Ch2 Fundamental Concepts

Defn :

1. Time Series : $Z_1, Z_2, \cdots, Z_n, \cdots$ (real value)

2. Stochastic process $Z_t(\omega) : t \in T =$ an index set,

   e.g. $T = \{0, \pm 1, \pm 2, \cdots\}$.

   $Z_t(\cdot)$ for fixed t, $Z_t(\cdot)$ is a r.v. defined on Ω.

   $Z.(\omega)$ for fixed ω, $Z.(\omega)$ is a realization of a time series.

3. n-dimensional distribution function of $\{Z_{t_1} Z_{t_2}, \cdots, Z_{t_n}\}$

$$F(z_{t_1}, \cdots, z_{t_n}) = P\{\omega : Z_{t_1} \leq z_{t_1}, \cdots, Z_{t_n} \leq z_{t_n}\}$$

4. If $F(Z_{t_1}, \cdots, Z_{t_n}) = F(Z_{t_1+k}, \cdots, Z_{t_n+k})$, $\forall (t_1, \cdots, t_n)$ and $k \in \mathbf{N}$, then the process is called nth order stationary in distribution.

   e.g. $n = 1$, $F(Z_{t_1}) = F(Z_{t_1+k})$ $k = 2, \cdots$

   If $\{Z_t\}_{t=1}^{n}$ is an nth order stationary process in distribution, $\forall n$, then $\{Z_t\}$ is called strictly stationary (strong stationary, completely stationary).

   Rmk: If $\{Z_t\}$ is nth order stationary in distribution, then $\{Z_t\}$ is also mth order stationary in distribution, $\forall m \leq n$.

   e.g. $Z_1, Z_2, \cdots, Z_n$ $i.i.d.$: strictly stationary process.

5. Mean function : $\mu_t = E(Z_t)$.

   Variance function : $\sigma_t^2 = E(Z_t - \mu)^2$.

Auto-Covariance function between Z_{t_1} and Z_{t_2},

$$r(t_1, t_2) = E(Z_{t_1} - \mu_1)(Z_{t_2} - \mu_2).$$

Auto-Correlation function between Z_{t_1} and Z_{t_2},

$$\rho(t_1, t_2) = \frac{r(t_1, t_2)}{\sqrt{\sigma_{t_1}^2}\sqrt{\sigma_{t_2}^2}}.$$

6. For strictly stationary process, assuming $E(|Z_t|) < \infty$ and $E(Z_t^2) < \infty$, then

$$r(t_1, t_2) = r(t_1 + h, t_2 + h) = r_{t_1 - t_2} = r_k, (|t_1 - t_2| = k)$$

and

$$\rho(t_1, t_2) = \rho(t_1 + h, t_2 + h) = \rho_{t_1 - t_2} = \rho_k.$$

i.e. r_k and ρ_k depends only on time lag k.

7. It is difficult to verify the joint distribution function of n random variables without stationarity assumption.
e.g. $X_1, \cdots, X_n \sim MN(\vec{\mu}, \Gamma)$, $\vec{\mu} = (\mu_1, \cdots, \mu_n)'$

$$\Gamma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \cdots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{n1}^2 & \cdots & \cdots & \sigma_{nn}^2 \end{pmatrix}$$

參數個數$=n + \frac{n(n+1)}{2} >$ sample size n. 然而就算是在strictly stationary假設下, 若僅假設 multi-variate normal,參數個數$=n+1$, 亦超過樣本個數, 故一般而言,

我們僅假設時序滿足下面的一種較弱的stationary assumption.

8. Covariance stationary (or weak stationary),

$$\begin{cases} Cov(Z_{t_1}, Z_{t_2}) = Cov(Z_{t_1+k}, Z_{t_2+k}), \quad \forall (t_1, t_2) \text{ and } k \\ Var(Z_{t_1}) < \infty, \quad \forall t_1 \end{cases}$$

<u>**Rmk**</u> A strictly stationary process with finite 2nd moment is also a weakly stationary process, 但若2nd moment不存在則未必。

e.g. $Z_1, \cdots, Z_n$ $i.i.d.$ Cauchy.

e.g. 丟die 3次independently, $\Omega = \{1, 2, 3, 4, 5, 6\}$ sample space, 其中$Z_i(\omega) = 2\omega$
$\because$ $i.i.d.$ $\therefore$ strictly stationary.

