x+y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\end{aligned}$$

Combinatorial Proof of the Binomial Theorem:

- Consider $(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$.
- How many of the 2^n terms in the sum will have as factors k of the x_i's and $(n - k)$ of the y_i's? Answer: $\binom{n}{k}$
- Set $x_i = x, y_i = y, i = 1, \ldots, n$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 1.4d. Expand $(x + y)^3$.

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$$\begin{aligned}
(x+y)^3 &= \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^3 \\
&= y^3 + 3xy^2 + 3x^2 y + x^3
\end{aligned}$$

Example 1.4e. How many subsets are there of a set consisting of n elements?

- $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$
- Hence the number of subsets that contain at least one element is $2^n - 1$.

1.5 Multinomial coefficients

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. There are
$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} =$
$\frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!}$
$= \frac{n!}{n_1!n_2!\cdots n_r!}$ different divisions.

Notation If $n_1 + n_2 + \cdots + n_r = n$, we defined $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$

Thus $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 1.5a. A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

- $\frac{10!}{5!2!3!} = 2520$ possible divisions.

Example 1.5b. Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

- $\frac{10!}{5!5!} = 252$ possible divisions.

Example 1.5c. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

- The desired answer is $\frac{10!/5!5!}{2!} = 126$.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

Multinomial coefficients

$$\binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Example 1.5d.

$$\begin{aligned}(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 \\ &+ \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 \\ &+ \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\end{aligned}$$

1.6 On the distribution of balls in urns

- There are r^n possible outcomes when n distinguishable balls are to be distributed into r distinguishable urns.
- Suppose that the n balls are indistinguishable from each other. In this case, how many different outcomes are possible?

- We can select $r-1$ of the $n-1$ spaces between adjacent objects as our dividing points.
- For example, $n=8$ and $r=3$:

$$ooo|ooo|oo$$

Proposition 6.1 There are $\binom{n-1}{r-1}$ distinct positive integer-valued vector $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n \quad x_i > 0, i = 1, \ldots, r$$

Proposition 6.2 There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vector $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n$$

Example 1.6a. How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

- $\binom{3+2-1}{2-1} = 4$ solutions: (0,3),(1,2), (2,1),(3,0).

Example 1.6b. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

- x_i: The number of thousands invested in investment number i.
- $x_1 + x_2 + x_3 + x_4 = 20 \quad xi \geq 0$.
- There are $\binom{23}{3} = 1771$ possible investment strategies.
- If not all of the money need be invested, then if let x_5 denote the amount kept in reserve.
- $x_1 + x_2 + x_3 + x_4 + x_5 = 20 \quad x_i \geq 0$.
- There are $\binom{24}{4} = 10,626$ possible strategies.

Example 1.6c. How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$?

- $n_1 + \cdots + n_r = n \quad n_i \geq 0$
- There are $\binom{n+r-1}{r-1}$ such terms.

Example 1.6d. Let us reconsider Example 4c,

- We have a set of n items, of which m are defective and the remaining $n - m$ are functional.
- x_1: Number of functional items to the left of the first defective.
- x_2: Number of functional items between the first two defectives.
- x_{m+1}: Number of functional items to the right of the mth defective.
- $x_1 + \cdots + x_{m+1} = n - m \quad x_1 \geq 0, x_{m+1} \geq 0, x_i > 0, i = 2, \ldots, m$
- Let $y_1 = x_1 + 1, y_i = x_i, i = 2, \ldots, m, y_{m+1} = x_{m+1} + 1,$

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2 \quad y_i > 0$$

- There are $\binom{n-m+1}{m}$ such outcomes.
- Suppose that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional ones.
- $x_1 + \cdots + x_{m+1} = n - m \quad x_1 \geq 0, x_{m+1} \geq 0, x_i \geq 2, i = 2, \ldots, m$
- Let $y_1 = x_1 + 1, y_i = x_i, i = 2, \ldots, m, y_{m+1} = x_{m+1} - 1,$

$$y_1 + y_2 + \cdots + y_{m+1} = n - 2m + 3 \quad y_i > 0$$

- There are $\binom{n-2m+2}{m}$ such outcomes.

Summary

- $\binom{n}{i} = \frac{n!}{(n-i)!i!}$
- $(x + y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}$
- $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$