§2.3 Properties of Gaussian quadrature rules

The Gaussian quadrature rule, in addition to being optimal, has some interesting and useful properities.

- (i) all nodes t_k are real distinct, and contained in (a,b) (**Homework**: 31)
- (ii) all weights w_k are positive

  $0 < \int_a^b l_i^2(t)w(t)dt = \sum\limits_{k=1}^{n} w_k l_i^2(t_k) = w_i$, $l_i^2 \in P_{2n-2}$, $2n-2 < 2n-1$
- (iii) If $[a,b]$ is a finite interval, then Gauss formula converges for any continuous function, i.e. $E_n(f) \to 0$ as $n \to \infty$

  Proof:

  $$\begin{aligned}&|E_n(f)| = |E_n(f - \hat{p}_{2n-1})| \\ &= \left| \int_a^b (f - \hat{p}_{2n-1})w(t)dt - \sum_{k=1}^{n} w_k \left[f(t_k) - \hat{p}_{2n-1}(t_k)\right] \right| \\ &\le \int_a^b |f(t) - \hat{p}_{2n-1}(f;t)|\, w(t)dt + \sum_{k=1}^{n} w_k\, |f(t_k) - \hat{p}_{2n-1}(t_k)| \\ &\le \|f - \hat{p}_{2n-1}(f)\|_\infty \left[\int_a^b w(t)dt + \sum_{k=1}^{n} w_k\right]\end{aligned}$$

  Notice that $\int_a^b w(t)dt = \sum\limits_{k=1}^{n} w_k = \mu_0$

  $= 2\mu_0 \|f - \hat{p}_{2n-1}(f)\|_\infty \to 0$ as $n \to \infty$
- (iv) The nodes in the n-pt Gauss formula separate those of the (n+1)-pt formula. (**Homework**: 32)

Prove that all zeros of $\pi_n(\cdot; w)$ are real, simple, and contained in (a,b). Let $a < x_1 < \cdots < x_l < b$ are roots of w_n in (a,b) and of which π_n changes sign. Now let $q(x) = \prod\limits_{j=1}^{l}(x - x_j)$, then $w_n(x)q(x)$ does not change sign in $[a,b]$. So $(w_n, q) = \int_a^b w(x)w_n(x)q(x)dx \neq 0$ but if $l < n$ then $\rightarrow\leftarrow$ to Theorem 3.2.1 so $l = n$, i.e. all roots are distinct and with multiplicity 1.

The next property forms the basis of an efficient algorithm for computing Gaussian quadrature formulae. Let

$$\pi_{k+1}(t) = (t - \alpha_k) \cdot \pi_k(t) - \beta_k \pi_{k-1}(t)$$

$$\pi_0(t) = 1, \pi_{-1}(t) = 0, \beta_0 \equiv \int_a^b w(x)dt = \mu_0$$

The nth-order Jacobi matrix for the weight function w is a tridiagonal symmetric matrix defined by

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \ddots & 0 \\ & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

Proposition:
$J_n v^{(k)} = t_k v^{(k)}$, $v^{(k)T} v^{(k)} = 1$ for some $v^{(k)}$. And the weight w_k expressible in terms of the first component $v_1^{(k)}$ of the corresponding (normalized) eigenvectors by

$$w_k = \beta_0 (v_1^{(k)})^2, k = 1, 2 \ldots, n.$$

Thus, to compute the Gaussian formula, we must solve an eigenvalue/eigenvector problem for a symmetric tridiagonal matrix.

Proof

(1) claim: $\tilde{v}^{(i)} = \begin{bmatrix} 1 \cdot \pi_0(t_i) \\ \frac{1}{\sqrt{\beta_1}} \pi_1(t_i) \\ \frac{1}{\sqrt{\beta_1 \beta_2}} \pi_2(t_i) \\ \vdots \\ \frac{1}{\sqrt{\beta_1 \cdots \beta_{n-1}}} \pi_{n-1}(t_i) \end{bmatrix}$ are eigenvectors.