e.g. $Z_t = A\sin(\omega t + \theta)$, $\begin{cases} A: \text{ r.v.}, E(A) = 0, Var(A) = 1 \\ \theta: \text{ r.v.} \sim U(-\pi, \pi] \\ A \text{ and } \theta \text{ indep.} \end{cases}$

$E(Z_t) = 0$
$E(Z_t Z_{t+\theta})$
$= E(A^2)E\{\frac{1}{2}[cos(\omega k) - cos(\omega(2t+k) + 2\theta)]\}$
$= \frac{1}{2}cos(\omega k) - \frac{1}{4\pi}\int_{-\pi}^{\pi} cos(\omega(2t+k) + 2\theta)d\theta$
$= \frac{1}{2}cos(\omega k) - \frac{1}{8\pi}[sin(\omega(2t+k) + 2\theta)]|_{-\pi}^{\pi} = \frac{1}{2}cos(\omega k)$.
Covariance stationary.

e.g. $Z_t = \begin{cases} X_t \sim N(0,1), t : \text{odd} \\ Y_t \sim P(Y_t = 1) = P(Y_t = -1) = \frac{1}{2}, t : \text{even} \end{cases}$, $X_t \perp Y_t$

$$E(Z_t) = 0, E(Z_t^2) = 1, \forall t$$

$$E(Z_t Z_s) = \begin{cases} 0, & t \neq s \\ 1, & t = s \end{cases}$$

$$\rho(t,s) = \frac{E(Z_t, Z_s)}{\sqrt{E(Z_t^2)}\sqrt{E(Z_s^2)}} = \begin{cases} 0, & t \neq s \\ 1, & t = s \end{cases} .$$

Covariance stationary, not strictly stationary.

<u>**Rmk**</u> Gaussian (or normal process) : the joint distribution of any finite dimensional is multivariate normal. For Gaussian process, strictly stationary = covariance stationary.

§ ACVF and ACF

1. $\{Z_t\}$ a stationary process,
   $E(Z_t) = \mu$, $Var(Z_t) = E(Z_t - \mu)^2 = \sigma^2$.
   ACVF (autocovariance function):
   $r_k = Cov(Z_t, Z_{t+k}) = E(Z_t - \mu)(Z_{t+k} - \mu)$.
   ACF (autocorrelation function) :
   $\rho_k = \frac{Cov(Z_t, Z_{t+k})}{\sqrt{Var(Z_t)}\sqrt{Var(Z_{t+k})}} = \frac{r_k}{r_0}$.

   (a) $\gamma_0 = Var(Z_t)$; $\rho_0 = 1$.

   (b) $|r_k| \leq r_0$; $|\rho_k| \leq 1$,
   (Cauchy-Schwarz inequality:
   $|E(Z_t - \mu)(Z_{t+k} - \mu)| \leq \sqrt{E(Z_t - \mu)^2}\sqrt{E(Z_{t+k} - \mu)^2} = \gamma_0$.)

   (c) $r_k = r_{-k}$, $\rho_k = \rho_{-k}$, $\forall k$, r_k is an even function.

   (d) $\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|} \geq 0$,
   for any set of $(t_1, t_2, \cdots, t_n)$, any $\alpha_i \in \Re$, $i = 1, 2, \cdots, n$.
   γ_k : a positive semidefinite (non-negative definite) function. Similarly, we can prove that ρ_k is a positive semidefinite function.

   pf: Define $X = \sum_{i=1}^{n} \alpha_i Z_{t_i}$, where $Z_{t_i} = X_{t_i} - E(X_{t_i})$ then

$$
\begin{aligned}
0 \leq Var(X) &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j Cov(Z_{t_i}, Z_{t_j}) \\
&= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|}
\end{aligned}
$$

§ PACF

1. $\{Z_t\}$: stationary process, $E(Z_t)=0$.

