§ 3.4 Interpolation by cubic splines

The most widely used splines are cubic splines, in particular, cubic interpolants. Continuity of the first derivative of any cubic spline interpolant $s_3(f;\cdot)$ can be enforced by prescribing the values of the first derivative at each point $x_i, i = 1, 2 \ldots, n$. Thus let $m_1, m_2, \ldots, m_n$ be arbitrary given numbers, and denote

$$s_3(f :, \cdot)|_{[x_i, x_{i+1}]} = p_i(x), \quad i = 1, 2, \ldots, n-1.$$

Then we enforce $s_3^{'}(f; x_i) = m_i, i = 1, 2, \ldots, n$, by selecting each piece p_i of $s_3^{'}(f; \cdot)$ to be the unique solution of a Hermite interpolation problem; namely

$$\begin{aligned}
&x_i, \quad i = 1, 2, \cdots, n \\
&s_3(f;\ x)|_{[x_i,\ x_{i+1}]} = p_i(x),\ \text{enforce}\ s_3'(f; x_i) = m_i, \\
&p_i(x_i) = f_i, \quad p_i(x_{i+1}) = f_{i+1}, \\
&p_i'(x_i) = m_i, \quad p_i'(x_{i+1}) = m_{i+1}, \quad i = 1, 2, \cdots, n-1.
\end{aligned}$$

x	f			
x_i	f_i			
x_i	f_i	m_i		
x_{i+1}	f_{i+1}	$[x_i,\ x_{i+1}]f$	$\frac{[x_i,\ x_{i+1}]f - m_i}{\triangle x_i}$	
x_{i+1}	f_{i+1}	m_{i+1}	$\frac{m_{i+1} - [x_i,\ x_{i+1}]f}{\triangle x_i}$	$\frac{m_{i+1} + m_i - 2[x_i,\ x_{i+1}]f}{(\triangle x_i)^2}$

So in Newton's form

$$\begin{aligned}
p_i(x) = \quad & f_i + (x - x_i)m_i + (x - x_i)^2 \tfrac{[x_i,\ x_{i+1}]f - m_i}{\triangle x_i} \\
& + (x - x_i)^2 (x - x_{i+1}) \tfrac{m_{i+1} + m_i - 2[x_i,\ x_{i+1}]f}{(\triangle x_i)^2}
\end{aligned}$$

also, in Taylor's form
$p_i(x) = c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3$
$c_{i,0} = f_i$
$c_{i,1} = m_i$
$c_{i,2} = \frac{[x_i,\ x_{i+1}]f - m_i}{\triangle x_i} - c_{i,3} \cdot \triangle x_i$
$c_{i,3} = \frac{m_{i+1} + m_i - 2[x_i,\ x_{i+1}]f}{(\triangle x_i)^2}$
Since $x - x_{i+1} = x - x_i - \triangle x$

Remark:
dim $S^k_m(\triangle) = (m-k)\cdot(N-2)+m+1$
$\Rightarrow$ dim $s^1_3(\triangle) = 2N,$ dim $S^2_3(\triangle) = N+2$
We now discuss some possible choices of parameters
$m_1,\ m_2,\ \cdots,\ m_n$

1° Piecewise cubic Hermite interpolation. $m_i = f'(x_i)$.
This gives rise to a strictly local scheme, and the error is as
$f(x) - p_i(x) = (x-x_i)^2(x-x_{i+1})^2[x_i,\ x_i,\ x_{i+1},\ x_{i+1},\ x]f,$
for $x_i \le x \le x_{i+1}$ hence, if $f \in C^4[a,\ b]$

$$|f(x) - p_i(x)| \le (\frac{1}{2}\triangle x_i)^4 \max_{[x_i,\ x_{i+1}]} \frac{|f^{(4)}|}{4!}, \quad x_i \le x \le x_{i+1}$$

$$\|f - s_3(f;\cdot)\|_\infty \le \frac{1}{384}|\triangle|^4\|f^{(4)}\|_\infty$$

if $\triangle x_i = h,\ h = \frac{1}{N}$ then

$$\|f - s_3(f)\|_\infty = O(h^4) \quad \text{as } h \ \to \ \infty$$

However, we need $f'(x_i)$ at nodes.

Question: How do we approximate m_i?

