Numerical Analysis **Answers to Assignment #8**

1. (a) If n is even, then $T(2/n)$ is the trapezoidal rule applied to [–1,0] plus the trapezoidal rule applied to [0,1], each being exact.

   (b) Denoting by $T(h)[a,b]$ the composite trapezoidal rule applied to $[a,b]$ with steplength h, we have, when n is odd,

$$
\begin{aligned}
T(2/n) &= T(2/n)[-1,-1/n] + T(2/n)[1/n,1] + \frac{2}{n}\left[\frac{1}{2}\,\frac{1}{n} + \frac{1}{2}\,\frac{1}{n}\right] \\
&= \int_{-1}^{-1/n} |x|dx + \int_{1/n}^{1} |x|dx + \frac{2}{n}\cdot\frac{1}{n} \\
&= 2\int_{1/n}^{1} x dx + \frac{2}{n^2} = 2\left[\frac{1}{2}\,x^2\right]_{1/n}^{1} + \frac{2}{n^2} \\
&= 1 - \frac{1}{n^2} + \frac{2}{n^2} = 1 + \frac{1}{n^2}\ .
\end{aligned}
$$

   Thus, we still have $O(h^2)$ convergence, even though the integrand is only in $C[-1,1]$. This is so because what we see is essentially a local error, which is normally of $O(h^3)$.

2. (a) By Taylor's theorem, writing $x_{k+\frac{1}{2}} = x_k + \frac{1}{2}\,h$, we have

$$
f(x) = f(x_{k+\frac{1}{2}}) + (x - x_{k+\frac{1}{2}})f'(x_{k+\frac{1}{2}}) + \tfrac{1}{2}(x - x_{k+\frac{1}{2}})^2 f''(\xi(x)), \\
x_k \le x \le x_k + h,
$$

   where $\xi(x) \in [x_k, x_k + h]$. Integrating gives

$$
\int_{x_k}^{x_k+h} f(x)dx = hf(x_{k+\frac{1}{2}}) + f'(x_{k+\frac{1}{2}})\int_{x_k}^{x_k+h}(x - x_{k+\frac{1}{2}})dx \\
+\frac{1}{2}\int_{x_k}^{x_k+h}(x - x_{k+\frac{1}{2}})^2 f''(\xi(x))dx.
$$

   Here, the first integral on the right vanishes (by skew symmetry), and to the last integral we can apply the mean value theorem. This yields

$$
\int_{x_k}^{x_k+h} f(x)dx = hf(x_{k+\frac{1}{2}}) + \frac{1}{2}\ f''(\xi_k)\int_{x_k}^{x_k+h}(x - x_{k+\frac{1}{2}})^2 dx, \\
x_k < \xi_k < x_k + h.
$$

The integral on the right is most easily evaluated by making the change of variables $x = x_{k+\frac{1}{2}} + \frac{1}{2} th$, $-1 \le t \le 1$, giving

$$\int_{x_k}^{x_k+h} (x - x_{k+\frac{1}{2}})^2 dx = \int_{-1}^{1} \left(\frac{1}{2}\ th\right)^2 \frac{1}{2}\ hdt = \frac{h^3}{8} \int_{-1}^{1} t^2 dt = \frac{h^3}{12},$$

so that

$$\int_{x_k}^{x_k+h} f(x)dx = hf(x_{k+\frac{1}{2}}) + \frac{h^3}{24}\ f''(\xi_k), \quad x_k < \xi_k < x_k + h.$$

(b) Summing over all subintervals, we get

$$\int_a^b f(x)dx = h \sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) + \frac{h^2}{24} \frac{b-a}{n} \sum_{k=0}^{n-1} f''(\xi_k),$$

that is,

$$\int_a^b f(x)dx = h \sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) + \frac{h^2}{24}(b-a)f''(\xi), \quad a < \xi < b.$$

3. (a) Putting $f(x) \equiv 1$ and $f(x) \equiv x$ gives, respectively,

$$\begin{aligned} a_0 \quad +a_1 &= \tfrac{1}{4}\ , \\ a_1 &= \tfrac{1}{9}\ . \end{aligned}$$

