§2.4 Chebyshev polynomials and nodes

The choice of nodes, as we saw in the previous section, distinctly influences the convergence character of the interpolation process. We now discuss a choice of points – the Chebyshev points – which leads to very favorable convergence properties. These point are useful, not only for interpolation, but also for other purposes (integration, collocation, etc.)

Fact :

The cosine of a multiple argument is a polynomial in the consine of the simple argument.

$$\cos n\theta = T_n(\cos\theta) \ , \ T_n \in p_n$$

Example:

$$\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\,\cos k\theta$$

$$\therefore \ \cos(k+1)\theta = 2\cos\theta\,\cos k\theta - \cos(k-1)\theta$$

so we have

$$\begin{cases} T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x) \\ T_0(x) = 1, \qquad \text{Chebyshev polynomials} \\ T_1(x) = x, \qquad \text{(of the first kind)} \end{cases}$$

$$\begin{aligned} &T_2(x) = 2x^2 - 1 \qquad \text{over } [-1,1] \\ &T_3(x) = 4x^3 - 3x \\ &T_4(x) = 8x^4 - 8x^2 + 1 \end{aligned}$$

Remark: The identity is only good on $[-1,1]$

Proposition:

$$T_n = 2^{n-1}\,x^n + p_{n-1}(x)$$

Then we call monic Chebyshew polynominal $\mathring{T}_n$

$$\mathring{T}_n(x) = \frac{1}{2^{n-1}}\, T_n(x), \quad n \geq 1\ ,\ \mathring{T}_0 = T_0$$

Proposition: Suppose $x_k^{(n)}$ are zeros of T_n, then

$$x_k^{(n)} = \cos\, \theta_k^{(n)}, \quad \theta_k^{(n)} = \frac{2k-1}{2n}\, \pi$$

Proof:

$$\begin{aligned}
\cos\, n\theta = 0 \quad &\Rightarrow \quad n\theta = \frac{(2k-1)\pi}{2} \\
&\qquad\ \theta_k = \frac{(2k-1)\pi}{2n} \quad , \quad k = 1\,, 2\,, \cdots, n \\
x_k^{(n)} = \cos\, \frac{(2k-1)\pi}{2n}
\end{aligned}$$

Example:

$$\begin{aligned}
T_4 = 8x^4 - 8x^2 + 1 = \cos\, 4\theta = 0 \\
\cos\, 4\theta \quad \text{if} \quad 4\theta = \frac{\pi}{2}\,, \frac{3\pi}{2}\,, \frac{5\pi}{2}\,, \frac{7\pi}{2} \\
\theta = \frac{\pi}{8}\,, \frac{3\pi}{8}\,, \frac{5\pi}{8}\,, \frac{7\pi}{8}
\end{aligned}$$

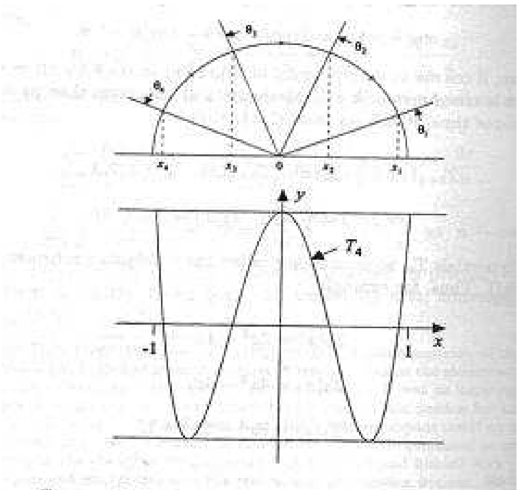

FIGURE 2.2.5. The Chebyshev polynomial $y = T_4(x)$

All zeros of T_n are thus real, distinct, and contained in the open interval (-1,1). They are projection onto the real line of equally spaced points on the unit circle.