Answer: Find the quadratic polynomial that interpolates $x_{i-1},\ x_i,\ x_{i+1}$

x	f		
x_{i-1}	f_{i-1}		
x_i	f_i	$\frac{f_i - f_{i-1}}{x_i - x_{i-1}}$	
x_{i+1}	f_{i+1}	$\frac{f_{i+1} - f_i}{x_{i+1} - x_i}$	$[x_{i-1},\ x_i,\ x_{i+1}]f$

$p_2(f; x_{i-1},\ x_i,\ x_{i+1})$
$= f_{i-1} + (x - x_{i-1})[x_{i-1},\ x_i]f + (x-x_{i-1})(x-x_i)[x_{i-1},\ x_i,\ x_{i+1}]f$
$p_2'(x_i) = [x_{i-1},\ x_i]f + \triangle x_i[x_{i-1},\ x_i,\ x_{i+1}]f$
Let $\tilde{m}_i = [x_{i-1},\ x_i]f + \triangle x_{i-1}[x_{i-1},\ x_i,\ x_{i+1}]f$
Facts:
For unequally spaced grid error maintains $e \ \sim \ O(|\triangle|^3)$
and or equally spaced grid $e \ \sim \ O(|\triangle|^4) = O(h^4)$

2° Cubic spline interpolation
$s_3(f;\cdot) \in S_3^2(\triangle)$
i.e. $p''_{i-1}(x_i) = p''_i(x_i), \quad i = 2, 3, \cdots, n-1$

Proposition: In terms of Taylor coefficients we have
$2c_{i-1,2} + 6c_{i-1,3} \cdot \triangle x_{i-1} = 2c_{i,2}$
And a calculation gives

$$
\begin{aligned}
&2(\frac{[x_{i-1},\ x_i]f - m_{i-1}}{\triangle x_{i-1}}) + 4(\frac{m_i + m_{i-1} - 2[x_{i-1},\ x_i]f}{(\triangle x_{i-1})^2})\triangle x_{i-1} \\
&\quad = 2(\frac{[x_i,\ x_{i+1}]f - m_i}{\triangle x_i}) - 2(\frac{m_{i+1} + m_i - 2[x_i,\ x_{i+1}]f}{(\triangle x_i)^2})\triangle x_i \\
&(\triangle x_i)(m_{i-1}) + 2(\triangle x_i + \triangle x_{i-1})m_i + (\triangle x_{i-1})m_{i+1} \\
&\quad = 3\{\triangle x_i[x_{i-1},\ x_i]f + (\triangle x_{i-1})[x_i,\ x_{i+1}]f\}
\end{aligned}
$$

So $m_1,\ m_n$ are left alone to be decided.
(dim $S_3^2(\triangle) = N + 2$)
(1) Complete splines:
$m_1 \ = \ f'(a)$
$m_n \ = \ f'(b)$

Facts:

$$\|f^{(r)} - s^{(r)}(f;\cdot)\|_\infty \le c_r|\triangle|^{4-r}\|f^{(4)}\|_\infty,\ r = 0, 1, 2, 3,\ \text{if } f \in C^4[a,\ b],$$

where $c_0 = \frac{5}{384},\ c_1 = \frac{1}{24},\ c_2 = \frac{3}{8}$.
Note that $c_0 = 5 \cdot \frac{1}{384}$, i.e. 5 times larger than the bound for piecewise cubic Hermite interpolant.
But we only need $f'(a)$ and $f'(b)$, which is easier to access.

(2) $s_3^{''}(f;a) = f^{''}(a),\ s_3^{''}(f;b) = f^{''}(b)$
Matching of 2^{nd} derivatives of the end points: Use Newton's form to write a cubic polynomial that fit $f_1,\ f_2,\ m_1,\ m_2$

$$\begin{aligned}
p_1(x) &= f_1 + (x-x_1)m_1 + (x-x_1)^2\frac{[x_1,\ x_2]f - m_1}{\triangle x_1} \\
&\quad +(x-x_1)^2(x-x_2)\frac{m_1+m_2-2[x_1,\ x_2]f}{(\triangle x_1)^2} \\
p_1^{''}(x_1) &= 2\frac{[x_1,\ x_2]f - m_1}{\triangle x_1} - 2\triangle x_1\frac{m_1+m_2-2[x_1,\ x_2]f}{(\triangle x_1)^2} \\
&= f^{''}(a)
\end{aligned}$$

$$\begin{aligned}
-(4m_1+2m_2) &= f^{''}(a)\triangle x_1 - 6[x_1,\ x_2]f \\
\text{i.e. } 2m_1+m_2 &= 3[x_1,\ x_2]f - \frac{1}{2}f^{''}(a)\triangle x_1 \\
m_{n-1}+2m_n &= 3[x_{n-1},\ x_n]f - \frac{1}{2}f^{''}(b)\triangle x_{n-1}
\end{aligned}$$

still tridiagonal matrix.

