Viscosity Approximation Methods for Equilibrium Problems and Fixed Point Problems of Nonlinear Semigroups

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Abstract. The purpose of this paper is to suggest and analyze a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a commutative family of nonexpansive mappings in a Hilbert space. Then we prove a strong convergence theorem which is connected with the results of Takahashi and Takahashi [13] and Yao and Noor [147]. Using this result, we obtain two corollaries which improve and extend their results.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $\Phi : C \times C \to \mathcal{R}$ be a bifunction, where $\mathcal{R}$ is the set of real numbers. The equilibrium problem for $\Phi : C \times C \to \mathcal{R}$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\Phi)$. Given a mapping $T : C \to H$, let $\Phi(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(\Phi)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Some methods have been proposed to solve the equilibrium problem; see, e.g., [2, 3].

A mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote by $F(S)$ the set of fixed points of S. If $C \subset H$ is bounded, closed and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; for instance, see [12]. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [4] proved the following strong convergence theorem.

Theorem 1.1. See Moudafi [4]. Let C be a nonempty closed convex subset of a Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S) \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and

$$x_{n+1} = \frac{1}{1+\varepsilon_n} S x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \quad \forall n \geq 1,$$

where $\{\varepsilon_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \ \text{ and } \ \lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0.$$

Then, $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Such a method for approximation of fixed points is called the viscosity approximation method. This approach is mainly due to Moudafi [4]; see also Xu [8]. Very recently, modified by Combettes and Hirstoaga [2], Moudafi [4], and Tada and Takahashi [7], Takahashi and Takahashi [13] introduced and studied an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Moreover, utilizing Opial's property of Hilbert space they proved a strong convergence theorem which is connected with the results of Combettes and Hirstoaga result [2] and Wittmann [11].

Theorem 1.2. See Takahashi and Takahashi [13, Theorem 3.2]. Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ satisfy (A1)-(A2):

(A1) $\Phi(x,x) = 0$ for all $x \in C$;

(A2) Φ is monotone, i.e., $\Phi(x,y) + \Phi(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0^+} \Phi(tz + (1-t)x, y) \leq \Phi(x,y);$$

(A4) for each $x \in C$, $y \mapsto \Phi(x,y)$ is convex and lower semicontinuous.

Let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(\Phi) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S u_n, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0 \ \text{ and } \ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(\Phi)$, where $z = P_{F(S)\cap EP(\Phi)} f(z)$.

Further, Ceng and Yao [10] investigated the problem of finding a common element of the set of solutions of a mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings in a Hilbert space. The authors' result is the improvements and extension of Takahashi and Takahashi Theorem 3.2 [13].

For recent years, viscosity approximation methods have been developed for finding a common fixed point of the family of nonlinear operators. Let G be an unbounded subset of $\mathcal{R}^+$ such that $s+t \in G$ whenever $s, t \in G$ (often $G = N$, the set of nonnegative integers of $\mathcal{R}^+$). Let X be a smooth Banach space, C a nonempty closed convex subset of X, and $\Gamma = \{T_s : s \in G\}$ a commutative family of nonexpansive self-mappings of C. Denote by $F(\Gamma)$ the set of common fixed points of Γ, i.e., $F(\Gamma) = \{x \in C : T_s x = x, \ \forall s \in G\}$. Throughout this paper we always assume that $F(\Gamma)$ is nonempty. Very recently, Yao and Noor [14] considered and analyzed the following viscosity iterative scheme for a commutative family of nonexpansive mappings:

Algorithm 1.1. See Yao and Noor [14, Algorithm 1]. Let $x_0 \in C$, $f : C \to C$ be a contraction on C, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{l_n\}$ be a sequence in G. Define a sequence $\{x_n\}$ recursively by the following explicit iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} x_n, \quad \forall n \geq 0. \tag{1.2}$$

In [14], Yao and Noor established the strong convergence of the sequence $\{x_n\}$ generated by (1.2) under some suitable conditions.