We can write the monic polynomial in factored form as

$$\mathring{T}_n(x) = \prod_{k=1}^{n}(x - x_k^{(n)})$$

$T_n(x)$ oscillates between $+1$ and -1 , attaining these extreme values at $y_k^{(n)}$.

i.e. $\cos n\theta = \pm 1 \quad \Rightarrow \quad n\theta = k\pi \qquad \therefore\ \theta = \dfrac{k\pi}{n}$

$$y_k^{(n)} = \cos\, n_k^{(n)} \quad , \quad n_k^{(n)} = k\,\frac{\pi}{n} \quad , \quad k = 0\,, \cdots, n$$

In summary ,

$$T_n(x_k^{(n)}) = 0 \quad , \quad x_k^{(n)} = \cos\, \frac{2k-1}{2n}\,\pi \quad , \quad k = 1\,,\,2\,, \cdots, n$$

$$T_n(y_k^{(n)}) = (-1)^k \quad , \quad y_k^{(n)} = \cos\, \frac{k}{n}\,\pi \quad , \quad k = 0\,,\,1\,, \cdots, n$$

Theorem: For an arbitrary monic polynomial $\mathring{p}_n$ of degree n

$$\Rightarrow \quad \max_{-1\le x\le 1} |\mathring{p}_n| \ge \max_{-1\le x\le 1} |\mathring{T}_n(x)| = \frac{1}{2^{n-1}} \quad , \quad n \ge 1$$

where $\mathring{T}_n(x)$ is the monic chebyshev poly of degree n

Proof: If not

$$\max_{-1\le x\le 1} |\mathring{p}_n| < \frac{1}{2^{n-1}}$$

Define $d_n(x) = \mathring{T}_n(x) - \mathring{p}_n(x)$

$$d_n(y_0^{(n)}) > 0 \quad , \quad d_n(y_1^{(n)}) < 0 \quad , \quad \cdots (-1)^n \, d_n(y_n^{(n)}) > 0$$

$d_n(x)$ changes sign at least n times. $\quad \rightarrow\leftarrow$

Remark:

(1) The best approximatuon to x^n from p_{n-1} is $x^n - \mathring{T}_n(x)$

$$\max_{-1\le x\le 1} |\, x^n - p^{n-1}| \ \ge \max_{-1\le x\le 1} |\mathring{T}_n(x)| = \max_{-1\le x\le 1} |\, x^n - (x^n - \mathring{T}_n(x))|$$

(2) $f(x) - p_n(f;x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \displaystyle\prod_{i=0}^{n} (x - x_i)$

The optimal nodes x_i are zeros of $T_{n+1}(x)$ Chebyshev points

$$\hat{x_i} = \cos \frac{2i-1}{2n+2}\,\pi \quad , \quad i = 0\,,\,1\,,\,2\,,\,\cdots,\,n$$

$$\|\, f - p_n(f)\|_\infty \;\leq\; \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \cdot \frac{1}{2^n}$$

(3) Fact: $p_n(f\,;\,\hat{x_0}^{(n)}\;,\;\cdots\;,\;\hat{x_n}^{(n)}\,;\,x) \to f(x)$ as $n \to \infty$

uniformly on $[-1,1]$ if $f \in C^1[-1,1]$

No need for analyticity of f.

(4) Let $d\lambda(x) = \dfrac{dx}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} T_k T_l \,\frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} T_k(\cos\,\theta) T_l(\cos\,\theta)\,d\theta$$

$$\int_0^{\pi} \cos\,k\theta\,\cos\,l\theta\,d\theta = \begin{cases} 0 & k \neq l \\ \pi & k = l = 0 \\ \dfrac{\pi}{2} & k = l > 0 \end{cases}$$

The Fourier expansion in Chebyshev polynomials is therefore given by

$$f(x) = \sum_{j=1}^{\infty} c_j T_j(x) = \frac{1}{2}\,c_0 + \sum_{j=1}^{\infty} c_j T_j(x),$$

where $c_j = \dfrac{2}{\pi} \displaystyle\int_{-1}^{1} f\,T_j(x)\,\frac{dx}{\sqrt{1-x^2}} \quad , \quad j = 0\,,\,1\,,\,2\,,\,\cdots$

And , let $\tau_n(x) = \sum_{j=1}^{n} c_j T_j(x)$ be the approximation with degree n

then the error is

$$f(x) - \tau_n(x) = \sum_{j=n+1}^{\infty} c_j T_j(x) \sim c_{n+1} T_{n+1}(x),$$

since $|c_j| \leq 2|f^{(k)}(x)|/j^k$, so error is bounded by $|c_{n+1}|$ and we can compare it to Taylor's expansion at 0 of $[-1, 1]$ (error is proportional to x^{n+1} on $[-1, 1]$)