2. Let $\hat{Z}_{t+k}$ be the best linear predictor of Z_{t+k} based on $\{Z_{t+1}, Z_{t+2}, \cdots, Z_{t+k-1}\}$, then

$$\hat{Z}_{t+k} = \alpha_1 Z_{t+k-1} + \alpha_2 Z_{t+k-2} + \cdots + \alpha_{k-1} Z_{t+1}$$

where $\alpha_i's$ are obtained by minimizing

$E(Z_{t+k} - \hat{Z}_{t+k})^2 = E(Z_{t+k} - \alpha_1 Z_{t+k-1} - \cdots - \alpha_{k-1} Z_{t+1})^2$

by differentiation we have

$\gamma_i = \alpha_1\gamma_{i-1} + \alpha_2\gamma_{i-2} + \cdots + \alpha_{k-1}\gamma_{i-k+1}, (1 \le i \le k-1)$

$\Rightarrow \rho_i = \alpha_1\rho_{i-1} + \alpha_2\rho_{i-2} + \cdots + \alpha_{k-1}\rho_{i-k+1},$

$$\Rightarrow \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{k-1} \end{bmatrix}$$

3. 同理, 令 $\hat{Z}_t$ be the best linear predictor of Z_t based on $\{Z_{t+1}, Z_{t+2}, \cdots, Z_{t+k-1}\}$, and $\hat{Z}_t = \beta_1 Z_{t+1} + \beta_2 Z_{t+2} + \cdots + \beta_{t-1} Z_{t+k-1}$ 其中$\beta_i's$ minimize

$$E(Z_t - \hat{Z}_t)^2 = E(Z_t - \beta_1 Z_{t+1} - \cdots - \beta_{k-1} Z_{t+k-1})^2$$

$$\Rightarrow \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k-1} \end{bmatrix}$$

故$\alpha_i = \beta_i$, $1 \leq i \leq k-1$.

4. Partial autocorrelation between Z_t and Z_{t+k}:
   the correlation between Z_t and Z_{t+k} after their mutual linear dependency on the interventing variables $Z_{t+1}, Z_{t+2}, \cdots$ and Z_{t+k-1} has been removed:
   $P_1 = Corr(Z_1, Z_2)$ and $P_k = Corr(Z_t - \hat{Z}_t, Z_{t+k} - \hat{Z}_{t+k}), k \geq 2$, i.e.

$$P_k = \frac{Cov[(Z_t - \hat{Z}_t), (Z_{t+k} - \hat{Z}_{t+k})]}{\sqrt{Var(Z_t - \hat{Z}_t)}\sqrt{Var(Z_{t+k} - \hat{Z}_{t+k})}}$$

$$\begin{aligned}
&Var(Z_{t+k} - \hat{Z}_{t+k}) \\
&= E[(Z_{t+k} - \alpha_1 Z_{t+k-1} - \cdots - \alpha_{k-1} Z_{t+1})^2] \\
&= E[Z_{t+k}(Z_{t+k} - \alpha_1 Z_{t+k-1} - \cdots - \alpha_{k-1} Z_{t+1})]
\end{aligned}$$

Hence

$$\begin{aligned}
Var(Z_{t+k} - \hat{Z}_{t+k}) &= Var(Z_t - \hat{Z}_t) \\
&= \gamma_0 - \alpha_1\gamma_1 - \cdots - \alpha_{k-1}\gamma_{k-1} \\
Cov[(Z_t - \hat{Z}_t), &(Z_{t+k} - \hat{Z}_{t+k})] \\
&= E[(Z_t - \alpha_1 Z_{t+1} - \cdots - \alpha_{k-1} Z_{t+k-1}) \\
&\qquad (Z_{t+k} - \alpha_1 Z_{t+k-1} - \cdots - \alpha_{k-1} Z_{t+1})] \\
&= E[(Z_t - \alpha_1 Z_{t+1} - \cdots - \alpha_{k-1} Z_{t+k-1}) Z_{t+k}] \\
&= \gamma_k - \alpha_1\gamma_{k-1} - \cdots - \alpha_{k-1}\gamma_1
\end{aligned}$$

$$\Rightarrow \quad P_k = \frac{\gamma_k - \alpha_1\gamma_{k-1} - \cdots - \alpha_{k-1}\gamma_1}{\gamma_0 - \alpha_1\gamma_1 - \cdots - \alpha_{k-1}\gamma_{k-1}} = \frac{\rho_k - \alpha_1\rho_{k-1} - \cdots - \alpha_{k-1}\rho_1}{1 - \alpha_1\rho_1 - \cdots - \alpha_{k-1}\rho_{k-1}}$$

