§2.5 Barycentric formula

Lagrange's formula is attractive more of theroetical purposes than for practical computational work. It can be rewritten, however, in a form that makes it efficient computationally, and that also allows additional interpolation nodes to be added with ease.

Lagrange formula:

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Alternative:

$$\begin{aligned}
&\lambda_0^{(0)} = 1 \\
&\lambda_0^{(1)} = \frac{1}{x_0 - x_1}, \qquad \lambda_1^{(1)} = \frac{1}{x_1 - x_0} \\
&\lambda_0^{(2)} = \frac{1}{x_0 - x_1} \cdot \frac{1}{x_0 - x_2}, \ \lambda_1^{(2)} = \frac{1}{x_1 - x_0} \cdot \frac{1}{x_1 - x_2}, \lambda_2^{(2)} = \frac{1}{x_2 - x_0} \cdot \frac{1}{x_2 - x_1} \\
&\quad \vdots \\
&\lambda_i^{(n)} = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}
\end{aligned}$$

Then the elementary Lagrange interpolation can be written as

$$\ell_i(x) = \frac{\lambda_i^{(n)}}{x - x_i} \cdot \omega_n(x), \ i = 0, \cdots, n,$$

where $\omega_n(x) = \prod_{j=0}^{n} (x - x_j)$.

Since $\sum_{i=0}^{n} \ell_i(x) = 1$, we could devide it out,

$$\begin{aligned}
p_n(f;x) &= \sum_{i=0}^{n} f_i \ell_i(x) = \frac{\sum_{i=0}^{n} f_i \ell_i(x)}{\sum_{i=0}^{n} \ell_i(x)} \\
&= \frac{\sum_{i=0}^{n} f_i \frac{\lambda_i^{(n)}}{x-x_i} \cdot \omega_n(x)}{\sum_{i=0}^{n} \frac{\lambda_i^{(n)}}{x-x_i} \cdot \omega_n(x)}, \ x \neq x_i \\
&= \frac{\sum_{i=0}^{n} f_i \dfrac{\lambda_i^{(n)}}{x-x_i}}{\sum_{i=0}^{n} \dfrac{\lambda_i^{(n)}}{x-x_i}}, \quad x \neq x_i \text{ for } i = 0, 1, \ldots, n.
\end{aligned}$$

This expresses the interpolation polynomial as a weighted average of the function values $f_i = f(x_i)$ and is, therefore, called the Barycentric formula – a slight misnomer, since the weights are not necessarily all positive.

First, note that

$$\ell_i(x) = \frac{\dfrac{\lambda_i^{(n)}}{x-x_i}}{\sum_{j=0}^{n} \dfrac{\lambda_j^{(n)}}{x-x_j}}$$

and

$$\lambda_i^{(k)} = \frac{\lambda_i^{(k-1)}}{x_i - x_k} \quad i = 0, 1, \cdots, k-1$$

Furthermore with $f \equiv 1$, then

$$p_k(f;x) = \sum_{i=0}^{k} \lambda_i^{(k)} \prod_{j=0, j\neq i}^{k} (x - x_j) \equiv 1,$$

since

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{k} \frac{x - x_j}{x_i - x_j} = \frac{\lambda_i^{(k)}}{(x-x_i)} w_k(x).$$

Comparing the coefficient of x^k yields

$$\sum_{i=0}^{k} \lambda_i^{(k)} = 0.$$

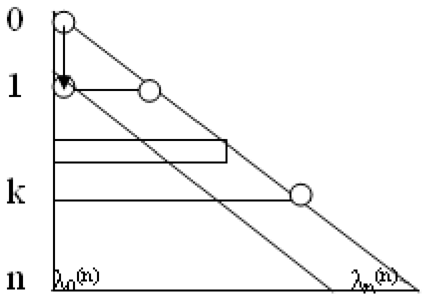
Therefore $\lambda_k^{(k)} = -\sum\limits_{i=0}^{k-1} \lambda_i^{(k)}$.

For $k = 1, n$ do

$\qquad \lambda_i^{(k)} = \frac{\lambda_i^{(k-1)}}{x_i - x_k} \quad i = 0, 1, \cdots, k-1$

$\qquad \lambda_k^{(k)} = -\sum\limits_{i=0}^{k-1} \lambda_i^{(k)}$

End Do



$\frac{1}{2}n(n+1)$ - division

$2 + 4 + \cdots + 2n$ - addtion

$= \frac{2n+2}{2} \cdot n = n(n+1)$

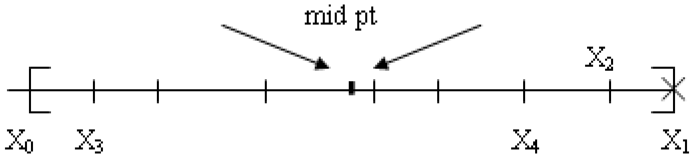
substraction

Lagrange formula $\sim O(n^3)$, $\sum_{i=0}^{n}(2i + 2i)i = O(n^3)$

multiplication

Problem: $|\lambda_k^{(k)}| \ll \max\limits_{0 \le i \le k-1} \lambda_i^{(k)}$

Solution:



