Numerical Analysis **Answers to Assignment #1**

1. PROGRAM (in fortran)

```
      double precision da,db,dc,deps
      a=4./3.
      b=a-1.
      c=b+b+b
      eps=abs(c-1.)
      da=4.d0/3.d0
      db=da-1.d0
      dc=db+db+db
      deps=dabs(dc-1.d0)
      write(*,1) eps,deps
    1 format(1x,e12.4,d12.4)
      stop
      end
```

RESULT (on a Sparc workstation)

```
0.1192E-06  0.2220D-15
```

2. (a) Let $x = u \cdot 2^e$. Since $x_{\text{next}} = (u + 2^{-t}) \cdot 2^e$, one has $d(x) = x_{\text{next}} - x = 2^{-t} \cdot 2^e$.

   (b) We have $\frac{d(x)}{x} = \frac{2^{-t} \cdot 2^e}{x}$. Since x is normalized, one has $\frac{1}{2} \cdot 2^e \leq x \leq (1 - 2^{-t})2^e$, hence $\frac{2^{-t}}{1-2^{-t}} \leq \frac{d(x)}{x} \leq 2 \cdot 2^{-t}$.

3. (a) Since

$$|x_1^* - x_2^*| = |x_1(1 + E_1) - x_2(1 + E_2)| = |x_1 - x_2 + x_1E_1 - x_2E_2|$$
$$\geq |x_1 - x_2| - |x_1|\ |E_1| - |x_2|\ |E_2| \geq |x_1 - x_2| - (|x_1| + |x_2|)E,$$

   we certainly have $x_1^* \neq x_2^*$ if

$$|x_1 - x_2| - (|x_1| + |x_2|)E > 0,$$

   that is, if

$$E < \frac{|x_1 - x_2|}{|x_1| + |x_2|}.$$

(b)(i) We have

$$\frac{1}{x_1^* - x_2^*} = \frac{1}{x_1(1+E_1) - x_2(1+E_2)} = \frac{1}{x_1 - x_2 + x_1 E_1 - x_2 E_2}$$

$$= \frac{1}{(x_1 - x_2)\left(1 + \frac{x_1}{x_1 - x_2}\, E_1 - \frac{x_2}{x_1 - x_2}\, E_2\right)}$$

$$\approx \frac{1}{x_1 - x_2}\left(1 - \frac{x_1}{x_1 - x_2}\, E_1 + \frac{x_2}{x_1 - x_2}\right.$$

so that the relative error, in absolute value, is approximately equal to

$$\left| - \frac{x_1}{x_1 - x_2}\, E_1 + \frac{x_2}{x_1 - x_2}\, E_2 \right| \le \frac{|x_1| + |x_2|}{|x_1 - x_2|}\, E.$$

(b)(ii) Additional errors caused by subtraction and division amount to multiplication by

$$\frac{1+\varepsilon_2}{1+\varepsilon_1} \approx (1+\varepsilon_2)(1-\varepsilon_1) \approx 1 - \varepsilon_1 + \varepsilon_2, \qquad |\varepsilon_i| \le \text{eps},$$

so that the result in (i) must be multiplied by $1 - \varepsilon_1 + \varepsilon_2$, giving

$$\left| - \frac{x_1}{x_1 - x_2}\, E_1 + \frac{x_2}{x_1 - x_2}\, E_2 - \varepsilon_1 + \varepsilon_2 \right| \le \frac{|x_1| + |x_2|}{|x_1 - x_2|}\, E + 2\,\text{eps}.$$

4. In IEEE arithmetic, $X_+ = .34028 \times 10^{39}$ for single precision, and $X_+ = .17977 \times 10^{309}$ for double precision. We seek the smallest n such that $n \ln x > \ln X_+$. We find

single precision			double precision	
x	$n \ge$	$n =$	$n \ge$	$n =$
10	38.53	39	308.25	309
20	29.61	30	236.93	237
30	26.08	27	208.69	209
40	24.05	25	192.41	193
50	22.67	23	181.44	182
60	21.66	22	173.36	174
70	20.88	21	167.07	168
80	20.24	21	161.98	162
90	19.71	20	157.74	158
100	19.26	20	154.13	155

5. Express e^x in terms of y and obtain

$$\sinh x = \frac{1}{2}\left(y + 1 - \frac{1}{y+1}\right) = \frac{1}{2}\, y\, \frac{y+2}{y+1}\,.$$