由3, 我們知

$$\alpha_i = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_1 & \cdots & \rho_{k-2} \\ \rho_1 & 1 & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \rho_{k-2} & \cdots & \cdots & \rho_{k-1} & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{i-1} & \cdots & \rho_{k-2} \\ \rho_1 & 1 & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \rho_{k-2} & \cdots & \cdots & \rho_{k-i-1} & \cdots & 1 \end{vmatrix}}$$

Recall

$$P_k = \frac{\rho_k - \alpha_1\rho_{k-1} - \cdots - \alpha_{k-1}\rho_1}{1 - \alpha_1\rho_1 - \cdots - \alpha_{k-1}\rho_{k-1}}$$

$$\Rightarrow \quad P_k = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_{k-1} & \rho_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-1} & \rho_{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

(對最後一行展行列式)

Z_{t+k}對$Z_{t+k-1}, Z_{t+k-2}, \cdots, Z_{t+1}, Z_t$做regression

i.e. $Z_{t+k} = \phi_{k1} Z_{t+k-1} + \phi_{k2} Z_{t+k-2} + \cdots + \phi_{kk} Z_t + \varepsilon_{t+k}$

其中ϕ_{kk}的值(=第k個迴歸係數), 即為Z_{t+k}與Z_t的PACF

。

e.g.$\{a_t\}$: $WN(0, \sigma_a^2)$

$$\Rightarrow \rho_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \Rightarrow \phi_{kk} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

§ Estimation of mean, ACVF and ACF, Z_t stationary time series

1. Sample mean $\bar{Z} = \frac{1}{n}\sum_{t=1}^{n} Z_t$

$$E(\bar{Z}) = \frac{1}{n} \cdot n\mu = \mu \text{ unbiased}$$

$$\begin{aligned} Var(\bar{Z}) &= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n} Cov(Z_t, Z_s) \\ &= \frac{\gamma_0}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n} \rho_{t,s} \\ &= \frac{\gamma_0}{n^2}\sum_{k=-(n-1)}^{n-1} (n-|k|)\rho_k \\ &= \frac{\gamma_0}{n}\sum_{k=-(n-1)}^{n-1} (1-\frac{|k|}{n})\rho_k \end{aligned}$$

若$\rho_k \to 0$, 則

$$\frac{1}{n}\sum_{k=-(n-1)}^{n-1} \rho_k \underset{n\to\infty}{\longrightarrow} 0 \text{ (Cesaro summable)}$$

故$Var(\bar{Z}) \to 0$ as $n \to \infty$

$\Rightarrow \bar{Z}_n \to \mu$ in mean square in probability

(by Chachy's inequality $P(|\bar{Z}_n - \mu| > \varepsilon), \frac{\sigma_n^2}{\varepsilon^2} \to 0$)

此亦稱$\{Z_t\}$是ergodic for the mean.

§ Sample ACVF

$$\hat{\gamma}_k = \frac{1}{n}\sum_{t=1}^{n-k}(Z_t - \bar{Z})(Z_{t+k} - \bar{Z})$$

$$\hat{\hat{\gamma}} = \frac{1}{n-k}\sum_{t=1}^{n-k}(Z_t - \bar{Z})(Z_{t+k} - \bar{Z})$$

$$\sum_{t=1}^{n-k}(Z_t - \bar{Z})(Z_{t+k} - \bar{Z})$$

$$= \sum_{t=1}^{n-k}[(Z_t - \mu) - (\bar{Z} - \mu)][(Z_{t+k} - \mu) - (\bar{Z} - \mu)]$$

$$= \sum_{t=1}^{n-k}(Z_t - \mu)(Z_{t+k} - \mu) - (\bar{Z} - \mu)\sum_{t=1}^{n-k}(Z_t - \mu)$$

$$-(\bar{Z} - \mu)\sum_{t=1}^{n-k}(Z_{t+k} - \mu) + (n-k)(\bar{Z} - \mu)^2$$

$$\approx \sum_{t=1}^{n-k}(Z_t - \mu)(Z_{t+k} - \mu) - (n-k)(\bar{Z} - \mu)^2$$

$$\Rightarrow \quad E(\hat{\gamma}_k) \approx \gamma_k - \frac{k}{n}\gamma_k - \frac{n-k}{n}Var(\bar{Z})$$

$$= \gamma_k - Var(\bar{Z}) + \frac{k}{n}(Var(\bar{Z}) - \gamma_k)$$

$$E(\hat{\hat{\gamma}}_k) \approx \gamma_k - Var(\bar{Z})$$

$\Rightarrow \hat{\gamma}_k$ has a larger bias than $\hat{\hat{\gamma}}$ especially when k is large w.r.t. n.