(2)

$$\begin{bmatrix} \pi_0(t_1) & \pi_0(t_2) & \cdots & \pi_0(t_n) \\ \frac{1}{\sqrt{\beta_1}}\pi_1(t_1) & \frac{1}{\sqrt{\beta_1}}\pi_1(t_2) & \cdots & \frac{1}{\sqrt{\beta_1}}\pi_1(t_n) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}\pi_{n-1}(t_1) & \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}\pi_{n-1}(t_2) & \cdots & \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}\pi_{n-1}(t_n) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} (\pi_0, \pi_0) \\ \frac{1}{\sqrt{\beta_1}}(\pi_1, \pi_0) \\ \vdots \\ \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}(\pi_{n-1}, \pi_0) \end{bmatrix} = \begin{bmatrix} (\pi_0, \pi_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ since } \pi_0 \equiv 1.$$

Eigenvectors of symmetric matrices for distinct eigenvalues are orthogonal. Therefore, multiplying by $\tilde{v}^{(i)}$ from the left yields

$$\tilde{v}^{(i)T}\tilde{v}^{(i)}w_i = (\pi_0, \pi_0) \cdot \tilde{v}_1^{(i)}$$

$$\Rightarrow (\tilde{v}^{(i)})^T\tilde{v}^{(i)}w_i = (\pi_0, \pi_0)$$

and $\tilde{v}^{(i)} = \frac{1}{v_1^{(i)}}v^{(i)}$, since $\tilde{v}_1^{(i)} = 1$, with $(v^{(i)})^T v^{(i)} = 1$

Multiplying $(v_1^{(i)})^2$ we have $v^{(i)T}v^{(i)}w_i = (v_1^{(i)})^2(\pi_0, \pi_0)$

$\Rightarrow w_i = (v_1^{(i)})^2(\pi_0, \pi_0) = (v_1^{(i)})^2\beta_0$

Proof of claim:

$$\begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \ddots & 0 \\ & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} \pi_0(t_i) \\ \frac{1}{\sqrt{\beta_1}}\pi_1(t_i) \\ \frac{1}{\sqrt{\beta_1\beta_2}}\pi_2(t_i) \\ \vdots \\ \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}\pi_{n-1}(t_i) \end{bmatrix}$$

(i) $\alpha_0\pi_0(t_i) + \sqrt{\beta_1} \cdot \frac{1}{\sqrt{\beta_1}}\pi_1(t_i) = \alpha_0\pi_0(t_i) + \pi_1(t_i)$
$= \alpha_0\pi_0(t_i) + (t_i - \alpha_0)\pi_0(t_i) = t_i \cdot 1 \cdot \pi_0(t_i)$

(ii) For $j = 1, \cdots, n-2$
$\sqrt{\beta_j} \cdot \frac{1}{\sqrt{\beta_1\cdots\beta_{j-1}}}\pi_{j-1}(t_i) + \alpha_j \cdot \frac{1}{\sqrt{\beta_1\cdots\beta_j}}\pi_j(t_i) + \sqrt{\beta_{j+1}} \cdot \frac{1}{\sqrt{\beta_1\cdots\beta_{j+1}}}\pi_{j+1}(t_i)$
$= \frac{1}{\sqrt{\beta_1\cdots\beta_j}}(\beta_j\pi_{j-1}(t_i) + \alpha_j\pi_j(t_i) + (t_i - \alpha_j)\pi_j(t_i) - \beta_j\pi_{j-1}(t_i))$
$= \frac{1}{\sqrt{\beta_1\cdots\beta_j}} \cdot t_i \cdot \pi_j(t_i)$

(iii)
$\sqrt{\beta_{n-1}} \cdot \frac{1}{\sqrt{\beta_1\cdots\beta_{n-2}}}\pi_{n-2}(t_i) + \alpha_{n-1} \cdot \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}}\pi_{n-1}(t_i)$
$= \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}} \cdot (\beta_{n-1}\pi_{n-2}(t_i) + \alpha_{n-1}\pi_{n-1}(t_i))$
$\because 0 = \pi_n(t_i) = (t_i - \alpha_{n-1})\pi_{n-1} - \beta_{n-1}\pi_{n-2}(t_i)$
$= \frac{1}{\sqrt{\beta_1\cdots\beta_{n-1}}} \cdot t_i \cdot \pi_{n-1}(t_i)$

