Numerical Analysis Assignment #9 Due: January 5, 2000

1. Let $\pi_n(\,\cdot\,;w)$ be the nth-degree orthogonal polynomial with respect to the weight function w on $[a,b]$, $t_1, t_2, \ldots, t_n$ its n zeros, and $w_1, w_2, \ldots, w_n$ the n Gauss weights.

   (a) Assuming $n > 1$, show that the n polynomials $\pi_0, \pi_1, \ldots, \pi_{n-1}$ are also orthogonal with respect to the *discrete* inner product $(u,v) = \sum_{\nu=1}^{n} w_\nu u(t_\nu) v(t_\nu)$.

   (b) With $\ell_i(t) = \prod_{k \neq i}[(t-t_k)/(t_i - t_k)]$, $i = 1, 2, \ldots, n$, denoting the elementary Lagrange interpolation polynomials associated with the nodes $t_1, t_2, \ldots, t_n$, show that
   $$\int_a^b \ell_i(t)\ell_k(t)w(t)dt = 0 \quad \text{if} \quad i \neq k.$$

2. Consider a quadrature formula of the type
   $$\int_0^\infty e^{-x} f(x)dx = af(0) + bf(c) + E(f).$$

   (a) Find a, b, c such that the formula has degree of exactness $d = 2$. Can you identify the formula so obtained? {*Point of information*: $\int_0^\infty e^{-x}x^r dx = r!$}

   (b) Let $p_2(x) = p_2(f; 0, 2, 2; x)$ be the Hermite interpolation polynomial interpolating f at the (simple) point $x = 0$ and the double point $x = 2$. Determine $\int_0^\infty e^{-x} p_2(x)dx$ and compare with the result in (a).

   (c) Obtain the remainder $E(f)$ in the form $E(f) = \text{const} \cdot f'''(\xi)$, $\xi > 0$.

3. Show that $\int_0^1 (1-t)^{-1/2} f(t)dt$, when f is smooth, can be computed accurately by Gauss-Legendre quadrature. {*Hint*: Substitute $1 - t = x^2$.}