(註. 故在一般計算中, 我們計算$\hat{\gamma}_k$, 只求到$k \le \frac{n}{4}$)

若$\rho_k \to 0$ as $k \to \infty \Rightarrow$ the process is ergodic for the mean and $Var(\bar{Z}) \longrightarrow 0$ as $n \to \infty$

$\hat{\gamma}$ and $\hat{\hat{\gamma}}$ are asymptotically unbiased ($\because$ 兩者均為biased)
(可比較$MSE(\hat{\gamma})$與$MSE(\hat{\hat{\gamma}})$)

- $\hat{\gamma}$ is always positive semidefine but $\hat{\hat{\gamma}}_k$ is not necessary so

$$\because \ \hat{\Gamma}_k = n^{-1}TT'$$

$$= \frac{1}{n}\begin{bmatrix} 0 & \cdots & 0 & 0 & Z_1-\bar{Z} & \cdots & \cdots & Z_k-\bar{Z} \\ 0 & \cdots & 0 & Z_1-\bar{Z} & Z_2-\bar{Z} & \cdots & Z_k-\bar{Z} & 0 \\ \vdots & & & & & & \vdots & \vdots \\ 0 & Z_1-\bar{Z} & & & & & & \end{bmatrix}_{k\times 2k}$$

$$\cdot \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & Z_1-\bar{Z} \\ 0 & 0 & & \vdots \\ 0 & Z_1-\bar{Z} & & Z_k-\bar{Z} \\ Z1-\bar{Z} & \vdots & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & Z_k-\bar{Z} & & \vdots \\ Z_k-\bar{Z} & 0 & \cdots & 0 \end{bmatrix}_{2k\times k}$$

$$\therefore \ \vec{a}'\hat{\Gamma}_k\vec{a} = n^{-1}(\vec{a}T)(T'\vec{a}) = n^{-1} \parallel T'\vec{a} \parallel \geq 0.$$

- If $\{Z_t\}$ stationary Gaussian (Bartlett)

$$\begin{cases} Cov(\hat{\gamma}_k, \hat{\gamma}_{k+j}) \approx \frac{1}{n}\sum_{i=-\infty}^{\infty}(\gamma_i\gamma_{i+j} + \gamma_{i+k+j}\gamma_{i-k}) \\ \\ Var(\hat{\gamma}_k) \approx \frac{1}{n}\sum_{i=-\infty}^{\infty}(\gamma_i^2 + \gamma_{i+k}\gamma_{i-k}) \end{cases}$$

$$\begin{cases} Cov(\hat{\hat{\gamma}}_k, \hat{\hat{\gamma}}_{k+j}) \approx \frac{1}{n-k}\sum_{i=-\infty}^{\infty}(\gamma_i\gamma_{i+j} + \gamma_{i+k+j}\gamma_{i-k}) \\ \\ Var(\hat{\hat{\gamma}}_k) \approx \frac{1}{n-k}\sum_{i=-\infty}^{\infty}(\gamma_i^2 + \gamma_{i+k}\gamma_{i-k}) > Var(\hat{\gamma}_k) \end{cases}$$

especially when k is large.

- If $\sum_{i=-\infty}^{\infty}|\gamma_i| < \infty$, then $\hat{\gamma}_k \to \gamma_k$ in mean square.

$$(E(\hat{\gamma}_k - \gamma_k)^2 \underset{n\to\infty}{\longrightarrow} 0)$$

$\Rightarrow \hat{\gamma}_k \to \gamma_k$ in probability.

$\Rightarrow \{Z_t\}$ is ergodic for ACVF.