$w(t)$	$[a,b]$	Orth. Pol.	Notation	α_k	β_k
1	$[-1,1]$	Legendre[7]	P_n	0	$2\ (k=0)$ $(4-k^{-2})^{-1}\ (k>0)$
$(1-t^2)^{-\frac{1}{2}}$	$[-1,1]$	Chebyshev #1	T_n	0	$\pi\ (k=0)$ $\frac{1}{2}\ (k=1)$ $\frac{1}{4}\ (k>1)$
$(1-t^2)^{\frac{1}{2}}$	$[-1,1]$	Chebyshev #2	U_n	0	$\frac{1}{2}\pi\ (k=0)$ $\frac{1}{4}\ (k>0)$
$(1-t)^\alpha\ (1+t)^\beta$ $\alpha>-1, \beta>-1$	$[-1,1]$	Jacobi[8]	$P_n^{(\alpha,\beta)}$	known	known
$t^\alpha e^{-t}, \alpha>-1$	$[0,\infty]$	Laguerre[9]	$L_n^{(\alpha)}$	$2k+\alpha+1$	$\Gamma(1+\alpha)\ (k=0)$ $k(k+\alpha)\ (k>0)$
e^{-t^2}	$[-\infty,\infty]$	Hermite[10]	H_n	0	$\sqrt{\pi}\ (k=0)$ $\frac{1}{2}k\ (k>0)$

Different approach: Let

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t)$$
$$\pi_{-1}(t) = 0, \quad \pi_0(t) = 1$$
$$\alpha_k = \frac{(t\pi_k,\ \pi_k)}{(\pi_k,\ \pi_k)}, \quad \beta_k = \frac{(\pi_k,\ \pi_k)}{(\pi_{k-1},\ \pi_{k-1})}$$

We want to find $\pi_2(t)$ using equations defined above,

$$\pi_1(t) = (t-\alpha_0)\pi_0(t) - \beta_0\pi_{-1}(t)$$

$$\alpha_0 = \frac{(t\pi_0,\ \pi_0)}{(\pi_0,\ \pi_0)} = \frac{\int_0^1 t\cdot t^{-\frac{1}{2}}dt}{\int_0^1 1\cdot 1\cdot 1\cdot t^{-\frac{1}{2}}dt} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

$$\begin{aligned}
\pi_1(t) &= (t-\frac{1}{3}) \\
\pi_2(t) &= (t-\alpha_1)\pi_1(t) - \beta_1\pi_0(t) \\
\alpha_1 &= \frac{(t\pi_1,\ \pi_1)}{(\pi_1,\ \pi_1)} = \frac{\int_0^1 t\cdot(t-\frac{1}{3})\cdot(t-\frac{1}{3})\cdot t^{-\frac{1}{2}}dt}{\int_0^1 (t-\frac{1}{3})\cdot(t-\frac{1}{3})\cdot t^{-\frac{1}{2}}dt} \\
&= \frac{\int_0^1 t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{9}t^{\frac{1}{2}}dt}{\int_0^1 t^{\frac{3}{2}} - \frac{2}{3}t^{\frac{1}{2}} + \frac{1}{9}t^{-\frac{1}{2}}dt} \\
&= \frac{\frac{2}{7} - \frac{2}{3}\cdot\frac{2}{5} + \frac{1}{9}\cdot\frac{2}{3}}{\frac{2}{5} - \frac{2}{3}\cdot\frac{2}{3} + \frac{1}{9}\cdot 2} = \frac{11}{21} \\
\beta_1 &= \frac{(\pi_1,\ \pi_1)}{(\pi_0,\ \pi_0)} = \frac{\int_0^1 (t-\frac{1}{3})^2 t^{-\frac{1}{2}}dt}{\int_0^1 t^{-\frac{1}{2}}dt} = \frac{4}{45} \\
\pi_2(t) &= (t-\frac{11}{21})(t-\frac{1}{3}) - \frac{4}{45}\cdot 1 \\
&= t^2 - \frac{6}{7}t + \frac{3}{35}
\end{aligned}$$

This agrees with the previous compuations.

