AN ALGEBRAIC APPROACH TO THE BANACH-STONE THEOREM FOR SEPARATING LINEAR BIJECTIONS

HWA-LONG GAU, JYH-SHYANG JEANG AND NGAI-CHING WONG

ABSTRACT. Let X be a compact Hausdorff space and $C(X)$ the space of continuous functions defined on X. There are three versions of the Banach-Stone theorem. They assert that the Banach space geometry, the ring structure, and the lattice structure of $C(X)$ determine the topological structure of X, respectively. In particular, the lattice version states that every disjointness preserving linear bijection T from $C(X)$ onto $C(Y)$ is a weighted composition operator $Tf = h \cdot f \circ \varphi$ which provides a homeomorphism φ from Y onto X. In this note, we manage to use basically algebraic arguments to give this lattice version a short new proof. In this way, all three versions of the Banach-Stone theorem are unified in an algebraic framework such that different isomorphisms preserve different ideal structures of $C(X)$.

Let X be a compact Hausdorff space and $C(X)$ the vector space of continuous (real or complex) functions on X. It is a common interest to see how the topological structure of X can be recovered from $C(X)$. If we look at $C(X)$ as a Banach space then the classical Banach-Stone theorem states that whenever there is a surjective linear isometry T between $C(X)$ and $C(Y)$ for some other compact Hausdorff space Y, T induces a homeomorphism between X and Y (see e.g. [3, p.172]). Here is a sketch of the proof. The dual map T^* of T preserves extreme points of the dual balls, which are exactly those linear functionals in the form of $\lambda\delta_x$ for some unimodular scalar λ and point mass δ_x at some point $x \in X$. Thus $T^*\delta_y = h(y)\delta_{\varphi(y)}$ defines a scalar-valued function h on Y and a map $\varphi : Y \to X$. In other words,

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall y \in Y, \forall f \in C(X). \tag{1}$$

It is then a routine work to verify that h is continuous and φ is a homeomorphism. Operators in the form of (1) are called *weighted composition operators*.

We are interested in the algebraic character of the Banach-Stone Theorem. The above argument merely shows that a surjective isometry T between the *rings* $C(X)$ and $C(Y)$ of continuous functions preserves maximal ideals. In fact, all maximal ideals of $C(X)$ are in the form of $M_x = \{f \in C(X) : f(x) = 0\}$. Thus, $TM_x = M_y$ where $x = \varphi(y)$. This is, of course, a well-known idea. In another situation, when T is a ring isomorphism from $C(X)$ onto $C(Y)$, T also induces a homeomorphism φ

Date: October 21, 2000, fix 1; November 27, 2000, fix 2 after submitted to Taiwanese J. Math.
1991 *Mathematics Subject Classification.* 46J10,47B30,47B53.
Key words and phrases. Banach-Stone theorem, separating maps, disjointness preserving maps.
Partially supported by Taiwan National Science Council under grant: NSC 90-2115-M-110-002.

from Y onto X (see e.g. [5, p.57]). In this case, T preserves *all* ideals of the rings and $Tf = f \circ \varphi, \forall f \in C(X)$.

A (not necessarily continuous) linear bijection $T : C(X) \longrightarrow C(Y)$ is said to be *separating*, or *disjointness preserving*, if $TfTg = 0$ whenever $fg = 0$. If T is onto, then the inverse of T also preserves disjointness (see e.g. [1, Theorem 1] and also [2]). In this case, T induces a homeomorphism between X and Y (see e.g. [6, 4, 7]). Readers are referred to [2] for more information of disjointness preserving operators.

For each x in X, let

$$I_x = \{f \in C(X) : f \text{ vanishes in a neighborhood of } x\}.$$

Note that the ideal I_x is neither closed, prime nor maximal. But it is contained in a unique maximal ideal M_x. Moreover, it is somehow 'prime' in the sense that $f \in I_x$ whenever $fg = 0$ *and* $g(x) \neq 0$. In fact, $|g(y)| > 0$ for all y in a neighborhood V of x and thus forces f vanishes in V. On the other hand, if I is any proper prime ideal of $C(X)$ then I must contains a unique I_x. In fact, x is the unique common point in the kernels of all functions in I. Let $\mathfrak{P}_x$ be the family of all prime ideals which contains I_x. Then, M_x is the union and I_x is the intersection of all prime ideals in $\mathfrak{P}_x$. Note also that $\bigcup_{x \in X} \mathfrak{P}_x$ consists of all proper prime ideals of $C(X)$.