Hence, $a_0 = 5/36$ and $a_1 = 1/9$, giving

$$\int_0^1 f(x) \cdot x \ln\ (1/x)dx \approx \frac{5}{36}\ f(0) + \frac{1}{9}\ f(1).$$

(b) Let $t = hx$, $dt = hdx$. Then

$$\int_0^h g(t) \cdot t \ln\ (1/t)dt = \int_0^1 g(hx) \cdot hx \cdot \ln\left(\frac{1}{hx}\right) \cdot hdx$$

$$= h^2 \left\{ \int_0^1 g(hx) \cdot x \ln\left(\frac{1}{x}\right) dx + \int_0^1 g(hx)x \cdot \ln\left(\frac{1}{h}\right) dx \right\}.$$

Evaluating the first integral as in (a), and the second by the trapezoidal rule (for example), one gets

$$\int_0^h g(t) \cdot t \ln\ (1/t)dt \approx h^2 \left(\frac{5}{36}\ g(0) + \frac{1}{9}\ g(h)\right) + \frac{1}{2}\ h^2 \ln\left(\frac{1}{h}\right) g(h).$$

4. (a) $s \in \mathbb{S}_3^0(\Delta)$.

   (b) Since the composite trapezoidal rule, with n even, can be interpreted as the sum of two composite trapezoidal rules, one over the interval $[-1,0]$, the other over $[0,1]$, the error is

$$E_n^T(s) = -\frac{h^2}{12}[s''(\xi_{-1}) + s''(\xi_1)]$$

   for some $-1 < \xi_{-1} < 0$, $0 < \xi_1 < 1$. Thus,

$$E_n^T(s) = -\frac{h^2}{12} \cdot 6(\xi_{-1} + 1 + 1 - \xi_1) = -\frac{h^2}{2}(2 + \xi_{-1} - \xi_1).$$

   Since $-2 < \xi_{-1} - \xi_1 < 0$, we can write this also as

$$E_n^T(s) = -\frac{h^2}{2}\xi, \quad 0 < \xi < 2.$$

   (c) If n is divisible by 4, the error is zero by a reasoning similar to the one in (b) but noting that the composite Simpson's rule integrates cubics exactly. If n is not divisible by 4, then the error is

$$\begin{aligned} E_n^S(s) &= \int_{-h}^{h} s(x)dx - \frac{h}{3}(s(-h) + 4s(0) + s(h)) \\ &= 2\int_0^h s(x)dx - \frac{2h}{3}(2s(0) + s(h)), \end{aligned}$$

   which by an elementary calculation becomes

$$E_n^S(s) = -h^2\left(1 - \tfrac{1}{6}h^2\right) = O(h^2).$$

   It is seen that Simpson's rule, in this case, has the same order of accuracy as the trapezoidal rule.

   (d) The error is zero since the Gauss-Legendre rule integrates cubics exactly.

5. Let $\pi_{n+1}^*(t) = \prod_{k=1}^{n+1}(t - t_k^*)$. Since $\prod_{k=1}^{n}(t - t_k^{(n)}) = \pi_n(t; w)$, and the node polynomial is $\omega_n(t) = \pi_n(t; w)\pi_{n+1}^*(t)$, the quadrature rule in question, by part (b) of Theorem 3.2.1, has degree of exactness $d = 2n + 1 - 1 + k = 2n + k$ only if

$$\int_a^b \pi_{n+1}^*(t)p(t)\pi_n(t; w)w(t)dt = 0, \quad \text{all}$$

If we want $d = 3n + 1$, this must hold for $k = n + 1$, i.e., $\pi^*_{n+1}(t)$ must be orthogonal to all lower-degree polynomials relative to the oscillating weight function $\pi_n(t; w)w(t)$. By part (a) of the same theorem, the quadrature rule must be interpolatory, which uniquely determines the weights w_k and w^*_k if the t^*_k are distinct among themselves and from the $t^{(n)}_k$.