Ch.10 Fourier Analysis

- **Orthogonal Functions**

  - Discrete-time (Continuous-time) functions:

$$\sum_{t\in D}\phi_k(t)\phi_j^*(t)\left(\int_D \phi_k(t)\phi_j^*(t)dt\right) \begin{array}{l} = 0, k \neq j \\ \neq 0, k = j \end{array}$$

$\phi_j^*(t)$: complex conjugate of $\phi_j(t)$

D : a discrete (continuous) set

- $\{\sin(\frac{2\pi kt}{n}), \cos(\frac{2\pi kt}{n}) : k = 0, 1, \cdots, [\frac{n}{2}]\}$

  exactly n nonzero functions

(i) when n is even, $[\frac{n}{2}] = \frac{n}{2}$

$\cos(2\pi 0t/n) = 1$

$\sin(2\pi kt/n)$ and $\cos(2\pi kt/n)$ for $k = 1, 2, \cdots, n/2 - 1$

$\cos(2\pi(n/2)t/n) = \cos(\pi t) = (-1)^t$

(ii) when n is odd, $[\frac{n}{2}] = \frac{(n-1)}{2}$

$\cos(2\pi 0t/n) = 1$

$\sin(2\pi kt/n)$ and $\cos(2\pi kt/n)$ for $k = 1, 2, \cdots, (n-1)/2$

- **A system of orthogonal functions**

$\{\sin(\frac{2\pi t}{n}), \cos(\frac{2\pi kt}{n}) : k = 0, 1, \cdots, [\frac{n}{2}]\}$

$$\begin{aligned}
\sum_{i=1}^{n} \cos(2\pi kt/n)\cos(2\pi jt/n) &= n, k = j = 0 \quad \text{or} \quad n/2(n \text{ even}) \\
&= n/2, k = j \neq 0 \quad \text{or} \quad n/2(n \text{ even}) \\
&= 0, k \neq j
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} \sin(2\pi kt/n)\sin(2\pi jt/n) &= n, k = j = 0 \quad \text{or} \quad n/2(n \quad \text{even}) \\
&= n/2, k = j \neq 0 \quad \text{or} \quad n/2(n \text{ even}) \\
&= 0, k \neq j
\end{aligned}$$

$$\sum_{i=1}^{n} \sin(2\pi kt/n)\cos(2\pi jt/n) = 0 \quad \text{for} \quad \text{all} \quad k, j$$

- **Complex form :** $e^{i2\pi kt/n}$

$-\frac{n}{2} + 1 \leq k \leq \frac{n}{2}$, if n is even

$-\frac{(n-1)}{2} \leq k \leq \frac{(n-1)}{2}$, if n is odd

$$\begin{aligned}
\sum_{t=1}^{n} e^{i2\pi kt/n} e^{-i2\pi jt/n} &= n, \;\; k = j \\
&= 0, \;\; k \neq j
\end{aligned}$$

- **Fourier representation of finite sequences**

  Any finite sequence of n numbers can be written as a linear combination of the **sine-cosine** sequences or the **complex exponentials**

  Any set of n orthogonal vectors forms a **basis**

- Fourier series of a sequence of n numbers $Z_1, Z_2, \cdots, Z_n$

$$Z_t = \sum_{k=0}^{n/2} [a_k \cos(\omega_k t) + b_k \sin(\omega_k t)], \ t = 1, 2, \cdots, n$$

  Fourier coefficients :a_k, b_k

  Fourier frequencies: $\omega_k = 2\pi k/n \ k = 0, 1, \cdots, [n/2]$

$$\begin{aligned} a_k &= \frac{1}{n}\sum_{t=1}^{n} Z_t \cos(2\pi kt/n), \ k = 0, \text{ and } k = n/2 \text{ if } n \text{ is even} \\ &= \frac{2}{n}\sum_{t=1}^{n} Z_t \cos(2\pi kt/n), \ k = 1, 2, \cdots, [\frac{n-1}{2}] \\ b_k &= \frac{2}{n}\sum_{t=1}^{n} Z_t \sin(2\pi kt/n), \ k = 1, 2, \cdots, [\frac{n-1}{2}]. \end{aligned}$$

- **Complex form**

$$\begin{aligned} Z_t &= \textstyle\sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i\omega_k t}, n \text{ odd} \\ &= \textstyle\sum_{k=-n/2+1}^{n/2} c_k e^{i\omega_k t}, n \text{ even} \end{aligned}$$

  $\omega_k = 2\pi k/n, k = 0, 1, \cdots, [n/2]$

  Fourier coefficients: $c_k = \frac{1}{n}\sum_{t=1}^{n} Z_t e^{-i\omega_k t}$

$$c_0 = a_0, \quad c_k = \frac{a_k - ib_k}{2}, \quad c_{-k} = c_k^* = \frac{a_k + ib_k}{2}$$

- **Fourier representation of periodic sequences**

- The discrete time function Z_t is periodic with period n, if $Z_{t+n} = Z_t$ for all integer t, its values are uniquely determined by its values at $t = 1, 2, \cdots, n$

- $Z_t, t = 1, 2, \cdots, n$ can be written as the linear combination of the orthogonal sine and cosine functions, or as the linear combination of complex exponentials.