$$
\begin{aligned}
J_2 &= \begin{bmatrix} \frac{1}{3} & \sqrt{\frac{4}{45}} \\ \sqrt{\frac{4}{45}} & \frac{11}{21} \end{bmatrix} = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} \\ \sqrt{\beta_1} & \alpha_1 \end{bmatrix} \\
|\lambda I - J_n| &= \left| \begin{bmatrix} \lambda - \frac{1}{3} & -\sqrt{\frac{4}{45}} \\ -\sqrt{\frac{4}{45}} & \lambda - \frac{11}{21} \end{bmatrix} \right| \\
&= (\lambda - \frac{1}{3})(\lambda - \frac{11}{21}) - \frac{4}{45}
\end{aligned}
$$

$t_1 = \frac{1}{7}(3 - 2\sqrt{\frac{6}{5}}),\ t_2 = \frac{1}{7}(3 + 2\sqrt{\frac{6}{5}})$

$\tilde{v}^{(1)} = \begin{pmatrix} 1 \\ 1/\sqrt{\frac{4}{45}}(t_1 - \frac{1}{3}) \end{pmatrix} \quad \tilde{v}^{(2)} = \begin{pmatrix} 1 \\ 1/\sqrt{\frac{4}{45}}(t_2 - \frac{1}{3}) \end{pmatrix}$

$\tilde{v}^{(k)} = \frac{\tilde{v}^{(k)}}{\|\tilde{v}^{(k)}\|}$

and $w_k = \beta_0 \cdot (v_1^{(k)})$

Remark:
Which way is easier to compute?

§2.4 Some applications of Gauss quadrature rule

We begin with the easy exercise of transforming the Gauss-Jacobi quadrature rule from an arbitrary finite interval to the canonical interval [-1,1]
(i) Gauss-Jacobi formula on $[-1,1]$

$$\int_{-1}^{1} f(t)(1-t)^{\alpha}(1+t)^{\beta} dt,$$

where $\omega(t)=(1-t)^{\alpha}(1+t)^{\beta}$, $\alpha>-1$, $\beta>-1$ Note that at $t=-1,1$ there are algebraic singularities.
Goal:

$$\int_{a}^{b} (b-x)^{\alpha}(x-a)^{\beta} g(x) dx$$

A linear transformation of variables.

$$x=\frac{1}{2}(b-a)t+\frac{1}{2}(b+a), \qquad dx=\frac{1}{2}(b-a)dt$$

maps $[a,b]\to[-1,1]$ and the integral becomes

$$\int_{-1}^{1}(b-\frac{1}{2}(b-a)t-\frac{1}{2}(b+a))^{\alpha}(\frac{1}{2}(b-a)t+\frac{1}{2}(b+a)-a)^{\beta}$$

$$g(\frac{1}{2}(b-a)t+\frac{1}{2}(b+a))\frac{1}{2}(b-a)dt$$

$$=\int_{-1}^{1}(\frac{1}{2}(b-a)(1-t))^{\alpha}(\frac{1}{2}(b-a)(1+t))^{\beta} f(t)\frac{1}{2}(b+a)dt$$

$$=[\frac{1}{2}(b-a)]^{\alpha+\beta+1}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta} f(t)dt,$$

where $f(t)=g(\frac{1}{2}(b-a)t+\frac{1}{2}(b+a)) \quad -1\le t\le 1$

$$\int_{a}^{b}(b-x)^{\alpha}(x-a)^{\beta} g(x)dx$$

$$=[\frac{1}{2}(b-a)]^{\alpha+\beta+1}\sum_{k=1}^{n} w_K^J g(\frac{1}{2}(b-a)t_K^J+\frac{1}{2}(b+a))+E_n(g)$$

t_K^J, w_K^J are (standard) Gauss-Jacobi nodes & weights
and exact for $g\in P_{2n-1}$

(ii) Iterated integrals

$$(Ig)(t) := \int_0^t g(\tau)d\tau$$

And I^p is the pth power. Then

$$(I^{p+1}g)(1) := \int_0^1 (I^p g)(t)dt$$

From calculus that we have

$$(I^{p+1}g)(1) = \int_0^1 \frac{(1-t)^p}{p!} g(t)dt$$

For p =1, we have

$$(I^2g)(1) = \int_0^1 \left(\int_0^t g(\tau)d\tau \right) dt = \int_0^1 \int_\tau^1 g(\tau)dt\, d\tau = \int_0^1 (1-\tau)g(\tau)d\tau.$$