§ Sample ACF

- $\hat{\rho}_k = \frac{\hat{r}_k}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{n-k}(Z_t-\bar{Z})(Z_{t+k}-\bar{Z})}{\sum_{t=1}^{n}(Z_t-\bar{Z})^2}$, $k = 0, 1, 2, \cdots$

- Plot of $\hat{\rho}_k$ v.s. k is called sample correlogram.

- $\{Z_t\}$ stationary Gaussian (Bartlett).

- $Cov(\hat{\rho}_k, \hat{\rho}_{k+j})$
  $$\approx \frac{1}{n}\sum(\rho_i\rho_{i+j} + \rho_{i+k+j}\rho_{i-k} - 2\rho_k\rho_i\rho_{i-k-j} - 2\rho_{k+j}\rho_i\rho_{i-k} + 2\rho_k\rho_{k+j}\rho_i^2), \quad k > 0, k+j > 0$$

- for large n, $\hat{\rho}_k \xrightarrow{D} N(\rho_k, Var(\hat{\rho}_k))$
  $$Var(\hat{\rho}_k) \approx \frac{1}{n}\sum(\rho_i^2 + \rho_{i+k}\rho_{i-k} - 4\rho_k\rho_i\rho_{i-k} + 2\rho_k^2\rho_i^2)$$

- If $\rho_k = 0$, for $k > m$, then
  $$Var(\hat{\rho}_k) \approx \frac{1}{n}(1 + 2\rho_1^2 + \cdots + 2\rho_m^2)$$

- In practice, 以$\hat{e}_i$取代ρ_i, 可得$\hat{\rho}_k$的大樣本stdev.
  $$S_{\hat{\rho}_k} = \sqrt{\frac{1}{n}(1 + 2\hat{\rho}_1^2 + \cdots + 2\hat{\rho}_m^2)}.$$

- 若$\rho_k = 0$, for $k \geq 1$, i.e. white noise process $S_k = \sqrt{\frac{1}{n}}$.

§ Sample PACF

Recall

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_1 \\ \vdots & & & \vdots & \vdots \\ \rho_{k-1} & \cdots & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \vdots & & & \vdots & \vdots \\ \rho_{k-1} & \cdots & \cdots & \rho_1 & 1 \end{vmatrix}},$$

k 大時, 計算過於繁瑣 。

Durbin derive

$$\begin{cases} \hat{\phi}_{k+1,k+1} = \frac{\hat{\rho}_{k+1} - \sum_{i=1}^{k} \hat{\phi}_{k,j} \hat{\rho}_{k+1-j}}{1 - \sum_{j=1}^{k} \hat{\phi}_{k,j} \hat{\rho}_j} \\ \hat{\phi}_{k+1,j} = \hat{\phi}_{k,j} - \hat{\phi}_{k+1,k+1} \hat{\phi}_{k,k+1-j} \quad j = 1, 2, \cdots, k \end{cases}$$

其中$\hat{\phi}_{ij}$為best regression coefficient of

$$Z_{t+k} = \phi_{k1} Z_{t+k-1} + \cdots + \phi_{kk} Z_t + \varepsilon.$$

(註:此recersive method對theoretical PACF亦成立 。)

- 若$\{Z_t\}$為WN, 則$Var(\hat{\phi}_{kk}) \approx \frac{1}{n}$, 故$\pm\frac{2}{\sqrt{n}}$可做為$\hat{\phi}_k$ 的0.95 critical limits.

§ MA(∞) and AR(∞) representation

- Linear process

$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

(infinite MA representation).

其中$\psi_0 = 1$, $\{a_t\} \sim WN(0, \sigma_a^2)$ and

$$\sum \psi_j^2 < \infty \text{ (or } \sum |\psi| < \infty)$$

由$\sum \psi_j^2 < \infty$的條件可推得

$$E[(Z_t - \mu - \sum_{j=0}^{n} \psi_j a_{t-j})^2] \to 0 \text{ as } n \to \infty.$$

i.e. $\mu + \sum_{j=0}^{\infty} \psi_j a_{t-j} \to Z_t$ in m.s.

- Define backshift operator $B^j X_t = X_{t-j}$, 亦可將$\{Z_t\}$寫為 $Z_t - \mu = \psi(B) a_t$, 其中$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$.

