Answers to Assignment #6

1. Let $\mathring{T}_n(x) = x^n - t_{n-1}(x)$ be the monic Chebyshev polynomial of degree n. Since $\| \mathring{T}_n \|_\infty \le 2^{-(n-1)}$, we have

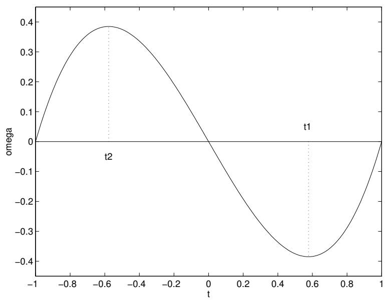
$$\| x^n - t_{n-1}(x) \|_\infty \le 2^{-(n-1)},$$

showing that $t_{n-1}(x)$ is a linear combination of $1, x, \ldots, x^{n-1}$ of the kind desired.

2. (a) Exploiting symmetry, let $x = x_1 + th$, $-1 \le t \le 1$. Then

$$(x - x_0)(x - x_1)(x - x_2) = (1 + t)h \cdot th \cdot (t - 1)h = t(t^2 - 1)h^3.$$

The function $\omega(t) = t(t^2 - 1)$ has two extrema at



$$t_{1,2} = \pm\sqrt{1/3}, \text{ with values}$$
$$|\,\omega(t_{1,2})| = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}}.$$

Therefore,

$$|(x - x_0)(x - x_1)(x - x_2)| \le \frac{2}{3\sqrt{3}}\, h^3 \quad \text{for} \quad x_0 \le x \le x_2,$$

and

$$|f(x) - p_2(f; x)| = |(x - x_0)(x - x_1)(x - x_2)\,| \left|\frac{f'''(\xi)}{3!}\right|$$
$$\le \frac{2}{3\sqrt{3}}\, h^3 \frac{M_3}{3!} = \frac{M_3}{9\sqrt{3}}\, h^3.$$

(b) The Chebyshev points on $[x_0, x_2]$ are

$$\hat{x}_i = x_1 + \hat{t}_i h, \quad \hat{t}_i = \cos\ (2i+1)\frac{\pi}{6}, \quad i = 0, 1, 2.$$

Letting (as in (a)) $x = x_1 + th$, we get

$$(x - \hat{x}_0)(x - \hat{x}_1)(x - \hat{x}_2) = (t - \hat{t}_0)h \cdot (t - \hat{t}_1)h \cdot (t - \hat{t}_2)h$$

$$= (t - \hat{t}_0)(t - \hat{t}_1)(t - \hat{t}_2)h^3 = \mathring{T}_3(t)h^3 = \frac{1}{4}\ T_3(t)h^3.$$

Therefore,

$$|(x - \hat{x}_0)(x - \hat{x}_1)(x - \hat{x}_2)| \leq \frac{1}{4}\ h^3 \quad \text{for} \quad x_0 \leq x \leq x_2,$$

as opposed to $\leq \dfrac{2}{3\sqrt{3}}\ h^3 = .38490\ldots h^3$ in the equally spaced case.

3. (a) One has

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\ \prod_{k=0}^{n}(x - x_k), \quad -1 < \xi < 1.$$

Now, for $f(x) = \ln(2 + x)$, the derivatives are

$$f'(x) = \frac{1}{2+x}, \ f''(x) = -\frac{1}{(2+x)^2}, \ldots, f^{(n+1)}(x) = \frac{(-1)^n n!}{(2+x)^{n+1}}\ .$$

Therefore,

$$f(x) - p_n(x) = (-1)^n \frac{n!}{(2+\xi)^{n+1}(n+1)!}\ \prod_{k=0}^{n}(x - x_k)$$

$$= \frac{(-1)^n}{(2+\xi)^{n+1}(n+1)} \cdot \frac{1}{2^n} T_{n+1}(x).$$

There follows, since $\xi > -1$ and $|T_{n+1}(x)| \leq 1$,

$$\| f - p_n \|_\infty \ \leq \ \frac{1}{(n+1)2^n}\ .$$

(b) From calculus (or the theory of Hermite interpolation),

$$f(x) - t_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}, \quad |\xi| \in (0, |x|).$$

Hence, for $|x| \leq 1$,

$$\| f - t_n \|_\infty \;\leq\; \frac{n!|x|^{n+1}}{(2+\xi)^{n+1}(n+1)!} \leq \frac{1}{n+1}\;.$$

The bound is larger by a factor of 2^n compared to the bound in (a). However, by using the full Taylor expansion of f, one gets

$$\begin{aligned}\| f - t_n \|_\infty \;&\leq\; \sum_{k=n+1}^{\infty} \left| \frac{f^{(k)}(0)}{k!} \right| \;=\; \sum_{k=n+1}^{\infty} \frac{(k-1)!}{k!2^k} \\ &= \sum_{k=n+1}^{\infty} \frac{1}{k \cdot 2^k} \;\leq\; \frac{1}{(n+1)2^n},\end{aligned}$$

which is the same bound as in (a). The comparison, nevertheless, is somewhat unfair, since more powerful series expansion techniques are used than the simple technique in part (a).

4. Put $x = \cos\,\theta$. Then

$$T_n(T_m(\cos\,\theta)) = T_n(\cos\, m\theta) = \cos(n \cdot m\theta) = T_{nm}(\cos\,\theta).$$

Therefore,

$$T_n(T_m(x)) = T_{nm}(x).$$