We do not give new results in this note. Instead, we demonstrate with *new proofs* that the above three Banach-Stone Theorems can be unified in an algebraic setting. In fact, T inherits algebraic properties from $C(X)$ to $C(Y)$ of different strength in different cases. When T is a ring isomorphism, it preserves all ideals. When T is an isometry, it preserves maximal ideals; namely, $TM_x = M_y$. When T is separating, we will see that it preserves all those ideals I_x; namely, $TI_x = I_y$. As consequences of these ideal preserving properties, T can be written as a weighted composition operator $Tf = h \cdot f \circ \varphi$ in all three cases. Here, $\varphi : Y \longrightarrow X$ is always a homeomorphism, but the property of the continuous weight function h differs. It is the constant function $h(y) \equiv 1$ if T is a ring isomorphism. It is unimodular, i.e., $|h(y)| \equiv 1$, if T is an isometry. And h is just non-vanishing when T is separating. In this sense, these three Banach-Stone type theorems are unified.

We would like to thank K. I. Beidar for helpful discussions. Some of the new algebraic idea utilized in the following proofs originate in a series of algebra seminars co-organized by him.

Lemma 1. *Let $T : C(X) \longrightarrow C(Y)$ be a separating linear bijection. Then for each x in X there is a unique y in Y such that*

$$TI_x = I_y.$$

Moreover, this defines a bijection φ from Y onto X by $\varphi(y) = x$.

Proof. For each x in X, denote by $\ker T(I_x)$ the set $\bigcap_{f \in I_x} (Tf)^{-1}(0)$. We first claim that $\ker T(I_x)$ is non-empty. Suppose on contrary that for each y in Y, there were an f_y in I_x with $Tf_y(y) \neq 0$. Thus, an open neighborhood U_y of y exists such that Tf_y is nonvanishing in U_y. Since $Y = \cup_{y \in Y} U_y$ and Y is compact, $Y = U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_n}$

for some $y_1, y_2, \cdots, y_n$ in Y. Let V be an open neighborhood of x such that $f_{y_i}|_V = 0$ for all $i = 1, 2, \cdots, n$. Let $g \in C(X)$ such that $g(x) = 1$ and g vanishes outside V. Then $f_{y_i} g = 0$, and thus $Tf_{y_i} Tg = 0$ since T preserves disjointness. This forces $Tg|_{U_i} = 0$ for all $i = 1, 2, \cdots, n$. Therefore, $Tg = 0$ and hence $g = 0$ by the injectivity of T, a contradiction! We thus prove that $\ker T(I_x) \neq \emptyset$.

Let $y \in \ker T(I_x)$. For each $f \in I_x$, we want to show that $Tf \in I_y$. If there exists a g in $C(X)$ such that $Tg(y) \neq 0$ and $fg = 0$, then we are done by the disjointness preserving property of T. Suppose there were no such g; that is, for any g in $C(X)$ vanishing outside $V = f^{-1}(0)$, we have $Tg(y) = 0$. Let $W \subset V$ be a compact neighborhood of x and $k \in C(X)$ such that $k|_W = 1$ and k vanishes outside V. Then for any g in $C(X)$, $g = kg + (1-k)g$. Since $(1-k)|_W = 0$, $(1-k)g \in I_x$. This implies $T((1-k)g)(y) = 0$ as $y \in \ker T(I_x)$. On the other hand, kg vanishes outside V. Hence $T(kg)(y) = 0$ by the above assumption. It follows that $Tg(y) = Tkg(y) + T(1-k)g(y) = 0$ for all g in $C(X)$. This conflicts with the surjectivity of T. Therefore, $TI_x \subseteq I_y$. Similarly, $T^{-1}(I_y) \subseteq I_{x'}$ for some x' in X since T^{-1} is also separating. It follows that $I_x \subseteq T^{-1}(I_y) \subseteq I_{x'}$. Consequently, $x = x'$ and $T(I_x) = I_y$. The bijectivity of φ is also clear now. □

Theorem 2. *Two compact Hausdorff spaces X and Y are homeomorphic whenever there is a separating linear bijection T from $C(X)$ onto $C(Y)$.*

Proof. We show that the bijection φ given in Lemma 1 is a homeomorphism. It suffices to verify the continuity of φ since Y is compact and X is Hausdorff. Suppose on contrary that there exists a net $\{y_\lambda\}$ in Y converging to y but $\varphi(y_\lambda) \to x \neq \varphi(y)$. Let U_x and $U_{\varphi(y)}$ be disjoint open neighborhoods of x and $\varphi(y)$, respectively. Now for any f in $C(X)$ vanishing outside $U_{\varphi(y)}$, we shall show that $Tf(y) = 0$. In fact, $\varphi(y_\lambda)$ belongs to U_x for large λ. Since $f|_{U_x} = 0$ and U_x is also a neighborhood of $\varphi(y_\lambda)$, we have $f \in I_{\varphi(y_\lambda)}$. By Lemma 1, $Tf \in I_{y_\lambda}$ and in particular $Tf(y_\lambda) = 0$ for large λ. This implies $Tf(y) = 0$ by the continuity of Tf. Let $k \in C(X)$ such that $k|_V = 1$ and k vanishes outside $U_{\varphi(y)}$, where $V \subset U_{\varphi(y)}$ is a compact neighborhood of $\varphi(y)$. Then $g = kg + (1-k)g$ for every g in $C(X)$. Since kg vanishes outside $U_{\varphi(y)}$, we have $T(kg)(y) = 0$. On the other hand, we have $(1-k)g \in I_{\varphi(y)}$ since $(1-k)|_V = 0$. By Lemma 1, $T((1-k)g) \in I_y$ and thus $T((1-k)g)(y) = 0$. It follows that $Tg(y) = T(kg)(y) + T((1-k)g)(y) = 0$. This is a contradiction since T is onto. Hence φ is a homeomorphism. □