§2.6 Newton's formula

This is another way of organizing the work in §2.5. We denote

$$p_n(x) = p_n(f; x_0, x_2, \cdots, x_n : x), \; n = 1, 2, \cdots .$$

We clearly have

$$\begin{aligned} p_0(x) &= a_0, \\ p_n(x) &= p_{n-1}(x) + a_n(x - x_0)\cdots(x - x_{n-1}), \quad n = 1, 2, 3, \ldots, \end{aligned}$$

for some constants $a_0, a_1, a_2, \ldots$. Note that

$$p_{n-1}(x_i) = f(x_i), \; \forall i = 0, \cdots, n-1.$$

We can then write

$$\begin{aligned} p_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &+ \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

-which is called Newton's form.
The constants involved can be determined, in principle, by the interpolation conditions.

$$\begin{aligned} f_0 &= a_0 \\ f_1 &= a_0 + a_1(x_1 - x_0) \\ f_2 &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \end{aligned}$$

$$\begin{aligned} a_0 &= f_0 \\ a_1 &= \frac{f_1 - f_0}{x_1 - x_0} \\ a_2 &= \frac{f_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

$$
f(x) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & x_1 - x_0 & \vdots & \vdots & \vdots \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & \vdots & . \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}
$$

det A=$(1)(x_1 - x_0)(x_2 - x_0)(x_2 - x_1) \cdots (x_n - x_0) \cdots (x_n - x_{n-1})$
Therefore A is a nonsingular $\Longrightarrow$ $(a_0, \cdots, a_n)$ is solvable.

Remark:
Evidently $\{a_n\}$ is a linear combination of $f_0, \cdots, f_n$. We use the notation

$$
a_n = [x_0, x_1, \cdots, x_n]f, \ n = 0, 1, 2, \cdots
$$

$\uparrow$

nth divided difference of f relative to the nodes $x_0, x_1, \cdots, x_n$

Proposition: $[x_0, x_1, \cdots, x_n]f$ is a symmetric function w.r.t. $x_0, x_1, \cdots, x_n$
Proof:
Since f is invariant w.r.t. the order of $[x_i]$

Proposition:

$$
[x_0, x_1, \cdots, x_k]f = \frac{[x_1, x_2, \cdots, x_k]f - [x_0, x_1, \cdots, x_{k-1}]f}{x_k - x_0}
$$

Proof: Let,

$$
r(x) = p_{k-1}(f; x_1, x_2, \cdots, x_k : x)
$$
$$
s(x) = p_{k-1}(f; x_0, x_1, \cdots, x_{k-1} : x)
$$

Then
$$
p_k(f; x_0, x_1, \cdots, x_k : x) = r(x) + \frac{x - x_k}{x_k - x_0}[r(x) - s(x)]
$$

By uniqueness of the interpolation polynomial, and equating the leading coefficients.

Example:

x	f			
x_0	f_0			
x_1	f_1	$[x_0, x_1]f$		
x_2	f_2	$[x_1, x_2]f$	$[x_0, x_1, x_2]f$	
x_3	f_3	$[x_2, x_3]f$	$[x_1, x_2, x_3]f$	$[x_1, x_2, x_3, x_4]f$

$n(n+1)$-addition
$\frac{n}{2}(n+1)$- division

x	f			
0	3			
1	4	$\frac{4-3}{1-0} = 1$		
2	7	$\frac{7-4}{2-1} = 3$	$\frac{3-1}{2-0} = 1$	
4	19	$\frac{19-7}{4-2} = 6$	$\frac{6-3}{4-1} = 1$	$\frac{1-1}{4-0} = 0$

$$p_3 = 3 + 1x + 1x(x-1) + 0x(x-1)(x-2) = 3 + x^2$$

Newton's formula also yields a new representation for the error term in Lagrange interpolation. Let t temporarily denote an arbitrary "node" $\neq x_i$. Then we have

$$p_{n+1}(f; x_0, x_1, \cdots, x_{n+1}, t : x) = p_n(f; x) + [x_0, x_1, \cdots, x_n, t]f \cdot \prod_{i=0}^{n}(x - x_i)$$

Now put $x = t$, then we have

$$f(t) = p_n(f; t) + [x_0, x_1, \cdots, x_n, t]f \cdot \prod_{i=0}^{n}(t - x_i)$$

$$f(t) - p_n(f; t) = [x_0, x_1, \cdots, x_n, t]f \cdot \prod_{i=0}^{n}(t - x_i)$$

Writing again x for t, we find

$$f(x) = p_n(f; x) + [x_0, x_1, \cdots, x_n, x]f \cdot \prod_{i=0}^{n}(x - x_i)$$

Compare to earlier formula we have

$$[x_0, x_1, \cdots, x_n, x]f = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}$$

or,

$$[x_0, x_1, \cdots, x_n]f = \frac{f^{(n)}(\xi(x))}{n!}$$

Thus, for any $n+1$ distinct points in [a,b] and any $f \in C^n[a, b]$, the divided difference of f of order n is the nth scaled derivative of f at some (unknown) intermediate point.

Definition

$$\underbrace{[x_0, x_0, \cdots, x_0]}_{n+1\ times} f = \frac{f^{(n)}(x_0)}{n!}.$$