Definition:
$(cond_{\nu\mu}\ f)(x) := \left| \frac{x_\mu \frac{\partial f_\nu}{\partial x_\mu}}{f_\nu(x)} \right| = \Gamma_{\nu\mu}$
This gives us a whole matrix $\Gamma(x) = [\gamma_{\nu\mu}(x)] \in \mathcal{R}_+^{n\times m}$ of condition numbers. To obtain a single condition number, we can take any convenient measure of the "magnitude" of the matrix $\Gamma(x)$ such as one of the matrix norms

$$(cond\ f)(x) = \|\Gamma(x)\|, \Gamma(x) = [\gamma_{\nu\mu}(x)].$$

The condition so defined, of course, depends on the choice of norm, but the order of magnitude should be more or less the same for any reasonable norm.
Example:

$$f(x) = \begin{bmatrix} \frac{1}{x_1} + \frac{1}{x_2} \\ \frac{1}{x_1} - \frac{1}{x_2} \end{bmatrix} = \frac{1}{x_1 x_2} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Df = \begin{bmatrix} \frac{-1}{x_1^2} & \frac{-1}{x_2^2} \\ \frac{-1}{x_1^2} & \frac{1}{x_2^2} \end{bmatrix} = \frac{-1}{x_1^2 x_2^2} \begin{bmatrix} x_2^2 & x_1^2 \\ x_2^2 & -x_1^2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \left| \frac{x_1 \cdot \frac{-1}{x_1^2}}{\frac{x_1+x_2}{x_1 x_2}} \right| & \left| \frac{x_2 \cdot \frac{-1}{x_2^2}}{\frac{x_1+x_2}{x_1 x_2}} \right| \\ \left| \frac{x_1 \cdot \frac{-1}{x_1^2}}{\frac{x_2-x_1}{x_1 x_2}} \right| & \left| \frac{x_2 \cdot \frac{1}{x_2^2}}{\frac{x_2-x_1}{x_1 x_2}} \right| \end{bmatrix} = \begin{bmatrix} \left| \frac{x_2}{x_1+x_2} \right| & \left| \frac{x_1}{x_1+x_2} \right| \\ \left| \frac{x_2}{x_2-x_1} \right| & \left| \frac{x_1}{x_2-x_1} \right| \end{bmatrix}$$

$\Gamma(x)$ indicates ill-conditioning if either $x_1 \approx x_2$ or $x_1 \approx -x_2$ and $|x_1|$ (hence $|x_2|$) is not small.

However, if we use the definition

$$(cond\, f)(x) := \frac{\|x\|_1 \|Df\|_1}{\|f\|_1}$$

$$\|Df\|_1 = \frac{2}{x_1^2 x_2^2} \max(x_1^2, x_2^2), \|x\|_1 = |x_1| + |x_2|$$

$$\|f\|_1 = \frac{1}{|x_1 x_2|}(|x_1 + x_2| + |x_1 - x_2|)$$

then

$$(cond\ f)(x) = 2\frac{|x_1| + |x_2|}{|x_1 x_2|}\frac{\max(x_1^2, x_2^2)}{|x_1 + x_2| + |x_1 - x_2|}$$

Hence

$$x_1 \approx x_2 \quad or \quad x_1 \approx -x_2 \Longrightarrow (cond\ f)(x) = 2$$

$$\begin{aligned}
\|Df\|_\infty &= (\frac{1}{x_1^2 x_2^2}) \max(x_1^2 + x_2^2, x_1^2 + x_2^2) = \frac{x_1^2 + x_2^2}{x_1^2 x_2^2} \\
\|f\|_\infty &= \frac{1}{x_1 x_2} \max(|x_1 + x_2|, |x_1 - x_2|) \\
\|x\|_\infty &= \max(x_1, x_2)
\end{aligned}$$

$$\begin{aligned}
(cond\ f)_\infty \quad &= \frac{\max(x_1, x_2)|x_1 x_2|}{\max(|x_1 + x_2|, |x_1 - x_2|)}(\frac{x_1^2 + x_2^2}{x_1^2 x_2^2}) \\
&\text{if } {}_{x_1 \approx x_2} \frac{|x_1||x_1|^2}{2|x_1|}(\frac{2x_1^2}{x_1^4}) \sim 1
\end{aligned}$$

Not too bad if $x_1 \neq 0$.
These are obviously misleading.