- Since $\sin \omega_k t$ and $\cos \omega_k t$ and $e^{i\omega_k t}$ are periodic with period n, $Z_{t+jn} = Z_t$ for all t and j

- The smallest positive value of n for which the Fourier seies representaion holds is call the **fundamental period** and the corresponding value $2\pi/n$ is called the **fundamental frequency**

- The term for $k = j$ and $k = -j$ both have the frequency $j(2\pi/n)$ and are referred to as the j**th harmonic component**

- Z_t :periodic sequence of period n

  $\sum_{t=1}^{n} Z_t^2$: **energy** associated with the sequence **in one period**

  $1/n \sum_{t=1}^{n} Z_t^2$: **energy per unit time**

- **Parseval's relation for Fourier series**

$$
\begin{aligned}
\sum_{t=1}^{n} Z_t^2 &= na_0^2 + \frac{n}{2}\sum_{k=1}^{(n-1)/2}(a_k^2+b_k^2),\ n \text{ odd} \\
&= na_0^2 + \frac{n}{2}\sum_{k=1}^{n/2-1}(a_k^2+b_k^2) + na_{\frac{n}{2}}^2,\ n \text{ even} \\
\sum_{t=1}^{n} Z_t^2 &= n\sum_{k=-(n-1)/2}^{(n-1)/2}|c_k|^2,\ n \text{ odd} \\
&= n\sum_{k=-n/2+1}^{n/2}|c_k|^2,\ n \text{ even}
\end{aligned}
$$

where $|c_k|^2 = c_k c_k^*$

- **Periodic sequence**: Total energy over the whole time horizon $t = 0, 1, 2, \cdots$ is **infinite**.

  Hence we consider its **energy per unit (power)**.

- **Power spectrum**: contribution to the total power at the **frequency** $\omega_k = 2\pi k/n$.

  $k = 1, 2, \cdots, (n-1)/2(n \text{ odd}), n/2-1(n \text{ even})$

$$
\begin{aligned}
f_0 &= c_0^2 = a_0^2 \\
f_k &= |c_{-k}|^2 + |c_k|^2 = 2|c_k|^2 = \frac{1}{2}(a_k^2+b_k^2) \\
f_{\frac{n}{2}} &= a_{\frac{n}{2}}^2, n \text{ even}
\end{aligned}
$$

- Example.

  $Z_1 = 1, Z_2 = 2, Z_3 = 3$

  $Z_{t+3j} = Z_t, t = 1, 2, 3, j = \pm 1, \pm 2, \pm 3, \cdots$

  Period $n = 3$, $[n/2] = 1$

  Fourier representation:

  (i) $Z_t = a_0 + a_1 \cos(2\pi t/3) + b_1 \sin(2\pi t/3)$

  $a_0 = 2, a_1 = 1, b_1 = -0.5773503$

  (ii) $Z_t = \frac{1}{2}(1 - 0.5773503i)e^{-i2\pi t/3} + 2 + \frac{1}{2}(1 + 0.5773503i)e^{i2\pi t/3}$

  Power spectrum of the sequence is

$$
\begin{aligned}
f_k &= 4, k = 0 \\
&= \frac{2}{3}, k = 1, \\
&= 0, \text{otherwise}
\end{aligned}
$$

- **Fourier representation of nonperiodic sequences-**
  The discrete-time Fourier transform

- Z_t : nonperiodic discrete-time function

  $Z_t = 0$, for $|t| > M$

  (i) Let $Y_{t+jn} = Z_t, -\frac{(n-1)}{2} \leq t \leq \frac{(n-1)}{2}, j = 0, \pm 1, \pm 2, \cdots$

  Y_t is periodic with period n

  (ii) $Y_t = \sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i2\pi kt/n}$

  (iii)

$$
\begin{aligned}
c_k &= \frac{1}{n} \sum_{t=-(n-1)/2}^{(n-1)/2} Y_t e^{-i2\pi kt/n} \\
&= \frac{1}{n} \sum_{t=-(n-1)/2}^{(n-1)/2} Z_t e^{-i2\pi kt/n} \\
&= \frac{1}{n} \sum_{t=-\infty}^{\infty} Z_t e^{-i(\frac{2\pi}{n})kt}
\end{aligned}
$$

  (iv) Let $f(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t}$

  $\Rightarrow c_k = \frac{2\pi}{n} f(k\Delta\omega)$, where $\Delta\omega = \frac{2\pi}{n}$

  $\Rightarrow$

$$
\begin{aligned}
Y_t &= \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{2\pi}{n} f(k\Delta\omega) e^{ik\Delta\omega t} \\
&= \sum_{k=-(n-1)/2}^{(n-1)/2} f(k\Delta\omega) e^{ik\Delta\omega t} \Delta\omega \\
&\to^{n\to\infty} \int_{-\pi}^{\pi} f(\omega) e^{i\omega t} d\omega
\end{aligned}
$$

(v) **Fourier transform pair**

(a) discrete inverse Fourier transform of $f(\omega)$

$$Z_t = \int_{-\pi}^{\pi} f(\omega) e^{i\omega t} d\omega, t = 0, \pm 1, \pm 2, \cdots$$

(b) discrete Fouier transform of Z_t

$$f(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t}, -\pi \leq \omega \leq \pi$$