Section 1.3 Least square error; convergence

There are many ways we can select a basis π_j in Φ_n and, therefore, many ways the solution $\hat{\varphi_n}$ can be represented. Nevertheless, it is always one and the same function, where

$$\|f-\hat{\varphi}_n\|_{2,d\lambda} = \min_{\varphi\in\Phi_n} \|f-\varphi\|_{2,d\lambda},$$

and $\hat{\varphi}$ is unique once Φ_n has been selected. Thereore, without restricting generality, we assume $\{\ \Phi_n\ \}$ is an orthogonal system, therefore

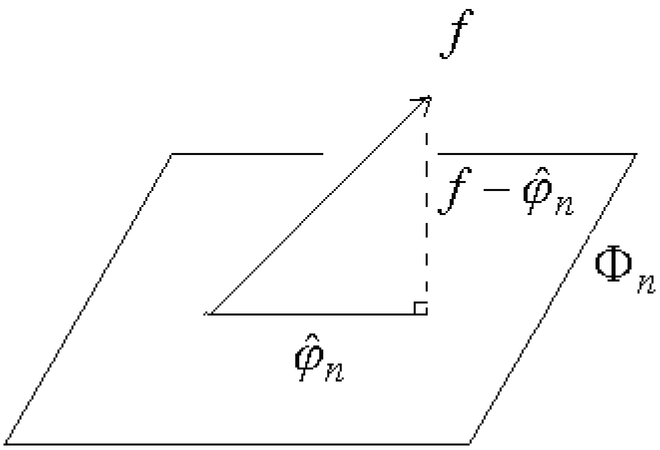
$$\hat{\varphi}_n(t) = \sum_j^n \frac{(\pi_j, f)}{(\pi_j, \pi_j)}\ \pi_j(t)$$

Proposition: $(f-\hat{\varphi}_n, \varphi) = 0\ \forall\varphi\in\Phi_n$

Proof:

Suffice to show it for $\pi_k\,\forall k$

$$(f-\hat{\varphi}_n, \pi_k) = (f, \pi_k) - \frac{(\pi_k, f)}{(\pi_k, \pi_k)} \cdot (\pi_k, \pi_k) = 0$$



In particular, $(f-\hat{\varphi}_n, \hat{\varphi}_n) = 0$

$$\begin{aligned}
\therefore \|f\|^2 &= \|f-\hat{\varphi}_n+\hat{\varphi}_n\|^2 = \|f-\hat{\varphi}_n\|^2 + \|\hat{\varphi}_n\|^2 \\
&= \|f-\hat{\varphi}_n\|^2 + \left\|\sum_{j=1}^n \hat{c}_j\pi_j\right\|^2 \\
&= \|f-\hat{\varphi}_n\|^2 + \sum_{j=1}^n |\hat{c}_j|^2\|\pi_j\|^2
\end{aligned}$$

Therefore,

$$\|f-\hat{\varphi}_n\| = (\|f\|^2 - \sum_{j=1}^{n} |\hat{c}_j|^2 \|\pi_j\|^2)^{\frac{1}{2}}$$

Remark: This error expression is not a good practical equation. The expression could be negative. Computing the error from the right-hand side cannot produce anything smaller than ($\sqrt{esp} > eps$) and they may be even produce a negative result.

Instead, we use $\|f-\varphi_n\| = \left\{ \int\limits_{\mathbb{R}} |f-\hat{\varphi}_n|^2 d\lambda \right\}^{\frac{1}{2}} \geqslant 0.$

Now if $\Phi_1 \subset \Phi_2 \subset \cdots$.

Definition: If $\lim\limits_{n\to\infty} \|f-\hat{\varphi}_n\| = 0$ then we can call the process converges, i.e: (completeness relation)

$$\sum_{j=1}^{\infty} |\hat{c}_j|^2 \|\pi_j\|^2 = \|f\|^2 \;\Leftrightarrow\; \forall f,\; \|f\| < \infty$$

i.e,

$$f \in L_{2,d\lambda}$$

$$\forall \varepsilon > 0 \; \exists n (= n_\varepsilon) \; s.t \; \exists \varphi^* \in \Phi_n$$

and

$$\|\varphi^* - f\| \leq \varepsilon$$

and we call the class $\{\Phi_n\}$ is a complete space in the norm $\|\cdot\| = \|\cdot\|_{2,d\lambda}$

Section 1.4 Example of orthogonal system

There are many orthogonal system in use. The prototype of them all is the system of trigonometric functions known from Fourier analysis. Other widely used systems involve algebraic polynomials.