Theorem 3. *Let X and Y be compact Hausdorff spaces. Then every separating linear bijection $T : C(X) \longrightarrow C(Y)$ is a weighted composition operator*

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall f \in C(X), \forall y \in Y.$$

Here φ is a homeomorphism from Y onto X and h is a nonvanishing continuous scalar function on Y. In particular, T is automatically continuous.

Proof. By Theorem 2, we have a homeomorphism φ from Y onto X such that $T(I_x) = I_y$ where $\varphi(y) = x$. We claim that $TM_x \subseteq M_y$. If this is true then $\ker \delta_x \subseteq \ker \delta_y \circ T$. Consequently, there is a scalar $h(y)$ such that $\delta_y \circ T = h(y)\delta_x$. Equivalently, $Tf(y) =$

$h(y)f(\varphi(y))$ for all f in $C(X)$ and y in Y. Since $h = T1$ and T is onto, h is continuous and non-vanishing.

To verify the claim, suppose on contrary $f \in M_x$ but $Tf(y) \neq 0$. If x belongs to the interior of $f^{-1}(0)$, then $f \in I_x$ and thus $Tf(y) = 0$. Therefore, we may assume there is a net $\{x_\lambda\}$ in X converging to x and $f(x_\lambda)$ is never zero. Let y_λ in Y such that $\varphi(y_\lambda) = x_\lambda$. Clearly, y_λ converges to y and we may assume there is a constant ϵ such that $|Tf(y_\lambda)| \geq \epsilon > 0$ for all λ. For $n = 1, 2, \ldots$, set

$$V_n = \{z \in X : \frac{1}{2n+1} \leq |f(z)| \leq \frac{1}{2n}\}$$

and

$$W_n = \{z \in X : \frac{1}{2n} \leq |f(z)| \leq \frac{1}{2n-1}\}.$$

Then at least one of the unions $V = \bigcup_{n=1}^{\infty} V_n$ and $W = \bigcup_{n=1}^{\infty} W_n$ contains a subnet of $\{x_\lambda\}$. Without loss of generality, we assume that all x_λ belong to V. Let V'_n be an open set containing V_n such that $V'_n \cap V'_m = \emptyset$ if $n \neq m$. Let g_n in $C(X)$ be of norm at most $1/2n$ such that g_n agrees with f on V_n and vanishes outside V'_n for each n. Then $g_n g_m = 0$ for all $m \neq n$. Let $g = \sum_{n=1}^{\infty} 2n g_n \in C(X)$. Note that g agrees with $2nf$ on each V_n. Moreover, each x_λ belongs to a unique V_n and $n \to \infty$ as $\lambda \to \infty$. Therefore, $g - 2nf \in I_{x_\lambda}$. This implies $T(g - 2nf) \in I_{y_\lambda}$ and thus $Tg(y_\lambda) = 2nTf(y_\lambda) \to \infty$ as $\lambda \to \infty$. But the limit should be $Tg(y)$, a contradiction. This completes the proof. $\square$

References

[1] Y. A. Abramovich and A. K. Kitover, A solution to a problem on invertible disjointness preserving operators, *Proc. Amer. Math. Soc.* 126(1998), 1501-1505.

[2] Y. A. Abramovich and A. K. Kitover, Inverses of disjointness preserving operators, *Mem. Amer. Math. Soc.* **143** (2000), no. 679, 1–162.

[3] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer-Verlag, New York, 1990.

[4] J. J. Font and S. Herández, On separating maps between locally compact spaces, *Arch. Math. (Basel)*, **63** (1994), 158–165.

[5] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1976.

[6] K. Jarosz, Automatic continuity of separating linear isomorphisms, *Canad. Math. Bull.*, **33** (1990), 139–144.

[7] Jyh-Shyang Jeang and Ngai-Ching Wong, Weighted composition operators of $C_0(X)$'s, *J. Math. Anal. Appl.*, **201** (1996), 981–993.

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan, 804, Republic of China

E-mail address: gauhl, jeangjs, wong@math.nsysu.edu.tw