Theorem 1.3. See Yao and Noor [14, Theorem 1]. Let C be a nonempty closed convex subset of a reflexive Banach space X with a weakly sequentially continuous duality mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{l_n\}$ be a sequence in G. Let $\{\alpha_n\}$ satisfy the control conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, and (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} r_n = \infty$;

(iv) Γ is a semigroup (i.e., $T_r T_s = T_{r+s}$ for all $r, s \in G$) and satisfies the uniformly asymptotic regularity condition

$$\lim_{r\in G, r\to\infty} \sup_{x\in\widetilde{C}} \|T_s T_r x - T_r x\| = 0, \quad \text{uniformly in } s \in G, \tag{UARC}$$

where $\widetilde{C}$ is any bounded subset of C. If there exists $Q(f) \in F(\Gamma)$ which solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0,$$

then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $Q(f) \in F(\Gamma)$.

In this paper, inspired by Combettes and Hirstoaga [2], Wittmann [11], Moudafi [4], Tada and Takahashi [7], Xu [8], Takahashi and Takahashi [13], Yao and Noor [14], and Ceng and Yao [10], we introduce and consider a new iterative scheme

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} u_n, \ \forall n \geq 0 \end{cases}$$

by the viscosity approximation method for finding a common element of the set of solutions of (1.1) and the set of common fixed points of a commutative family of nonexpansive mappings in a Hilbert space. Then we prove a strong convergence theorem which is connected with the results of Takahashi and Takahashi [13] and Yao and Noor [14]. Using this result, we obtain two corollaries which improve and extend their results.

Throughout the rest of this paper, we denote by "$\to$" and "$\rightharpoonup$" the strong convergence and weak convergence, respectively.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. It is well known that there holds the identity

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \lambda \in [0,1].$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \Leftrightarrow \quad \langle x - z, z - y \rangle \geq 0, \forall y \in C.$$

It is also known that H satisfies Opial's property, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$; see [5] for more details.

Before starting the main results of this paper, we include some lemmas. The following lemma appears implicitly in [1].

Lemma 2.1. See [1]. Let C be a nonempty closed convex subset of H and let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in [2].

Lemma 2.2. See [2]. Assume that $\Phi : C \times C \to \mathcal{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:

$$S_r(x) = \{z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) S_r is single-valued;

(2) S_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle;$$

(3) $F(S_r) = EP(\Phi)$;

(4) $EP(\Phi)$ is closed and convex.

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. In a real Hilbert space H, there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.4. See [9]. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5. Demiclosedness Principle. See [12]. Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.

Lemma 2.6. See [8]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Strong Convergence Theorem

In this section, we deal with an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of a commutative family of nonexpansive mappings in a Hilbert space.

Theorem 3.1. Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4) and let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{l_n\}$ be a sequence in G. Let $\{\alpha_n\}$ satisfy the control conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, and (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that
(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lim_{n\to\infty} l_n = \infty$;
(iv) Γ is a semigroup (i.e., $T_r T_s = T_{r+s}$ for $r, s \in G$) with $F(\Gamma) \cap EP(\Phi) \neq \emptyset$ and satisfies the uniformly asymptotic regularity condition

$$\lim_{r\in G, r\to\infty} \sup_{x\in \widetilde{C}} \|T_s T_r x - T_r x\| = 0, \quad \text{uniformly in } s \in G, \qquad \text{(UARC)}$$

where $\widetilde{C}$ is any bounded subset of C.

Let $f : C \to C$ be a contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} u_n, \ \forall n \geq 0, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies

$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(\Gamma) \cap EP(\Phi)$, where $z = P_{F(\Gamma)\cap EP(\Phi)} f(z)$.

Proof. Let $Q = P_{F(\Gamma)\cap EP(\Phi)}$. Then Qf is a contraction of C into itself. In fact, there exists $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$. So, we have that

$$\|Qf(x) - Qf(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|$$

for all $x, y \in C$. So, Qf is a contraction of C into itself. Since C is complete, there exists a unique element $z \in C$ such that $z = Qf(z)$.

For the remainder of the proof, we proceed with the following steps.

