Numerical Analysis **Answers to Assignment #5**

1. (a) We have

$$\omega_n(n-x) = \prod_{k=0}^{n}(n-x-k) \stackrel{(j=n-k)}{=} \prod_{j=0}^{n}(j-x) = (-1)^{n+1}\prod_{j=0}^{n}(x-j).$$

This implies symmetry (when n is odd) or skew symmetry (when n is even) with respect to the line $x = \frac{1}{2}n$.

(b) One has

$$\begin{aligned}\omega_n(x+1) &= \prod_{k=0}^{n}(x+1-k) = \prod_{j=-1}^{n-1}(x-j)\\ &= \frac{x+1}{x-n}\prod_{j=0}^{n}(x-j) = \frac{x+1}{x-n}\,\omega_n(x).\end{aligned}$$

Therefore,

$$|\omega_n(x)| = \frac{|n-x|}{x+1}\,|\omega_n(x+1)| < |\omega_n(x+1)|$$

since $\dfrac{|n-x|}{x+1} < 1$ if $x > \dfrac{n-1}{2}$.

(c) Let

$$|\omega_n(\xi_k)| = \max_{k\le x\le k+1}|\omega_n(x)| = M_k, \quad 0 \le k < n.$$

Then, using (b) and $\xi_k > \dfrac{n-1}{2}$ if $k \ge \dfrac{n-1}{2}$, we have

$$M_k = |\omega_n(\xi_k)| < |\omega_n(\xi_k+1)| \le \max_{k+1\le x\le k+2}|\omega_n(x)| = M_{k+1},$$

whenever $\frac{n-1}{2} \le k < n-1$.

2. Since $f(x) = x^{-1}$, $f''(x) = 2x^{-3}$ and $p_1(f;x) = (2-x)f(1) + (x-1)f(2) = 2 - x + \frac{1}{2}(x-1) = \frac{1}{2}(3-x)$, we have

$$f(x) - p_1(f;x) = \frac{1}{x} - \frac{1}{2}\,(3-x) = \frac{(x-1)(x-2)}{2x} = (x-1)(x-2)\frac{1}{\xi^3(x)}.$$

The last equality gives

$$\xi^3(x) = 2x, \quad \xi(x) = (2x)^{1/3}.$$

Furthermore,

$$\max_{1\le x\le 2} \xi(x) = 4^{1/3} = 1.5874\ldots\ ; \quad \min_{1\le x\le 2} \xi(x) = 2^{1/3} = 1.2599\ldots\ .$$

3. (a) We have

$$e^x - p_n(f;x) = \frac{e^{\xi(x)}}{(n+1)!} \prod_{k=0}^{n} \left(x - \frac{k}{n}\right), \qquad 0 < \xi(x) < 1.$$

Here we use

$$\prod_{k=0}^{n} \left|x - \frac{k}{n}\right| = \sqrt{\prod_{i=0}^{n} \left|x - \frac{i}{n}\right| \left|x - \frac{n-i}{n}\right|}$$

and the fact that the maximum of $\left|(x - \frac{i}{n})(x - \frac{n-i}{n})\right|$ must occur either at $x = 0$ or at $x = \frac{1}{2}$, and hence is the larger of $\frac{i(n-i)}{n^2}$ and $\frac{(n-2i)^2}{4n^2}$. The former attains its maximum at $i = \frac{n}{2}$, the latter at $i = 0$ (and $i = n$). Either one equals $\frac{1}{4}$. Thus,

$$\max_{0\le x\le 1} \left|(x - \frac{i}{n})(x - \frac{n-i}{n})\right| \le \frac{1}{4} \quad \text{for} \quad i = 0, 1, \ldots, n.$$

There follows

$$\max_{0\le x\le 1} |e^x - p_n(f;x)| \le \frac{e}{(n+1)!} \left(\frac{1}{4}\right)^{\frac{n+1}{2}} = \frac{e}{2^{n+1}(n+1)!}.$$

The smallest n making the upper bound $\le 10^{-6}$ is $n = 7$.

(b) From Taylor's formula,

$$e^x - t_n(x) = \frac{e^{\xi(x)}}{(n+1)!}\, x^{n+1}, \qquad 0 < \xi(x) < 1.$$

Thus,

$$|e^x - t_n(x)| \le \frac{e}{(n+1)!}.$$

For the bound to be $\le 10^{-6}$ now requires $n = 9$.

4. PROGRAM

```
      eps=.5e-5
      do 20 n=5,30,5
        ommin=1.e30
        ommax=0.
        do 10 k=0,(n-1)/2
          xlow=float(k)+.01
          xhigh=float(k+1)-.01
          ylow=omder(n,xlow)
          yhigh=omder(n,xhigh)
          slow=sign(1.,ylow)
          shigh=sign(1.,yhigh)
          if(slow*shigh.ge.0.) then
            write(*,2) slow,shigh
    2       format(1x,'slow=',f5.2,' shigh=',f5.2,' of the same sign')
            stop
          end if
c
c bisection method to find the zero of the derivative of omega
c
          i=0
          do while (xhigh-xlow.gt.eps .and. i.lt.100)
            i=i+1
            x=.5*(xlow+xhigh)
            s=sign(1.,omder(n,x))
            if(s*slow.ge.0.) then
              xlow=x
            else
              xhigh=x
            end if
          end do
          omabs=abs(om(n,x))
          if(omabs.lt.ommin) ommin=omabs
          if(omabs.gt.ommax) ommax=omabs
   10   continue
        ratio=ommax/ommin
```

```
        write(*,3) n,ratio,i
    3   format(1x,i8,e12.3,i5)
   20 continue
      stop
      end

      function om(n,x)
      p=1.
      do 10 k=0,n
        p=(x-float(k))*p
   10 continue
      om=p
      return
      end

      function omder(n,x)
      s=0.
      do 10 k=0,n
        s=s+1./(x-float(k))
   10 continue
      omder=s
      return
      end
```

OUTPUT

```
       n     ratio    no. it

        5   0.481E+01   18
       10   0.867E+02   18
       15   0.215E+04   18
       20   0.544E+05   18
       25   0.152E+07   18
       30   0.419E+08   18
```