§3. Approximation and Interpolation by Spline Functions

Our concern in §2 was with approximation of functions by a single polynomial over a finite interval [a,b]. When more accuracy was wanted, we simply increased the degree of the polynomial, and under suitable assumptions the approximation indeed can made as accurate as one wishes by choosing the degree of the approximating polynomial sufficiently large.

However, there are other ways to control accuracy.

Alternative: $\triangle$: subdivision. $a = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ and use low-degree polynomails on each $[x_i,\ x_{i+1}]$.

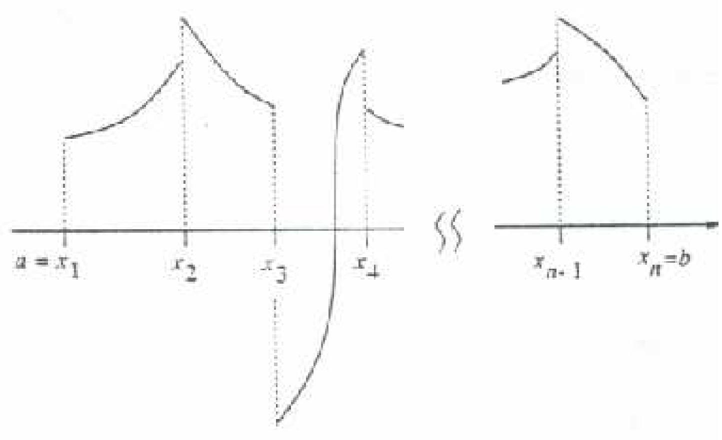
$$|\triangle| = \max_{1 \le i \le n-1} \triangle x_i,\ \triangle x_i = x_{i+1} - x_i.$$

Goal: keep the degree of polynomial pieces uniformly low and increase accuracy by decreasing $|\triangle|$.

Definition:

$S_m^k(\triangle) = \{s : s \in C^k[a,\ b],\ s|_{[x_i,\ x_{i+1}]} \in P_m,\ i = 1, 2, \cdots, n-1,\ m \ge 0,\ k \ge 0\}$.

We refer to $S_m^k(\triangle)$ as the spline functions of degree m and smoothness class k relative to the subdivision $\triangle$.



A function $s \in \mathbb{S}_m^{-1}$

$k = -1,\ S_m^{-1}(\triangle)$ piecewise disjoint m polynomials.

$k = m,\ S_m^m(\triangle) = P_m$. (uninteresting case).

§ 3.1 Interpolation by piecewise linear function

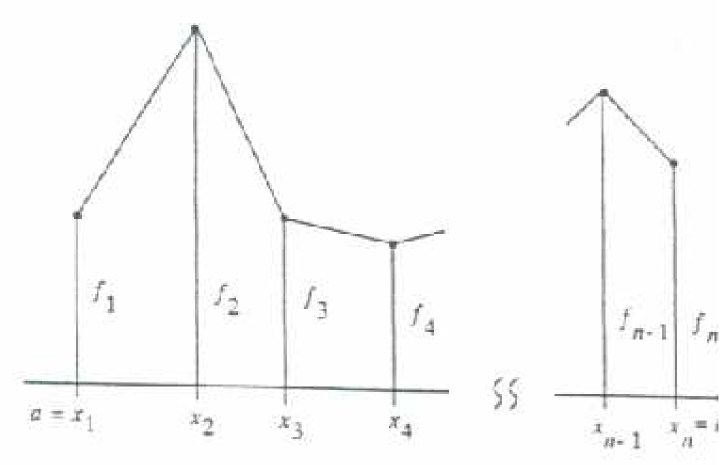
The problem here is to find $s \in S_1^0(\triangle)$, so that for a given function f on $[a,\ b]$, we have

$$s(x_i) = f_i,$$

where $f_i = f(x_i),\ i = 1, 2, \cdots, n$.
We conveniently let the interpolation nodes coincide with the points x_i of the subdivision $\triangle$. This simplifies matters, but is not necessary. The formula of linear interpolation gives

$$s_1(f; x) = f_i + (x - x_i)[x_i,\ x_{i+1}]f, \text{ for } x_i \le x \le x_{i+1}$$

x	f	
x_i	f_i	
x_{i+1}	f_{i+1}	$[x_i,\ x_{i+1}]f$



Piecewise linear interpolation

Error :
$f(x) - s_1(f;x) = (x - x_i)(x - x_{i+1})[x_i,\ x_{i+1},\ x]f$
for $x \in [x_i,\ x_{i+1}]$

Hence, if $f \in C^2[a,\ b]$,

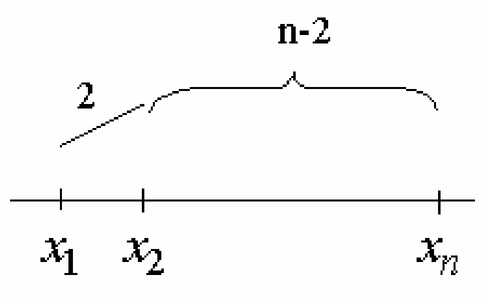
$$\begin{aligned}
|f(x) - s_1(f;x)| &\leq \frac{(\triangle x_i)^2}{8} \max_{[x_i,\ x_{i+1}]} |f''|, \quad x \in [x_i,\ x_{i+1}] \\
&\text{Therefore} \\
\|f - s_1(f)\|_\infty &\leq \frac{1}{8}|\triangle|^2 \|f''\|_\infty \ \rightarrow\ 0 \quad \text{as } |\triangle| \ \rightarrow\ 0
\end{aligned}$$