(1)Trigonometric function

$$
\begin{array}{llll}
1, & \cos t, & \cos 2t, & \cdots\cdots \\
\sin t, & \sin 2t, & \cdots\cdots &
\end{array}
$$

$$
d\lambda(t) = \begin{cases} dt, & \text{on } [0, 2\pi] \\ 0, & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
\int_0^{2\pi} \sin kt \cdot \sin lt \, dt &= -\frac{1}{2} \int_0^{2\pi} [\cos(k+l)t - \cos(k+l)t] dt \\
&= 0
\end{aligned}
$$

$$
\int_0^{2\pi} \sin kt \cdot \sin lt \, dt = \begin{cases} 0 & \text{if } k \neq l \\ \pi & \text{if } k = l \end{cases} \qquad k, \ l = 1, 2, 3, \cdots
$$

$$
\int_0^{2\pi} \cos kt \cdot \cos lt \, dt = \begin{cases} 0 & \text{if } k \neq l \\ 2\pi & \text{if } k = l = 0 \\ \pi & \text{if } k = l > 0 \end{cases} \qquad k, \ l = 0, 1, 2, \cdots
$$

$$
\int_0^{2\pi} \sin kt \cdot \cos lt \, dt = 0
$$

The theory of Fourier series is concerned with the expansion of a given 2π-periodic function in terms of those trigonometric functions, therefore we have

$$
\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} a_k \cos kt + \sum_{k=1}^{\infty} b_k \sin kt \\
a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(t) dt \\
a_k &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt \\
b_k &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt.
\end{aligned}
$$

Proposition: Three consecutive orthogonal polynomials are linearly related, i.e. $\exists\, \alpha_k = \alpha_k(d\lambda)$(depending on the measure$d\lambda$), $\beta_k = \beta_k(d\lambda)$ such that

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0, 1, 2, \ldots,$$

$$\pi_{-1}(t) = 0 \text{ do we need to have this? }, \quad \pi_0(t) = 1.$$

Proof:

Say π_{k-1}, π_k, π_{k+1} are 3 consecutive orthogonal monic polynomials. Since $\deg(\pi_{k+1} - t\,\pi_k) = k$, let's consider the following

$$(\pi_{k+1}(t) - t\pi_k(t) = -\alpha_k\pi_k(t) - \beta_k\pi_{k-1}(t) + \sum_{j=0}^{k-2}\gamma_{k,j}\,\pi_j(t),\, \pi_k)$$

$$(-t\pi_k, \pi_k) = -\alpha_k(\pi_k, \pi_k),\;\; \alpha_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}$$

Similarly,

$$\begin{array}{ccc} (-t\pi_k, \pi_{k-1}) & = & -\,\beta_k(\pi_{k-1}, \pi_{k-1}) \\ \| & & \\ \text{and } (\pi_k, \underbrace{-t\pi_{k-1}}) & = & (\pi_k, -\pi_k) \\ \| & & \\ & -\pi_k + \displaystyle\sum_{l=0}^{k-1} a_l\pi_l & \end{array}$$

then we have $\beta_k = \dfrac{(\pi_k, \pi_k)}{(\pi_{k-1}, \pi_{k-1})}$
similarly, we have $\gamma_{k,j} = 0$
So, $\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t)$

The recursion provides us with a practical scheme of generating orthogonal polynomials. This procedure is called Stieltjes's procedure.

Homework: If $\beta_0 = \int\limits_R d\lambda(t)$, then $\|\pi_k\|^2 = \beta_0\beta_1\cdots\beta_k$

Example: Legendre polynomials

$$\pi_k(t) = (-1)^k \frac{k!}{(2k)!}\frac{d^k}{dt^k}(1-t^2)^k,\ k = 0, 1, 2, \cdots,$$

which is known as the Rodrigues formula.