Example: This example deals with a map from the integers to reals, and therefore does not fit our concepts. However, we propose to compute I_n recursively by relating I_k to I_{k-1}.

$$I_n = \int_0^1 \frac{t^n}{t+5} dt$$

$$\int_0^1 \frac{1}{t+5} dt = \ln|t+5|\Big|_0^1 = \ln\frac{6}{5}$$

$$\int_0^1 \frac{t}{t+5} dt = \int_0^1 dt + 5\int_0^1 \frac{1}{t+5} dt = 1 - 5 \cdot \ln\frac{6}{5}$$

$$\text{So} \quad I_n \searrow 0 \quad n \longrightarrow \infty,$$

since t^n is decreasing to 0, as $n \to \infty$.

We have $f : N \longrightarrow \Re$ which doesn't fit the setting of the condition number

Let's try

$$(\frac{t}{t+5} = 1 - \frac{5}{t+5})t^{k-1}$$

$$\int_0^1 \frac{t^k}{t+5}dt = \int_0^1 t^{k-1}dt - 5\int_0^1 \frac{t^{k-1}}{t+5}dt$$

$$\text{So} \ , \qquad I_k = \frac{1}{k} - 5I_{k-1}$$

$$y_k = -5y_{k-1} + \frac{1}{k}$$

$$y_1 = -5y_0 + 1$$

$$y_2 = -5(-5y_0 + 1) + \frac{1}{2} = (-5)^2 y_0 - 5 + \frac{1}{2}$$

$$\vdots$$

$$y_n = (-5)^n y_0 + p_n, \text{ for some number } p_n$$

Then the problem changes to $y_0 \longrightarrow \boxed{f_n} \longrightarrow y_n$ where $f_n : \Re \longrightarrow \Re$ and

$$Cond\ f_n(y_0) = \left|\frac{|y_0|(-5)^n}{y_n}\right| > |\frac{y_0}{y_0}|5^n = 5^n$$

$\Longrightarrow$ ill-condition

Indeed, I_0 is not machine-representable, and must be rounded to I_0^*

before the recursion can be employed.

$$\left|\frac{I_n^* - I_n}{I_n}\right| = (Cond\ f_n)(I_0)\left|\frac{I_0^* - I_0}{I_0}\right| > 5^n\left|\frac{I_0^* - I_0}{I_0}\right|$$

i.e. the error grows as n increases.

Let's look at it in another way ,

$$5y_{k-1} = \frac{1}{k} - y_k$$

$$y_{k-1} = \frac{1}{5}\Big(\frac{1}{k} - y_k\Big) = \Big(\frac{-1}{5}\Big)y_k + \Big(\frac{1}{5}\Big)\frac{1}{k}$$

$$y_{k-2} = \frac{-1}{5}\Big(\frac{-1}{5}y_k\Big) + q_{k-2} = \Big(\frac{-1}{5}\Big)^2 y_k + q_{k-2}$$

$$\vdots$$

$$y_n = \Big(\frac{-1}{5}\Big)^{v-n} y_v + q_n$$

so, take $v > n$, we have

$$y_v \longrightarrow \boxed{g_n} \longrightarrow y_n$$

$$(Cond\ g_n)(y_n) = \Big|\frac{y_v(-\frac{1}{5})^{v-n}}{y_n}\Big| < (\frac{1}{5})^{v-n}\ ,\ v > n$$

$$\begin{aligned} \Big|\frac{I_n^* - I_n}{I_n}\Big| \quad & = (Cond\ g_n)(I_v)\Big|\frac{I_v^* - I_v}{I_v}\Big| \\ & < (\frac{1}{5})^{v-n}\Big|\frac{I_v^* - I_v}{I_v}\Big| \\ & < (\frac{1}{5})^{v-n},\ \text{if we take } I_v^* = 0 \text{ and } v > n \end{aligned}$$

$Let\ \ \varepsilon > 0,$ and we can find v,

so that $\left(\frac{1}{5}\right)^{v-n} \leq \varepsilon$

$(v-n)\ln\left(\frac{1}{5}\right) \leq \ln \varepsilon$

$(n-v)\ln(5) \leq \ln \varepsilon$

$n - v \leq \frac{\ln \varepsilon}{\ln\, 5}$

$i.e.\ n + \frac{\ln \frac{1}{\varepsilon}}{\ln\, 5} \leq v \quad \ldots\ldots(*)$

Now choose v satisfies $*$, than $\left|\frac{I_n^* - I_n}{I_n}\right| < \varepsilon$

Take v sufficient large s.t. $I_v^* = 0$

$I_{k-1}^* = \frac{1}{5}\left(\frac{1}{k} - I_k^*\right)\ ,\ k = v\ , v-1\ , \ldots\ n+1$

This will produce a sufficiently accurate $I_n^* \approx I_n$.