Proposition:

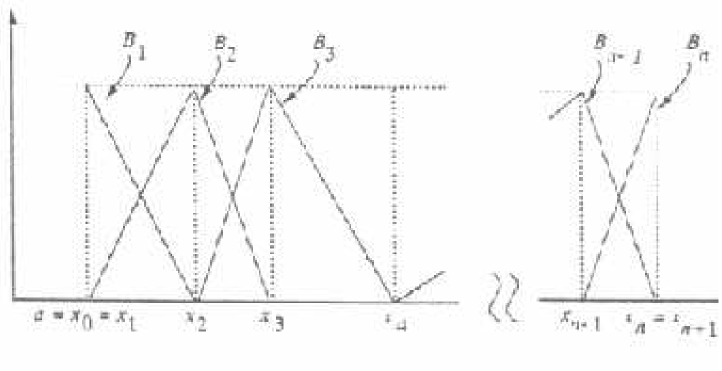
$$\begin{aligned}
\inf_{s \in S_1^0} \|f - s\|_\infty &\leq \|f - s_1(f)\|_\infty \leq 2 \inf_{s \in S_1^0} \|f - s\|_\infty \\
& \qquad \text{nearly} \quad \text{optimal} \\
\text{Since } \|f(\cdot) - s_1(f)\|_\infty &\leq \|f - s^*\| + \|s^* - s_1(f)\| \\
\text{and } \|s_1(g)\|_\infty &\leq \|g\|_\infty \quad \forall\, g \in C[a,\ b].(\#73).
\end{aligned}$$

In other words, the piecewise linear interpolant $s_1(f)$ is a nearly optimal approximation, its error differing from the error of the best approximant to f from S_1^0 by at most a factor of 2.

§ 3.2 A basis for $S_1^0(\triangle)$



Remark:
What is the dimension of the space $S_1^0(\triangle)$?
Answer:
dim $S_1^{-1}(\triangle) = 2n-2$
dim $S_1^0(\triangle) = 2+n-2 = n$.
Therefore, we expect a basis of $S_1^0(\triangle)$ to have n-basis functions.



The functions B_i

$$B_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}}, & \text{if } x_{i-1} \le x \le x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & \text{if } x_i \le x \le x_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition: B_i are basis of $S_1^0(\triangle)$.
$\Leftrightarrow$

1° $\{B_i\}_{i=1}^n$ are linearly independent.

2° they span the space $S_1^0(\triangle)$.

Proof:

1° $s(x) = \sum_{i=1}^n c_i B_i(x) \equiv 0 \;\Rightarrow\; c_i = 0$
$s(x_j) = c_j B_j(x_j) = c_j = 0,\ \forall j$ done.

2° $s(x) = \sum_{i=1}^n s(x_i) B_i(x)$

§ 3.3 Least squares approximation

We consider the problem of least squares approximation on [a,b] by functions in $S_1^0(\triangle)$. Since $\|f - s_1(f)\|_{\text{discrete, 2}} = 0$, we therefore consider only the continuous problem:
Given $f \in C[a,b]$, find

$$\hat{s_1} = \inf_{s \in S_1^0(\triangle)} \|f - s\|,$$

i.e.

$$\|f - \hat{s_1}\|_2 \le \|f - s\|_2, \quad \forall\, s \in S_1^0.$$

Writing

$$\hat{s_1} = \sum_{i=1}^n \hat{c}_i B_i(x),$$

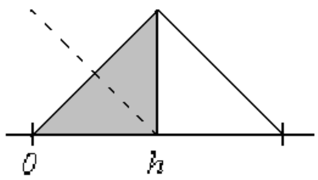
we know from the general theorey of §1 that the coefficients $\hat{c_i}$ must satisfy the normal equations

$$\sum_{j=1}^n (B_i,\ B_j)\hat{c_j} = (B_i,\ f),$$

where $(f,g) = \int_a^b f(x)\, g(x)\, dx$. Or, we have

$$\frac{1}{6}\triangle x_{i-1} \cdot \hat{c_{i-1}} + \frac{1}{3}(\triangle x_{i-1} + \triangle x) \cdot \hat{c_i} + \frac{1}{6}\triangle x_i \cdot \hat{c_{i+1}} = b_i,$$

where $b_i = \int_a^b B_i(x) f dx = \int_{x_{i-1}}^{x_{i+1}} B_i(x) f(x)$



Since

$$\int_0^h (\frac{1}{h}x)(-\frac{1}{h}x+1)dx = \frac{h}{6}$$

$$\int_0^h (\frac{1}{h}x)(\frac{1}{h}x)dx = \frac{1}{h^2}\frac{h^3}{3} = \frac{h}{3}$$

Matrix:

$$\begin{bmatrix} \frac{1}{3}\triangle x_1 & \frac{1}{6}\triangle x_1 & & & \\ \frac{1}{6}\triangle x_1 & \frac{1}{3}(\triangle x_1 + \triangle x_2) & \frac{1}{6}\triangle x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{6}\triangle x_{n-1} \\ & & & \frac{1}{6}\triangle x_{n-1} & \frac{1}{3}\triangle x_{n-1} \end{bmatrix}$$

Remark:

1° symmetry, positive definite, diagonally dominant

2° $\inf_{s\in S_1^0} \|f-s\|_\infty = \|f-\hat{s_1}(f)\|_\infty \le 4\inf_{s\in S_1^0}\|f-s\|_\infty$