For p =2, we have

$$(I^3g)(1) = \int_0^1 \left(\int_0^s \left(\int_0^t g(\tau)d\tau \right) dt \right) ds = \int_0^1 \int_0^s \int_\tau^s g(\tau)dt\, d\tau\, ds$$

$$= \int_0^1 \int_0^s (s-\tau)g(\tau)d\tau\, ds = \int_0^1 \int_\tau^1 (s-\tau)g(\tau)\, ds\, d\tau = \int_0^1 (s-\tau)^2/2 \;\; g(\tau)|_\tau^1\, d\tau$$

$$= \int_0^1 (1-\tau)^2/2 \;\; g(\tau)\, d\tau.$$

When $p << \infty$ we could use either trapezoidal or Simpson's rule. But as $p \to \infty$, $(1-t)^p$ approaches a discontinuous function. Use Jacobi weight on $[0,1]$, $\alpha = p$, $\beta = 0$

$$(I^{p+1}g)(1) = \frac{1}{2^{p+1}p!} \sum_{k=1}^n w_K^J g(\frac{1}{2}(1+t_K^J)) + E_n(g)$$

(iii) Integration over $\Re$

$$\int_{-\infty}^{\infty} F(x)dx, \qquad F(x) \sim e^{-ax^2}, \quad x \to \pm\infty, \quad a > 0.$$

Let

$$x = \frac{t}{\sqrt{a}}, \quad dx = \frac{1}{\sqrt{a}}dt, \quad f(t) = e^{t^2}F(\frac{t}{\sqrt{a}}), \quad f(\infty) = f(-\infty) = 1$$

$$\int_{-\infty}^{\infty} F(x)dx = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} F(\frac{t}{\sqrt{a}})dt$$
$$= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-t^2}\cdot(e^{t^2}F(\frac{t}{\sqrt{a}}))dt$$
$$= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-t^2}f(t)dt.$$

Use Gauss Hermite formula

$$= \frac{1}{\sqrt{a}}\sum_{k=1}^{n} w_K^H e^{(t_K^h)^2} F(\frac{t_K^H}{\sqrt{a}}) + E_n(f)$$

$$E_n(f) = 0, \text{ if } e^{t^2}F(\frac{t}{\sqrt{a}}) \in P_{2n-1}$$

i.e. $F(x) = e^{-t^2}\cdot p(x) = e^{-ax^2}\cdot p(x)$, $p(x) \in P_{2n-1}$

(iv)Integration over $\mathcal{R}_+$.
Compute

$$\int_0^\infty F(x)dx$$

Assuming that

$$F(x) \sim \begin{cases} x^p & \text{as } x \downarrow 0 \quad (p > -1) \\ x^{-q} & \text{as } x \to \infty \; (q < 1) \end{cases}$$

Similarly as in (iii), we define f by

$$F(x) = \frac{x^p}{(1+x)^{p+q+\lambda}} f(x), \quad \lambda = 0, 1, 2, \cdots, 2n-1.$$

So as to $f(x) \longrightarrow 1$ as $x \downarrow 0$ and $x \to \infty$. The change of variables

$$x = \frac{1+t}{1-t}, \quad (1-t)^\lambda = (\frac{2}{x+1})^\lambda, \quad dx = \frac{2dt}{(1-t)^2}$$

$$\Rightarrow \int_0^\infty F(x)dx = \int_{-1}^1 (\frac{1+t}{1-t})^p (\frac{2}{1-t})^{-(p+q)} f(\frac{1+t}{1-t}) \frac{2dt}{(1-t)^2}$$

$$= \frac{1}{2^{p+q-1}} \int_{-1}^1 (1-t)^{q-2}(1+t)^p g(t)dt, \quad \text{when } \lambda = 0$$

where $g(t) = f(\frac{1+t}{1-t})$
Gauss-Jacobi quadrature , $\alpha = q-2$, $\beta = p$

$$\int_0^\infty F(x)dx = \frac{1}{2^{p+q-1}} (\sum_{k=1}^n w_K^J g(t_K^J) + E_n(g))$$

$$2^{-p-q+1} \sum_{k=1}^n w_K^J \frac{2^{p+q} \frac{1}{(1-t_K^J)^{p+q}}}{(\frac{1+t_K^J}{1-t_K^J})^p} \cdot F(\frac{1+t_K^J}{1-t_K^J})$$

$$= 2 \sum_{k=1}^n w_K^J (1+t_K^J)^{-P} (1-t_K^J)^{-q} F(\frac{1+t_K^J}{1-t_K^J}) + 2^{-p-q+1} E_n(g)$$

