§2.7 Hermite interpolation

The general Hermite interpolation problem consists of the following:
Give $k+1$ distinct points $x_0, x_1, \cdots, x_k \in [a,b]$ and corresponding $m_k \geq 1$, and a given $f \in C^{M-1}[a,b]$, with $M = \max_m m_k$.
Find a polynomials p of lowest degree such that

$$p^{\mu}(x_k) = f_k^{(\mu)},\ \mu = 0, 1, \cdots, m_k - 1$$

The problem can be thought of as a limiting case of Lagrange interpolation if we consider x_k to be a point of multiplicity m_k, that is, obtained by a confluence of m_k distinct points into a single point x_k. We can imagine setting up the table of divided differences, and Newton's interpolation formula.
Example

x	f			
x_k	f_k			
x_k	f_k	$[x_k, x_k]f = f'(x_k)$		
x_k	f_k	$[x_k, x_k]f$	$[x_k, x_k, x_k]f = \frac{f''(x_k)}{2!}$	
x_k	f_k	$[x_k, x_k]f$	$[x_k, x_k, x_k]f$	$[x_k, x_k, x_k, x_k]f = \frac{f'''(x_k)}{3!}$

$$\underbrace{[x_0, x_0, \cdots, x_0]}_{n+1\ times} f = \frac{f^{(n)}(x_0)}{n!}$$

We obtain a table with $m_0 + m_1 + \cdots + m_k$ entries in the first column, and hence an interpolation polynomial of degree $\leq m_0 + \cdots + m_k - 1$, which as in the Lagrange case, is unique. The $n+1$ diagonal entries in the table give us the coefficients in the Newton's formula.
Example:
Find $p \in P_3$

$$p(x_0) = f_0,\ p'(x_0) = f_0',\ p''(x_0) = f_0'',\ p'''(x_0) = f_0'''$$
$$p(x) = f_0 + f_0'(x - x_0) + f_0''/2(x - x_0)^2 + f_0'''/6(x - x_0)^3$$

Error:

$$\frac{1}{24}(x - x_0)^4 f^4(\xi)$$

Example:

Find $p \in P_3$ such that

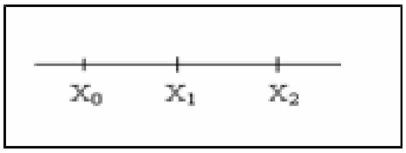
$$p(x_0) = f_0,\ p(x_1) = f_1,\ p'(x_1) = f'(x_1),\ p(x_2) = f_2$$

x	f			
x_0	f_0			
x_1	f_1	$[x_0, x_1]f$		
x_1	f_1	f_1'	$[x_0, x_1, x_1]f$	
x_2	f_2	$[x_1, x_2]f$	$[x_1, x_1, x_2]f$	$[x_0, x_1, x_1, x_2]f$

$$\begin{aligned} p(x) = f_0 \quad &+ \quad [x_0, x_1]f(x - x_0) + [x_0, x_1, x_1]f(x - x_0)(x - x_1) \\ &+ \quad [x_0, x_1, x_1, x_2]f(x - x_0)(x - x_1)^2 \end{aligned}$$

Error : $(x - x_0)(x - x_1)^2(x - x_2)\frac{f^{(4)}(\xi)}{4!},\ x_0 < \xi < x_2$

if $x_i - x_{i-1} = h,\ x = x_1 + th$



$|(x - x_0)(x - x_1)^2(x - x_2)| = |(t^2 - 1)t^2h^4| \le \frac{1}{4}h^4$

Therefore,

$$\|f - p\|_\infty \le \frac{1}{4}h^4\frac{\|f^{(4)}\|_\infty}{24} = \frac{h^4}{96}\|f^{(4)}\|_\infty$$