(3)$s^{''}(f;a) = 0 = s^{''}(f;b)$ Natural cubic spline:
The nice thing about this spline is that it requires only function values of f – no deivatives – but the price to pay is the degradation of the accuracy to $error \sim O(h^2)$.

(4)Not-a-knot spline

$$\text{Requires} = \begin{cases} p_1 & \equiv p_2 \\ p_{n-2} & \equiv p_{n-1} \end{cases}$$

$2c_{1,2} + 6c_{1,3}\triangle x_1 = 2c_{2,2}$
$\Rightarrow\ 6c_{1,3}\triangle x_1 = 0$
$\Rightarrow\ 6c_{1,3} = 0$
$m_2 + m_1 - 2[x_1,\ x_2]f = 0$
and $m_n + m_{n-1} - 2[x_{n-1},\ x_n]f = 0$

§ 3.5 Minimality properties of cubic spline interpolants

The complete and natural splines have interesting optimality properties.
Definition:
$\triangle' = a = x_0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = x_{n+1} = b.$
g is said to interpolates f on $\triangle'$ if

$$\begin{cases} g(x_i) = f(x_i), & i = 1, \cdots, n \\ g'(x_i) = f'(x_i), & i = 0, n+1 \end{cases}$$

Theorem:
For any function $g \in C^2[a,\ b]$ that interpolates f on $\triangle'$, there holds

$$\int_a^b [g''(x)]^2 dx \geq \int_a^b [s''_{compl}(f;x)]^2 dx$$

with equality holds $\Leftrightarrow\ g(\cdot) = s_{compl}(f;\cdot)$
where $s_{compl}(f;\cdot)$ is the complete cubic spline interpolant.

Remark:
Among all such interpolants, s_{compl} has the smallest 2^{nd} derivative L_2 norm, i.e. $|g|_{W^{2,2}}$.

Proof:
Let $s_{compl} = s$.

$$\begin{aligned} \int_a^b (g'')^2 dx &= \int_a^b [(g'' - s'') + s'']^2 dx \\ &= \int_a^b [(g'' - s'')]^2 dx + \int_a^b (s'')^2 dx + 2\int_a^b (g'' - s'')s'' dx. \end{aligned}$$

Theorem holds if $\int_a^b (g'' - s'')s'' dx = 0$
$\int_a^b \underbrace{(g'' - s'')}_{\substack{\parallel \\ dv}} \underbrace{s''}_{\substack{\parallel \\ u}} dx = s'' \underbrace{[g' - s']|_a^b}_{\substack{\parallel \\ 0}} - \int_a^b (g' - s')s''' dx$
Since $(g' - s')(a) = 0,\ (g' - s')(b) = 0$
$\int_a^b (g' - s')s''' dx = \sum s_i''' \int_{x_i}^{x_{i+1}} (g' - s') dx = \sum s_i''' (g - s)|_{x_i}^{x_{i+1}} = 0$

Theorem 2.3.2:
For any function $q \in C^2[a,\ b]$ that interpolates f on $\triangle$ there holds

$$\int_a^b [q^{''}]^2 dx \geq \int_a^b [s^{''}_{nat}(f;x)]^2 dx$$

with equality holds $\Leftrightarrow$ $q(\cdot) = s'_{nat}(f;\cdot)$.
Now $s^{''}[q' - s'] = 0$ since $s^{''}(a) = s^{''}(b) = 0$.

Remark:
$$\int_a^b [s^{''}_{compl}(f;x)]^2 dx \geq \int_a^b [s^{''}_{nat}(f;x)]^2 dx$$
Therefore, the natural cubic spline is the "smoothest" interpolant.
The property expressed in Theorem 2.3.2 is the orgion of the name “spline”.
A spline is a flexible stripe of wood used in drawing curves.