Numerical Analysis **Assignment #8**

1. Consider the integral $I = \int_{-1}^{1} |x|dx$, whose exact value is evidently 1. Suppose I is approximated (as it stands) by the composite trapezoidal rule $T(h)$ with $h = 2/n$, $n = 1, 2, 3, \ldots$.
   (a) Show (without any computation) that $T(2/n) = 1$ if n is even.
   (b) Determine $T(2/n)$ for n odd and comment on the speed of convergence.

2. (a) Derive the "midpoint rule" of integration
$$\int_{x_k}^{x_k+h} f(x)dx = hf(x_k + \tfrac{1}{2}h) + \frac{1}{24}h^3 f''(\xi), \quad x_k < \xi < x_k + h.$$
   {*Hint*: Use Taylor's theorem centered at $x_k + \frac{1}{2}h$.}
   (b) Obtain the composite midpoint rule for $\int_a^b f(x)dx$, including the error term, subdividing $[a, b]$ into n subintervals of length $h = \frac{b-a}{n}$.

3. (a) Construct the weighted Newton-Cotes formula
$$\int_0^1 f(x) \cdot x \ln(1/x)dx \approx a_0 f(0) + a_1 f(1).$$
   {*Hint:* Use $\int_0^1 x^r \ln(1/x)dx = (r + 1)^{-2}$, $r = 0, 1, 2, \ldots$.}
   (b) Discuss how the formula in (a) can be used to approximate $\int_0^h g(t) \cdot t \ln(1/t)dt$ for small $h > 0$. {*Hint*: Make a change of variables.}

4. Let s be the function defined by
$$s(x) = \begin{cases} (x + 1)^3 & \text{if } -1 \le x \le 0, \\ (1 - x)^3 & \text{if } \ 0 \le x \le 1. \end{cases}$$
   (a) With Δ denoting the subdivision of $[-1, 1]$ into the two subintervals $[-1, 0]$ and $[0, 1]$, to what class $\mathbb{S}_m^k(\Delta)$ does the spline s belong?
   (b) Estimate the error of the composite trapezoidal rule applied to $\int_{-1}^{1} s(x)dx$, when $[-1, 1]$ is divided into n subintervals of equal length $h = 2/n$ and n is even.
   (c) What is the error of the composite Simpson's rule applied to $\int_{-1}^{1} s(x)dx$, with the same subdivision of $[-1, 1]$ as in (b)?
   (d) What is the error resulting from applying the 2-point Gauss-Legendre rule to $\int_{-1}^{0} s(x)dx$ and $\int_0^1 s(x)dx$ separately and summing?

5. (Gauss-Kronrod rule) Let $\pi_n(\,\cdot\,; w)$ be the (monic) orthogonal polynomial of degree n relative to a nonnegative weight function w on $[a, b]$, and $t_k^{(n)}$ its zeros. Use the theorem proved in class to determine conditions on w_k, w_k^*, t_k^* for the quadrature rule
$$\int_a^b f(t)w(t)dt = \sum_{k=1}^{n} w_k f(t_k^{(n)}) + \sum_{k=1}^{n+1} w_k^* f(t_k^*) + E_n(f)$$
   to have degree of exactness at least $3n + 1$; that is, $E_n(f) = 0$ for all $f \in \mathbb{P}_{3n+1}$. {*Hint.* Replace in the theorem n by $2n + 1$.}