Numerical Analysis Assignment #5

1. Consider equidistant sampling points $x_k = k$ $(k = 0, 1, \ldots, n)$ and $\omega_n(x) = \prod_{k=0}^{n} (x-k)$.

   (a) Show that $\omega_n(x) = (-1)^{n+1}\omega_n(n - x)$. What kind of symmetry does this imply?

   (b) Show that $|\,\omega_n(x)| < |\,\omega_n(x + 1)|$ for nonintegral $x > (n - 1)/2$.

   (c) Show that the relative maxima of $|\,\omega_n(x)|$ increase monotonically (from the center of $[0, n]$ outward).

2. The error in linear interpolation of f at x_0, x_1 is known to be

$$f(x) - p_1(f; x) = (x - x_0)(x - x_1)\,\frac{f''(\xi(x))}{2}\,, \quad x_0 < x < x_1,$$

   if $f \in C^2[x_0, x_1]$. Determine $\xi(x)$ explicitly in the case $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$, and find $\max_{1\le x\le 2} \xi(x)$ and $\min_{1\le x\le 2} \xi(x)$.

3. (a) Let $p_n(f; x)$ be the interpolation polynomial of degree $\le n$ interpolating $f(x) = e^x$ at the points $x_i = i/n$, $i = 0, 1, 2, \ldots, n$. Derive an upper bound for

$$\max_{0\le x\le 1} |e^x - p_n(f; x)|,$$

   and determine the smallest n guaranteeing an error less than 10^{-6} on $[0, 1]$. {*Hint*: First show that for any integer i with $0 \le i \le n$ one has $\max_{0\le x\le 1} |(x - \frac{i}{n})(x - \frac{n-i}{n})| \le \frac{1}{4}$.}

   (b) Solve the analogous problem for the nth-degree Taylor polynomial $t_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, and compare the result with the one in (a).

4. With $\omega_n(x) = \prod_{k=0}^{n}(x - k)$, let M_n be the largest, and m_n the smallest, relative maximum of $|\,\omega_n(x)|$. Calculate M_n/m_n for $n = 5, 10, 15, \ldots, 30$, to 3 decimal places. {*Hint*. Use your method of choice to compute relative extrema, and state what it consists of.}