-
$$E(Z_t) = \mu,$$

$$Var(Z_t) = \sigma_a^2 \textstyle\sum_{j=0}^{\infty} \psi_j^2,$$

$$E(a_t Z_{t-j}) = \begin{cases} \sigma_a^2, & \text{for } j = 0 \\ 0, & \text{for } j > 0 \end{cases},$$

$\Rightarrow$

$$\begin{aligned} \gamma_k &= E(Z_t - \mu)(Z_{t+k} - \mu) \\ &= E(\textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j a_{t-i} a_{t+k-j}) \\ &= \sigma_a^2 \textstyle\sum_{i=0}^{\infty} \psi_i \psi_{i+k} \end{aligned}$$

$\Rightarrow$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{i=0}^{\infty} \psi_j^2}$$

Note that $|\gamma_k| \leq \gamma_0 = \sigma_a^2 \sum \psi_j^2 < \infty$ 故$\gamma_k < \infty \ \forall k$.

- Wold : A nondeterministic stationary process can always be expressed in the form of $MA(\infty)$+deterministic term. (註) A deterministic term is a process that can be predicted or forecast exactly from to past.

- ACF generating function

$$\gamma(B) = \sum_{k=-\infty}^{\infty} \gamma_k B^k$$

- $\{Z_t\}$ linear process, 其ACV generating function為

$$\begin{aligned}
\gamma(B) &= \textstyle\sum_{k=-\infty}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_{i+k} B^k \sigma_a^2 \\
& \qquad\qquad\qquad\qquad\qquad (j = i + k \text{ and} \\
&= \textstyle\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_j B^{j-i} \sigma_a^2 \qquad \psi_j = 0, \text{ for } j < 0) \\
&= \sigma_a^2 \psi(B) \psi(B^{-1}),
\end{aligned}$$

$\Rightarrow$

$$\rho(B) = \sum \rho_k Bk = \frac{\gamma(B)}{\gamma_0}.$$

- $AR(\infty)$ representation : useful in understanding the mechanism of forecasting $\Rightarrow$ the process is invertible.

$$\dot{Z}_t = \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \cdots + a_t \text{ 其中} \dot{Z}_t = Z_t - \mu$$

$\Rightarrow$

$$\pi(B) Z_t = a_t, \quad \pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j$$

(註)

1. An invertible process is stationary

$\Longleftrightarrow$

$$Z_t = \frac{1}{\pi(B)} a_t = \psi(B) a_t \ s.t. \ \sum_{j=0}^{\infty} \psi_j^2 < \infty$$

$\Longleftrightarrow$

$$\pi(B) = 0\text{的根落在單位圓外。}$$

2. A linear process is invertible

$\Longleftrightarrow$

$$a_t = \psi^{-1}(B) Z_t = \phi(B) Z_t \ s.t. \ \sum \phi_j^2 < \infty$$

$\Longleftrightarrow$

$$\psi(B) = 0\text{的根落在單位圓外。}$$

- $AR(\infty)$ or $MA(\infty)$ contain an infinite number of parameters不利於建模型。

- $AR(p)$ $\quad Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t$
  $MA(q)$ $\quad Z_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$
  or $ARMA(p,q)$
  $Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$

§ Linear Difference Equations

nth order linear difference equation

$$Z_t + C_1 Z_{t-1} + C_2 Z_{t-2} + \cdots + C_n Z_{t-n} = e_t$$

The function e_t is called forecasting function.

1. 若$e_t = 0$, 則稱為homogeneous equation.
2. 若$e_t \neq 0$, 則稱為nonhomogeneous equation.

令$C(B) : 1 + C_1 B + C_2 B^2 + \cdots + C_n B^n$

$\Rightarrow C(B) Z_t = e_t$

$C(B) = 0$ is called the auxiliary equation.

Lemma

1. If $C(Z_t^{(1)}) = 0$ and $C(Z_t^{(2)}) = 0$, then
$$C(b_1 Z_t^{(1)} + b_2 Z_t^{(2)}) = 0, \quad \forall b_1,\ b_2 \text{constants.}$$
2. If $C(Z_t^{(H)}) = 0$ and $C(Z_t^{(P)}) = e_t$, then $Z_t^{(H)} + Z_t^{(P)}$ is the general solution of the complete equation.