§2.4 Approximation of linear functionals: method of interpolation v.s method of undetermined coefficients

Up until now, we heavily relied on interpolation to obtain approximations to derivatives, integrals, and the like. That is not the only possible approach, however, to construct approximation formulae, and indeed not usually the simplest one. Another is the method of undetermined coefficients. Both approaches can be described in a vastly more general setting, in which a given linear functional is to be approximated by other linear functionals so as to have exactness on a suitable finite-dimensional function space. Although both approaches yield the same formulae, the mechanices involved are quite different. Let's see two simple examples.

Example1:

$$\int_0^1 f(x)dx \approx \int_0^1 p_1(x)dx = \int_0^1 ((1-x)f(0) + xf(1))dx = \frac{1}{2}[f(0) + f(1)]$$

Method of undetermined coefficients :

$$\int_0^1 f(x)dx = a_1 f(0) + a_2 f(1), \qquad \text{if } f \in P_1(x)$$

This gives

$$1 = a_1 + a_2$$
$$\frac{1}{2} = a_2$$

$$\Rightarrow a_1 = a_2 = \frac{1}{2}$$

Example2:

$$\int_0^1 \sqrt{x} f(x)dx \approx a_1 f(0) + a_2 \int_0^1 f(x)dx$$

Method of interpolation. "Interpolation" here means interpolation to the given data – the value of f at $x = 0$ and the integral of f from 0 and 1. We are thus seeking a polynomial $p \in P_1$ such that

$$p(0) = f(0), \quad \int_0^1 p(x)dx = \int_0^1 f(x)dx$$

Let $p = c_0 + c_1 x$, we have

$$c_0 = f(0), \quad c_0 + \frac{1}{2}c_1 = \int_0^1 f(x)dx$$

$$\Longrightarrow c_0 = f(0), \quad c_1 = 2\left\{\int_0^1 f(x)dx - f(0)\right\}$$

$$\begin{aligned}(c_0 + c_1)|_{x=0} &= f(0) = L_1(f), \\ \int_0^1 (c_0 + c_1 x)\,dx &= \int_0^1 f(x)\,dx = L_2(f).\end{aligned}$$

that is

$$\begin{bmatrix} 1 & 0 \\ \int_0^1 1\,dx & \int_0^1 x\,dx \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(0) \\ \int_0^1 f(x)\,dx \end{bmatrix} = \begin{bmatrix} L_1(f) \\ L_2(f) \end{bmatrix}$$

Therefore,

$$\int_0^1 \sqrt{x} f(x)dx \approx \int_0^1 \sqrt{x} p(x)dx = \int_0^1 \sqrt{x}(c_0 + c_1 x)dx$$

$$= \frac{2}{3}c_0 + \frac{2}{5}c_1 = \frac{2}{3}f(0) + \frac{2}{5}\cdot 2\left\{\int_0^1 f(x)dx - f(0)\right\}$$

$$\begin{aligned} &= \sum_{i=1}^{n} c_j L(\varphi_j) = \lambda^T G^{-1}\ell \\ &= [[G^T]^{-1}\lambda]^T \ell = \sum a_i L_i(f)\end{aligned}$$

i.e.

$$\int_0^1 \sqrt{x} f(x)dx \approx -\frac{2}{15}f(0) + \frac{4}{5}\int_0^1 f(x)dx$$

Method of undetermined Coefficients :
Equality of (2.40) for $f(x) = 1$ and $f(x) = x$ immediately yields

$$\frac{2}{3} = \int_0^1 \sqrt{x}\cdot 1dx = a_1 + a_2$$

$$\frac{2}{5} = \frac{1}{2}a_2$$

$$\Longrightarrow a_1 = -\frac{2}{15}, \quad a_1 = \frac{4}{5}$$

$$L\varphi_j = \sum_{i=1}^{n} a_i L_i \varphi_j$$

$$G^T = \begin{bmatrix} L_1(1) & L_2(1) \\ L_1(x) & L_2(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} L(1) \\ L(x) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \int_0^1 1\,dx \\ 0 & \int_0^1 x\,dx \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(f) \\ \lambda_2(f) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/5 \end{bmatrix}$$

