Numerical Analysis   Answers to Assignment #10

1. (a) Since by assumption $f^{(d+1)}$ is continuous only for $d = 0$, we must compute the Peano kernel K_0 for the functional E. Assuming $0 < t < 1$, we have

$$\begin{aligned}
K_0(t) &= \int_0^1 \sqrt{x}(x-t)_+^0\,dx + \frac{2}{15}\cdot 0 - \frac{4}{5}\int_0^1 (x-t)_+^0\,dx \\
&= \int_t^1 \sqrt{x}\,dx - \frac{4}{5}\int_t^1 dx = \frac{2}{3}\,x^{\frac{3}{2}}\Big|_t^1 - \frac{4}{5}(1-t) \\
&= \frac{2}{3}(1-t^{\frac{3}{2}}) - \frac{4}{5}(1-t) \\
&= -\frac{2}{15}(1-6t+5t^{\frac{3}{2}}).
\end{aligned}$$

(b) Let $q(t) = 1 - 6t + 5t^{\frac{3}{2}}$. Then, $q(0) = 1$, $q(1) = 0$, $q'(t) = -6 + \frac{15}{2}t^{\frac{1}{2}}$. Since $q'(1) > 0$, the function q changes sign on (0,1). In fact, q decreases from 1 to a negative value and then increases from there on to zero. Thus, q has exactly one zero, $t = t_0$, in (0,1). To compute it, let $t = u^2$, $0 < u < 1$. Then

$$q(u^2) = 1 - 6u^2 + 5u^3 = (u-1)(5u^2 - u - 1),$$

and this has exactly one zero in (0,1) given by $u_0 = (1+\sqrt{21})/10$. Therefore, $t_0 = u_0^2 = (11+\sqrt{21})/50 = .3116515138\ldots$. It now follows that

$$|Ef| \le \int_0^1 |K_0(t)|\,|f'(t)|dt \le c_0\|f'\|_\infty,$$

where

$$\begin{aligned}
c_0 &= \int_0^1 |K_0(t)|dt = \frac{2}{15}\left(\int_0^{t_0} q(t)dt - \int_{t_0}^1 q(t)dt\right) \\
&= \frac{2}{15}\cdot 2(t - 3t^2 + 2t^{\frac{5}{2}})_{t=t_0} = .03432397564\ldots\ .
\end{aligned}$$

2. Taking for $f(t)$ in turn $1,\ t,\ t^2$ gives

$$\begin{array}{rrrl} \alpha_{-1} & +2\alpha_0 & +\alpha_1 & = 4, \\ -\alpha_{-1} & & +\alpha_1 & = 0, \\ \alpha_{-1} & & +\alpha_1 & = \frac{16}{7}. \end{array}$$

The solution is

$$\alpha_{-1} = \alpha_1 = \frac{8}{7}, \quad \alpha_0 = \frac{6}{7}.$$

Since $E(t^3) = 0$ and $E(t^4) = \frac{2}{5} - 2 \cdot \frac{8}{7} \cdot \frac{1}{5} \cdot \frac{31}{32} = -\frac{3}{70} \neq 0$, we have $d = 3$.

3. By contradiction. Let $0 \leq \delta \leq d-1$, and suppose that K_δ is definite. Then, for any $f \in C^{\delta+1}[a,b]$,

$$E(f) = \int_a^b K_\delta(t) f^{(\delta+1)}(t) dt = f^{(\delta+1)}(\tau) \int_a^b K_\delta dt, \quad \tau \in (a,b),$$

where

$$\int_a^b K_\delta(t) dt = E\left(\frac{x^{\delta+1}}{(\delta+1)!}\right) = 0,$$

since $\delta + 1 \leq d$ and the formula has degree of exactness d. This is absurd: Take any function $f \in C^{\delta+1}[a,b]$ such that $f(x_k) = 0$, $k = 1, 2, \ldots, n$, and $f(x) \geq 0$ on $[a,b]$ with $f(x) \not\equiv 0$, to get the contradiction

$$0 = E(f) = \int_a^b f(x) dx > 0.$$

For example, $f(x) = \prod_{k=1}^n (x - x_k)^2$ will do.

4. (a) The formula is exact for quadratic polynomials if it is exact for the first three powers of x,

$$\begin{array}{rrl} a + & b & = 1, \\ & b & = \frac{1}{2}, \\ & b + \ 2c & = \frac{1}{3}, \end{array}$$

that is, if

$$a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = -\frac{1}{12}.$$

To get exactness for $f(t) = t^3$ requires $b + 6c\gamma = \frac{1}{4}$, giving

$$\gamma = \frac{1}{2}.$$

Since, with these choices of the parameters,

$$E(x^4) = \frac{1}{5} - \frac{1}{2} + \frac{1}{4} = -\frac{1}{20} \neq 0,$$

the precise degree of exactness is $d = 3$.

(b) With

$$E(f) = \int_0^1 f(x)dx - \frac{1}{2}\,f(0) - \frac{1}{2}\,f(1) + \frac{1}{12}\,f''(\frac{1}{2})$$

we have

$$6K_3(t) = E_{(x)}((x-t)_+^3) = \int_t^1 (x-t)^3 dx - \frac{1}{2}\,(1-t)^3 + \frac{1}{12}\cdot 6\,(\frac{1}{2} - t)_+.$$

Thus, if $0 \leq t \leq \frac{1}{2}$, then

$$6K_3(t) = \frac{1}{4}\,(1-t)^4 - \frac{1}{2}\,(1-t)^3 + \frac{1}{2}\,(\frac{1}{2} - t) = -\frac{1}{4}\,t^3(2-t),$$

and if $\frac{1}{2} \leq t \leq 1$,

$$6K_3(t) = -\frac{1}{4}\,(1-t)^3(2-(1-t)).$$

We see that K_3 is negative definite on $[0,1]$ and symmetric with respect to the midpoint. In particular,

$$E(f) = e_4 f^{(4)}(\tau), \qquad 0 < \tau < 1,$$

with

$$e_4 = E(\frac{x^4}{4!}) = -\frac{1}{4!}\frac{1}{20} = -\frac{1}{480} \quad \text{(cf. (a))}.$$