- The eventual behavior of a time series model is often governed by a homogeneous difference equation.

Lemma $Z_t = bt^j$, $j < m$ (integers), $b :$ constant, 為$(1 - B)^m Z_t = 0$的解 。

pf: $m=1$, $Z_t = bt^0 = b \;\Rightarrow\; (1-B)Z_t = b-b=0$,

假設$(1-B)^{m-1}Z_t = 0$, for $Z_t = bt^j$, $j < m-1$,

則對$Z_t = bt^j$, $j < m$

$$\begin{aligned}(1-B)^m Z_t &= (1-B)^{m-1}(1-B)bt^j \\ &= (1-B)^{m-1}b(t^j-(t-1)^j) \\ &= (1-B)^{m-1}[-b\textstyle\sum_{i=0}^{j-1}(-1)^{j-i}t^i] \\ &= 0,\end{aligned}$$

故由數學歸納法得證。

(註) 由上面Lemma(1)亦可知

$$(1-B)^m(\sum_{j=0}^{m-1} b_j t^j) = 0,\ \forall b_0, b_1, \cdots, b_{m-1}.$$

Lemma $Z_t = R^t t^j$, $j < m$是$(1-RB)^m Z_t = 0$的解。

pf:

$$\begin{aligned}(1-RB)Z_t &= (1-RB)R^t t^j \\ &= R^t t^j - RR^{t-1}(t-1)^j \\ &= R^t(1-B)t^j\end{aligned}$$

$\Rightarrow\ (1-RB)^m R^t t^j = R^t(1-B)^m t^j = 0$ (由上面的Lemma)。

Thm $C(B)Z_t = 0$, $C(B) = 1 + C_1B + C_2B^2 + \cdots + C_nB^n$
If $C(B) = \prod_{i=1}^{N}(1 - R_iB)^{m_i}$, where $\sum_{i=1}^{N} m_i = n$, then

$$Z_t = \sum_{i=1}^{N} R_i^t \sum_{j=0}^{m_i-1} b_{ij}t^j$$

In particular, if $m_i = 1$, $\forall i$, and R_i^{-1} are distinct, then

$$Z_t = \sum_{i=1}^{n} b_iR_i^t$$

(註) complex roots必成對發生

$$c \pm d_i = \alpha(cos\phi + isin\phi).$$
$$\alpha = \sqrt{c^2 + d^2},\ \phi = tan^{-1}(\tfrac{d}{c}).$$

$\Rightarrow \quad (c + d_i)^t = \alpha^t(cos\phi t + isin\phi t)$

$\Rightarrow$ solution of the homogeneous difference equation將包含

$$\alpha^t cos\phi t,\ \alpha^t sin\phi t,\ t\alpha^t cos\phi t,\ t\alpha^t sin\phi t,$$
$$\cdots\cdots\cdots, t^{m-1}\alpha^t cos\phi t,\ t^{m-1}\alpha^t sin\phi t.$$

E.g.

$$Z_t - 2Z_{t-1} + Z_{t-2} = 0$$

$$\because \quad (1 - 2B + B^2) = (1 - B)^2 = 0$$

$$\therefore \quad Z_t = 1^t(b_1 + b_2t) = b_1 + b_2t$$

E.g.

$$Z_t - 2Z_{t-1} + 1.5Z_{t-2} - 0.5Z_{t-3} = 0$$

$$\because \quad (1 - 2B + 1.5B^2 - 0.5B^3) = (1 - B + 0.5B^2)(1 - B)$$

$$B_1 = R_1^{-1} = 1$$

$$B_2 = R_2^{-1} = \tfrac{1+\sqrt{1-4(0.5)}}{2(0.5)} = 1 + i$$

$$B_3 = 1 - i$$

$$\therefore \quad Z_t = b_1 + b_2(\sqrt{\tfrac{1}{2}})^t cos(\tfrac{\pi}{4}t) + b_3(\sqrt{\tfrac{1}{2}})^t sin(\tfrac{\pi}{4}t)$$

E.g.

$$(1 - \phi B)(1 - B)^2 Z_t = 0$$

$$\because \quad R_1 = \phi, \; R_2 = 1(q)$$

$$\therefore \quad Z_t = b_1\phi^t + b_2 + b_3 t$$