Remark: In place of polynomials , we could have chosen other classes of functions, as long as we make sure that their dimension matches the number of " free parameters "

The essence of the two examples consists of showing how a centain linear functional can be approximated in terms of other (presumably simpler) linear functionals by forming a suitable linear combination of the latter.
Definition: A linear functional L on a linear space F is a map $L : F \to \mathbb{R}$ which satisfies

$$L(f+g) = Lf + Lg \ , \ \forall \ f, g \in F$$

$$L(cf) = cLf \ , \ \forall c \in \mathbb{R}, \ f \in F$$

Definition: F is a linear function space if

$$f, g \in F \Rightarrow f + g \in F$$

$$c \in \mathbb{R} \Rightarrow cf \in F$$

Example: Linear functional:
(i) $Lf = f(0); F = \{f : f \text{ is defined at } x = 0\}$
(ii) $Lf = f''(\frac{1}{2}); F = \{f : f \text{ has 2nd derivatives at } x = \frac{1}{2}\}$
(iii) $Lf = \int_0^1 f dx; F = C[0,1]$
(iv) $Lf = \int_0^1 fw dx; F = C[0,1]$ w is intergrable

Example: Nonlinear functional:
$Kf = |f(0)|$
$Kf = \int_0^1 f^2 dx$

General approximation problem:

Given a linear functional L on $\mathbb{F}$, n special linear functionals $L_1, \ldots, L_n$ on $\mathbb{F}$ and $\ell_i = L_i f$, $i = 1, \ldots, n$, and a linear subspace $\Phi \subset \mathbb{F}$ with dim $\Phi = n$

Goal: We want to find an approximation formula, so that $Lf \approx \sum_{i=1}^{n} a_i L_i f$ and is exact whenever $f \in \Phi$

Assumption: The "interpolation problem"

Find $\varphi \in \Phi$ s.t. $L_i\varphi = s_i, i = 1, 2, \ldots, n$ has a unique solution, $\varphi(\cdot) = \varphi(s; \cdot)$
$\forall\ s = [s_1, \ldots, s_n]^T$

Let $\varphi = \sum_{j=1}^{n} c_j \varphi_j$, where $\{\varphi_1, \ldots, \varphi_n\}$ is a basis of Φ
and $L_i\varphi = \sum_{j=1}^{n} c_j L_i \varphi_j = s_i$

$$G = \begin{bmatrix} L_1\varphi_1 & L_1\varphi_2 & \cdots & L_1\varphi_n \\ L_2\varphi_1 & L_2\varphi_2 & \cdots & L_2\varphi_n \\ \cdots & \cdots & \cdots & \cdots \\ L_n\varphi_1 & L_n\varphi_2 & \cdots & L_n\varphi_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

$Gc = s$ is solvable iff $|G| \neq 0$, the matrix G is called "Gram matrix."

Method of interpolation:

Given

$$\ell = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \end{bmatrix}, \quad \ell_i = L_i f$$

find $\varphi(\ell; \cdot) \in \Phi$ so that $Lf \approx L\varphi$ i.e. $Gc = l$ and $\varphi = \sum c_i \varphi_i,\ Lf = L\varphi$ if $f \in \Phi$

Writing $\lambda_j = L\varphi_j, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$

We have $L\varphi = \sum_{j=1}^n c_j L\varphi_j = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \lambda^T G^{-1} \ell = [(G^T)^{-1}\lambda]^T \ell$

i.e. $L\varphi = \sum_{i=1}^n a_i L_i f,\ a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (G^T)^{-1}\lambda$

Method of undetermined coefficients:

$L\varphi_j = \sum_{i=1}^n a_i L_i \varphi_j$

$G^T = \begin{bmatrix} L_1\varphi_1 & L_2\varphi_1 & \cdots & L_n\varphi_1 \\ L_1\varphi_2 & L_2\varphi_2 & \cdots & L_n\varphi_2 \\ \cdots & \cdots & \cdots & \cdots \\ L_1\varphi_n & L_2\varphi_n & \cdots & L_n\varphi_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

i.e. $\therefore \quad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (G^T)^{-1}\lambda$

so those two methods are mathematically equivalent.

