§4 The Condition of an Algorithm

We again sssume that we are dealing with a problem f given by

$$f : \Re^m \to \Re^n, \qquad y = f(x).$$

Along with the problem f, we are also given an alorithm A that "solves" the problem. That is, given a machine vector $x \in \Re^m(t,s)$, the algorithm A produces a vector y_A (in machine arithmetic) that is supposed to approximate $y = f(x)$. Thus we have another map f_A describing how the problem f is solved by the algorithm A,

$$f_A : R^m(t,s) \to R^n(t,s),\ y_A = f_A(x),\ \text{where } x \in R^m(t,s)$$

Assumption: $\forall\, x \in \Re^m(t,s),\ \exists\, x_A \in \Re^m,\ s.t.\ f_A(x) = f(x_A)$

$$\begin{array}{lccl} i.e. & x_A \xrightarrow{f} & y_A & \\ & & \uparrow & f_A \\ & & x & \end{array}$$

Remark: x_A is not necessarily a machine vector nor uniquely dertemined.

Remark: The closer we can find an x_A to x, the more confidence we should place in the algorithm A.

We then define the condition of A by

Definition:

$(CondA)(x) = \dfrac{\inf_{x_A} \frac{\|x_A - x\|}{\|x\|}}{eps}$, or an upper bound

$(CondA)(x) \leq \frac{\|x_A - x\|}{\|x\| eps}$, for some x_A.

Example: Suppose a library routine for logarithm function satisfies $y_A = [\ln x](1+\varepsilon), |\varepsilon| \leq 5eps$

$$\text{Let } \; x_A = x^{1+\varepsilon}, \text{ then } \; y_A = \ln x_A$$

$$|\frac{x^{1+\varepsilon} - x}{x}| = |x^{\varepsilon} - 1| \underset{\substack{\uparrow \\ \text{Taylor's expension w.r.t. } \epsilon \text{ (at } \epsilon \;=\; 0)}}{\approx} |\varepsilon \ln x| \leq 5|\ln x| eps$$

Good condition when x away from zero and not big.

Example: $f_x : \Re^n \rightarrow \Re$

$$x_1 x_2 .... x_n \quad y$$

$$p_1 = x_1$$

$$A : p_k = fl(x_k p_{k-1})$$

$$y_A = p_n$$

$p_1 = x_1$

$p_k = x_k p_{k-1}(1+\varepsilon)$, $\varepsilon_k \leq eps$

$\therefore p_n = x_1 x_2 \cdots x_n (1+\varepsilon_2) \cdots (1+\varepsilon_n)$

Simply take $x_A = [x_1, x_2(1+\varepsilon_2), \cdots, x_n(1+\varepsilon_n)]^T$

$\frac{\|x_A - x\|_\infty}{\|x\|_\infty eps} = \frac{\|(0, x_2\varepsilon_2, \ldots, x_n\varepsilon_n)\|_\infty}{\|(x_1, x_2, \ldots, x_n)\|_\infty eps} \leq \frac{\|x\|_\infty eps}{\|x\|_\infty eps} = 1$

$\therefore (condA)(x) \leq 1$, well-conditioned

§5 Computer Solution of a Problem

We again sssume that we are dealing with a problem f given by

$$f : \Re^m \to \Re^n, \qquad y = f(x).$$

This is the mathematical (idealized) problem, where the data are exact real numbers, and the solution is mathematically exact solution. When solving such a problem on a computer, in floating-point arithmetic with precision eps, and using some algorithm A, one first of all rounds the data, and applies to these rounded data not f, but f_A:

$x^* =$ rounded data, $\frac{\|x^*-x\|}{\|x\|} = \varepsilon$,

Here ε represents the rounding error in the data, and could be due to sources other than rounding, e.g., measurement.

$y_A^* = f_A(x^*) = f(x_A^*)$

$y^* = f(x^*)$.

$$\begin{aligned}\frac{\|y_A^* - y\|}{\|y\|} &\le \frac{\|y_A^* - y^*\|}{\|y\|} + \frac{\|y^* - y\|}{\|y\|} \quad \because \text{ assuming } \|y\| \sim \|y^*\| \\ &\sim \frac{\|y_A^* - y^*\|}{\|y^*\|} + \frac{\|y^* - y\|}{\|y\|} = (1) + (2)\end{aligned}$$

$$
\begin{aligned}
(1) \;=\; & \frac{\|y_A^* - y^*\|}{\|y^*\|} = \frac{\|f_A(x^*) - f(x^*)\|}{\|f(x^*)\|} = \frac{\|f(x_A^*) - f(x^*)\|}{\|f(x^*)\|} \\
= \; & (\mathrm{cond}\, f)(x^*) \cdot \frac{\|x_A^* - x^*\|}{\|x^*\|} \\
\leq \; & (\mathrm{cond}\, f)(x^*) \cdot (\mathrm{cond}\, A)(x^*) \cdot eps
\end{aligned}
$$

$$
\begin{aligned}
(2) = \frac{\|y^* - y\|}{\|y\|} = \frac{\|f(x^*) - f(x)\|}{\|f(x)\|} \;\leq\; & (\mathrm{cond}\, f)(x) \cdot \frac{\|x^* - x\|}{\|x\|} \\
= \; & (\mathrm{cond}\, f)(x) \cdot \varepsilon
\end{aligned}
$$

Assume that $(\mathrm{Cond}\, f)(x^*) \approx (\mathrm{Cond}\, f)(x)$

$$
\frac{\|y_A^* - y\|}{\|y\|} \leq (\mathrm{Cond}\, f)(x)\{\underset{\substack{\uparrow \\ \text{data error}}}{\varepsilon} + (\mathrm{cond}\, A)(x^*) \cdot \underset{\substack{\uparrow \\ \text{machine precision}}}{eps}\}
$$

This show how the data error and machine precision contribute toward the total error: both are amplified by the condition of the problem, but the latters is futher amplified by the condition of the algorithm.