Numerical Analysis **Answers to Assignment #9**

1. (a) Use the Gauss quadrature rule and the fact that $\pi_k \cdot \pi_\ell \in \mathbb{P}_{2n-2}$ for $k, \ell = 0, 1, \ldots, n-1$, to obtain

$$\sum_{\nu=1}^{n} w_\nu \pi_k(t_\nu)\pi_\ell(t_\nu) = \int_a^b \pi_k(t)\pi_\ell(t)w(t)dt = 0, \quad k \neq \ell.$$

(b) Apply Gaussian quadrature: since $\ell_i \cdot \ell_k \in \mathbb{P}_{2n-2}$, we get

$$\begin{aligned}\int_a^b \ell_i(t)\ell_k(t)w(t)dt &= \sum_{\nu=1}^{n} w_\nu \ell_i(t_\nu)\ell_k(t_\nu)\\ &= \sum_{\nu=1}^{n} w_\nu \delta_{i\nu}\delta_{k\nu} = w_k\delta_{ik} = 0, \quad i \neq k.\end{aligned}$$

2. (a) Putting $f(x) = x^r$, $r = 0, 1, 2$, one obtains

$$\begin{aligned} a \;\; +b &= 1,\\ bc &= 1,\\ bc^2 &= 2.\end{aligned}$$

Clearly, $bc \neq 0$. Dividing the third equation by the second gives $c = 2$, hence $b = \frac{1}{2}$ and $a = \frac{1}{2}$. Thus,

$$\int_0^\infty e^{-x} f(x)dx = \frac{1}{2}\,[f(0) + f(2)] + E(f).$$

This is the 2-point Gauss-Radau formula for the weight function e^{-x} on $[0,\infty]$.

(b) ¿From the table of divided differences

x	f		
0	f_0		
2	f_2	$\frac{1}{2}\,(f_2 - f_0)$	
2	f_2	f_2'	$\frac{1}{2}\left(f_2' - \frac{f_2 - f_0}{2}\right)$

one gets

$$\begin{aligned} p_2(x) &= f_0 + \frac{1}{2}\,(f_2 - f_0)x + \frac{1}{2}\left(f_2' - \frac{f_2 - f_0}{2}\right)x(x-2)\\ &= \left(1 - x + \frac{1}{4}\,x^2\right)f_0 + \left(x - \frac{1}{4}\,x^2\right)f_2 + \frac{1}{2}\,x(x-2)f_2',\end{aligned}$$

hence

$$\begin{aligned}
\int_0^\infty e^{-x} p_2(x)dx &= f_0 \int_0^\infty e^{-x}(1 - x + \frac{1}{4}\, x^2)dx \\
+f_2 \int_0^\infty e^{-x}(x - \frac{1}{4}\, x^2)dx &+ f_2' \int_0^\infty e^{-x}\frac{1}{2}\, x(x-2)dx \\
= \left(1 - 1 + \frac{1}{4} \cdot 2\right) f_0 + \left(1 - \frac{1}{4} \cdot 2\right) f_2 &+ \frac{1}{2}\, (2-2) f_2' = \frac{1}{2}\, f_0 + \frac{1}{2}\, f_2,
\end{aligned}$$

the same formula as the one obtained in (a).

(c) Using the remainder term for Hermite interpolation, one gets

$$\begin{aligned}
E(f) &= \int_0^\infty x(x-2)^2\, \frac{f'''(\xi(x))}{6}\, e^{-x}dx \\
&= \frac{1}{6}\, f'''(\xi) \int_0^\infty (x^3 - 4x^2 + 4x)e^{-x}dx \\
&= \frac{1}{6}\, f'''(\xi)(6 - 4 \cdot 2 + 4) = \frac{1}{3}\, f'''(\xi).
\end{aligned}$$

3. Following the hint, put $1 - t = x^2,\ \ dt = -2xdx$. Then

$$\begin{aligned}
\int_0^1 (1-t)^{-1/2} f(t)dt &= -\int_1^0 \frac{1}{x}\, f(1-x^2) \cdot 2xdx \\
&= 2 \int_0^1 f(1-x^2)dx.
\end{aligned}$$

Here, the last integral can be accurately computed by Gauss-Legendre quadrature on $[0,1]$, since the integrand is smooth.

4. The composite trapezoidal rule is

$$I_p \approx h\{\tfrac{1}{2}\, f(0) + \sum_{k=1}^{n-1} (1-t_k)^p f(t_k)\},$$

where $h = \frac{1}{n},\ t_k = kh$. The Gauss-Jacobi formula is applied, after a preliminary transformation to the interval $[-1,1]$, in the form

$$I_p = \tfrac{1}{2^{p+1}} \int_{-1}^1 (1-t)^p f(\tfrac{1+t}{2})dt \approx \tfrac{1}{2^{p+1}} \sum_{k=1}^n w_k^J f(\tfrac{1+t_k^J}{2}),$$

where $t_k^J,\ w_k^J$ are the nodes and weights of the n-point Gauss-Jacobi formula with parameters $\alpha = p,\ \beta = 0$.

PROGRAM

```
   dimension a(5),b(5),x(5),w(5),e(5),fit(5),fig(5)
   write(*,1)
1  format(/)
   eps=r1mach(3)
   do 60 i=1,4
     ip=5*i
     p=float(ip)
     do 20 n=10,50,10
       h=1./float(n)
       sum=.5*f(0.)
       do 10 k=1,n-1
         tk=h*float(k)
         sum=sum+((1.-tk)**ip)*f(tk)
10     continue
       fit(n/10)=h*sum
20   continue
     call recur(5,6,p,0.,a,b,ierr)
     do 40 n=1,5
       call gauss(n,a,b,eps,x,w,ierr,e)
       sum=0.
       do 30 k=1,n
         sum=sum+w(k)*f(.5*(1+x(k)))
30     continue
       fig(n)=sum/(2.**(ip+1))
40   continue
     do 50 in=1,5
       write(*,2) 10*in,fit(in),in,fig(in)
 2     format(1x,i5,e17.7,i8,e17.7)
50   continue
     write(*,1)
60 continue
   stop
   end
```

```
      function f(x)
      f=tan(x)
      return
      end
```

RESULTS

```
   n      trapez         n     gauss

  10   0.2370035E-01     1  0.2397282E-01    p=5
  20   0.2431732E-01     2  0.2450806E-01
  30   0.2443262E-01     3  0.2452452E-01
  40   0.2447307E-01     4  0.2452510E-01
  50   0.2449179E-01     5  0.2452512E-01

  10   0.6865513E-02     1  0.7593351E-02    p=10
  20   0.7455963E-02     2  0.7660925E-02
  30   0.7569821E-02     3  0.7661936E-02
  40   0.7610015E-02     4  0.7661961E-02
  50   0.7628675E-02     5  0.7661957E-02

  10   0.2947504E-02     1  0.3680710E-02    p=15
  20   0.3495462E-02     2  0.3698230E-02
  30   0.3606878E-02     3  0.3698386E-02
  40   0.3646654E-02     4  0.3698387E-02
  50   0.3665201E-02     5  0.3698390E-02

  10   0.1479825E-02     1  0.2165997E-02    p=20
  20   0.1973725E-02     2  0.2172409E-02
  30   0.2081782E-02     3  0.2172445E-02
  40   0.2120975E-02     4  0.2172446E-02
  50   0.2139362E-02     5  0.2172447E-02
```

The Gauss rule is incomparably faster than the trapezoidal rule and seems to gain in speed as p increases, unlike the trapezoidal rule.