Proposition: If $d\lambda(t) = dt$ then $(\pi_k, \pi_l) = 0\ \forall\ k \neq l$
For any l with $0 \leq l < k$

$$\begin{aligned}
&\int_{-1}^{1} \frac{d^k}{dt^k}(1-t^2)^k \cdot t^l\, dt \\
&= \frac{d^{k-1}}{dt^{k-1}}(1-t^2)^k \cdot t^l\Big|_{-1}^{1} \qquad \underbrace{-\int_{-1}^{1} \frac{d^{k-1}}{dt^{k-1}}(1-t^2)^k \cdot l \cdot t^{l-1} dt}_{\substack{(-1)^1 \cdot l \cdot (\frac{d^{k-2}}{dt^{k-2}}(1-t^2)^k \cdot t^{l-1}\Big|_{-1}^{1} + \\ (-1)^2 \cdot l \cdot (l-1)\int_{-1}^{1} \frac{d^{k-2}}{dt^{k-2}}(1-t^2)^k \cdot t^{l-2} dt}} \\
&= \sum_{m=0}^{l} (-1)^m \cdot l \cdot (l-1)\cdots(l-m-1) \cdot t^{l-m} \cdot \frac{d^{k-m-l}}{dt^{k-m-l}} \cdot (1-t^2)^k\Big|_{-1}^{1} \\
&= 0
\end{aligned}$$

since $0 \leq k-m-l < k$.
Therefore,

$$(\pi_k, p) = 0\ \forall\ p \in P_{k-1}$$

and

$$(\pi_k, \pi_l) = 0\ \forall\ k \neq l.$$

$$\begin{aligned}
\pi_0 &= (-1)^0 \cdot \frac{0!}{(0)!} \cdot 1 &&= 1 \\
\pi_1 &= (-1)^1 \cdot \frac{1!}{2!} \cdot (-2t) &&= t \\
\pi_2 &= (-1)^2 \cdot \frac{2!}{4!} \cdot (-4 + 12t^2) &&= \frac{-4+12t^2}{12} \\
&\vdots
\end{aligned}$$

Writing (by symmetry to zero),

$$\pi_k(t) = t^k + \mu_k t^{k-2} + \cdots, \quad k \geq 2.$$

Or, $\pi_{k+1}(t) = t^{k+1} + \mu_{k+1} t^{k-1} + \cdots$
but we know

$$\begin{aligned}
\pi_{k+1}(t) &= (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t) \\
&= t\pi_k(t) - \alpha_k \pi_k(t) - \beta_k \pi_{k-1}(t)
\end{aligned}$$

so $\quad \beta_k = \dfrac{t\pi_k(t) - \pi_{k+1}(t)}{\pi_{k-1}(t)} \qquad \forall t$, since $\alpha_k = 0$.
In particular, as $t \to \infty$,

$$\beta_k = \lim_{t\to\infty} \frac{t\pi_k - \pi_{k+1}}{\pi_{k-1}(t)} = \lim_{t\to\infty} \frac{(\mu_k - \mu_{k+1})t^{k-1}}{\pi_{k-1}(t)} = \mu_k - \mu_{k+1}$$

and from the formula

$$\begin{aligned}
\pi_k &= \frac{k!}{(2k)!} \cdot \frac{d^k}{dt^k}(t^{2k} - kt^{2k-2} + \cdots) \\
&= \frac{k!}{(2k)!}(2k(2k-1)) \cdots (k+1) \cdot t^k - k(2k-2)(2k-3) \cdots (k-1) \cdot t^{k-2} + \cdots) \\
&= t^k - \frac{k(2k-2)(2k-3) \cdots k(k-1) \cdot k \cdot (k-1) \cdots}{(2k)(2k-1)(2k-2) \cdots 1} t^{k-2} \\
&= t^k - \frac{k(k-1)}{2(2k-1)} t^{k-2} + \cdots
\end{aligned}$$

i.e. $\quad \mu_k = -\dfrac{k(k-1)}{2(2k-1)} \qquad k \geq 2$

$\mu_1 = 0$. Therefore,

$$\begin{aligned}
\beta_k &= \mu_k - \mu_{k+1} = -\frac{k(k-1)}{2(2k-1)} + \frac{(k+1)k}{2(2k+1)} = \frac{k}{2} \frac{2k}{(2k+1)(2k-1)} \\
&= \frac{2k^2}{2(4k^2-1)} = \frac{1}{(4 - \frac{1}{k^2})} \qquad k \geq 1
\end{aligned}$$

$\pi_0 = 1$
$\pi_1 = t$
$\pi_2 = t \cdot \pi_1 - \beta_1 \pi_0 \qquad \beta_1 = \frac{1}{3}$
$\pi_3 = t \cdot \pi_2 - \beta_2 \pi_1 \qquad \beta_2 = \frac{4}{3}$

$\|\pi_k\|^2 = \beta_0 \beta_1 \cdots \beta_k$, where $\beta_0 = 1$