§2.6 Peano representation of linear functionals

It may be argued that the method of interpolation, at least in the case of polynomials is more powerful than the method of undetermined coefficients because it also yields an expression for the error term. The method of undetermined coefficients, in contrast, generates only the coefficients in the approximation and gives no clue as to the approximation error.
There is, however, a device due to Peano that allows us to discuss the error after the approximation has been found. The point is that the error,

$$Ef = Lf - \sum_{i=1}^{n} a_i L_i f$$

is itself a linear functional, one that annihilates all polynomials, say, of degree d,

$$Ep = 0 \ \forall \ p \in \mathbb{P}_d$$

Let $F = \mathbb{C}^{d+1}[a, b]$, then by Taylor's Theorem, we have

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + \cdots + (x-a)^d \frac{f^{(d)}(a)}{d!} + \frac{1}{d!}\int_a^x (x-t)^d f^{(d+1)}(t)dt$$

Now, let

$$(x-t)_+ = \begin{cases} x-t & \text{if } x-t \geq 0 \\ 0 & \text{if } x-t < 0 \end{cases}$$

Thus

$$\int_a^x (x-t)^d f^{(d+1)}(t)dt = \int_a^b (x-t)_+^d f^{(d+1)}(t)dt$$

And

$$Ef = \frac{1}{d!} E\{\int_a^b (x-t)_+^d f^{(d+1)}(t)dt\} = \frac{1}{d!}\int_a^b [E_{(x)}(x-t)_+^d] f^{(d+1)}(t)dt,$$

provided the interchange of E with the integral is legitimate. (For most functionals it is.) The subscript x is to indicate that E acts on the variable x (not t).

Definition:

$$K_d(t) = \frac{1}{d!}E_{(x)}(x-t)_+^d,\ t \in \mathbb{R}$$

Thus

$$Ef = \int_a^b K_d(t)f^{(d+1)}(t)dt$$

This is called the Peano representation of the functional E, and K_d the dth Peano kernel for E.

Definition:
E is definite of order d if its Peano kernel K_d doesn't change sign. (We then say K_d is definite).
By Mean Value Theorem of integration we have

$$Ef = f^{((d+1))}(\tau)\int_a^b K_d(t)dt, \quad a < \tau < b$$

Moreover, if we let

$$f = \frac{t^{d+1}}{(d+1)!}$$

then

$$E(\frac{t^{d+1}}{(d+1)!}) = \int_a^b K_d(t)dt$$

denote it by e_{d+1}, therefore

$$Ef = e_{d+1}f^{d+1}(\tau)$$

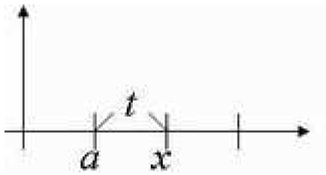
hence

$$|Ef| \le |e_{d+1}| \cdot ||f^{(d+1)}||_\infty$$

Example:

$$Ef = \int_0^1 \sqrt{x} f(x)dx - (-\frac{2}{15} f(0) + \frac{4}{5}\int_0^1 f(x)dx)$$

$$Ef = \int_0^1 K_1(x) f''(t)dt,\ K_1 = E_{(x)}(x-t)_+$$



Furthermore,

$$K_1(t) = \int_0^1 \sqrt{x}(x-t)_+ dx + \frac{2}{15}\cdot 0 - \frac{4}{5}\int_0^1 (x-t)_+ dx$$
$$= \int_t^1 \sqrt{x}(x-t)dx - \frac{4}{5}\int_t^1 (x-t)dt = \frac{2}{15} t(2t^{\frac{3}{2}} - 3t + 1)$$

Now the function in parenthese, say, $q(t)$, satisfies $q(0) = 1$, $q(1) = 0$, $q^{'}(t) = -3(1-t^{1/2}) < 0$ for $0 < t < 1$. Therefore $K_1(t)$ is(positive) definite and

$$e_2 = E(\frac{t^2}{2!}) = \int_0^1 \sqrt{x}\frac{x^2}{2} dx + \frac{2}{15}\cdot 0 - \frac{4}{5}\int_0^1 \frac{x^2}{2} dx = \frac{1}{105}$$

i.e. $Ef = \frac{1}{105} f''(\tau), 0 < \tau < 1$