Numerical Analysis **Answers to Assignment #6**

1. (a) One has

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (x - x_k), \quad -1 < \xi < 1.$$

Now, for $f(x) = \ln(2 + x)$, the derivatives are

$$f'(x) = \frac{1}{2+x}, \; f''(x) = -\frac{1}{(2+x)^2}, \ldots, f^{(n+1)}(x) = \frac{(-1)^n n!}{(2+x)^{n+1}} .$$

Therefore,

$$f(x) - p_n(x) = (-1)^n \frac{n!}{(2+\xi)^{n+1}(n+1)!} \prod_{k=0}^{n} (x - x_k)$$

$$= \frac{(-1)^n}{(2+\xi)^{n+1}(n+1)} \cdot \frac{1}{2^n} T_{n+1}(x).$$

There follows, since $\xi > -1$ and $|T_{n+1}(x)| \leq 1$,

$$\| f - p_n \|_\infty \;\leq\; \frac{1}{(n+1)2^n} .$$

(b) ¿From calculus (or the theory of Hermite interpolation),

$$f(x) - t_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}, \quad |\xi| \in (0, |x|).$$

Hence, for $|x| \leq 1$,

$$\| f - t_n \|_\infty \;\leq\; \frac{n! |x|^{n+1}}{(2+\xi)^{n+1}(n+1)!} \leq \frac{1}{n+1} .$$

The bound is larger by a factor of 2^n compared to the bound in (a). However, by using the full Taylor expansion of f, one gets

$$\| f - t_n \|_\infty \;\leq\; \sum_{k=n+1}^{\infty} \left| \frac{f^{(k)}(0)}{k!} \right| \;=\; \sum_{k=n+1}^{\infty} \frac{(k-1)!}{k! 2^k}$$

$$= \sum_{k=n+1}^{\infty} \frac{1}{k \cdot 2^k} \;\leq\; \frac{1}{(n+1)2^n},$$

which is the same bound as in (a). The comparison, nevertheless, is somewhat unfair, since more powerful series expansion techniques are used than the simple technique in part (a).

2. Put $x = \cos\theta$. Then

$$T_n(T_m(\cos\theta)) = T_n(\cos m\theta) = \cos(n \cdot m\theta) = T_{nm}(\cos\theta).$$

Therefore,

$$T_n(T_m(x)) = T_{nm}(x).$$