Step 1. $\{x_n\}$ and $\{u_n\}$ are bounded. Indeed, let $p \in F(\Gamma) \cap EP(\Phi)$. Then from $u_n = S_{r_n} x_n$, we have

$$\|u_n - p\| = \|S_{r_n} x_n - S_{r_n} p\| \le \|x_n - p\|, \quad \forall n \ge 0.$$

Put $M = \max\{\|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\}$. It is obvious that $\|x_0 - p\| \le M$. Suppose $\|x_n - p\| \le M$. Then, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} u_n - p\| \\
&\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|T_{l_n} u_n - p\| \\
&\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\
&\le \alpha\alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= [1 - (1 - \alpha)\alpha_n]\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= [1 - (1 - \alpha)\alpha_n]\|x_n - p\| + (1 - \alpha)\alpha_n \cdot \tfrac{1}{1-\alpha}\|f(p) - p\| \\
&\le [1 - (1 - \alpha)\alpha_n]M + (1 - \alpha)\alpha_n M = M.
\end{aligned}$$

So, by induction we have that $\|x_n - p\| \le M$ for all $n \ge 0$ and hence $\{x_n\}$ is bounded. We also know that $\{u_n\}$ and $\{f(x_n)\}$ are bounded. Since for each $s \in G$ we have

$$\|T_s x_n - p\| = \|T_s x_n - T_s p\| \le \|x_n - p\|,$$

it is known that the set $\{T_s x_n : s \in G \text{ and } n \ge 0\}$ is bounded, and so is $\{T_{l_n} x_n\}$.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Indeed, define a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n. \tag{3.1}$$

Then, observe that

$$\begin{aligned} y_{n+1} - y_n &= \tfrac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T_{l_{n+1}} u_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n f(x_n) + \gamma_n T_{l_n} u_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) - \tfrac{\alpha_n}{1-\beta_n} f(x_n) \\ &\quad + \tfrac{\gamma_{n+1}}{1-\beta_{n+1}} (T_{l_{n+1}} u_{n+1} - T_{l_{n+1}} u_n) + T_{l_{n+1}} u_n \\ &\quad - T_{l_n} u_n + \tfrac{\alpha_n}{1-\beta_n} T_{l_n} u_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} T_{l_{n+1}} u_n. \end{aligned}$$

It follows that

$$\begin{aligned} &\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} (\|f(x_{n+1})\| + \|T_{l_{n+1}} u_n\|) + \tfrac{\alpha_n}{1-\beta_n} (\|f(x_n)\| + \|T_{l_n} u_n\|) \\ &\quad + \tfrac{\gamma_{n+1}}{1-\beta_{n+1}} \|T_{l_{n+1}} u_{n+1} - T_{l_{n+1}} u_n\| + \|T_{l_{n+1}} u_n - T_{l_n} u_n\| - \|x_{n+1} - x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} (\|f(x_{n+1})\| + \|T_{l_{n+1}} u_n\|) + \tfrac{\alpha_n}{1-\beta_n} (\|f(x_n)\| + \|T_{l_n} u_n\|) \\ &\quad + \|u_{n+1} - u_n\| + \|T_{l_{n+1}} u_n - T_{l_n} u_n\| - \|x_{n+1} - x_n\|. \end{aligned} \tag{3.2}$$

On the other hand, from $u_n = S_{r_n} x_n$ and $u_{n+1} = S_{r_{n+1}} x_{n+1}$, we have

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C \tag{3.3}$$

and

$$f(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \quad \forall y \in C. \tag{3.4}$$

Putting $y = u_{n+1}$ in (3.3) and $y = u_n$ in (3.4), we have

$$f(u_n, u_{n+1} + \frac{1}{r_n}) \langle u_{n+1} - u_n, u_n - x_n \rangle \ge 0$$

and

$$f(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0.$$

So, from (A2) we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \ge 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \rangle \ge 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0, \ \forall n \ge 0$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\le \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \tfrac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \\ &\le \|u_{n+1} - u_n\| \{ \|x_{n+1} - x_n\| + |1 - \tfrac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \} \end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \tfrac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \tfrac{1}{b}|r_{n+1}-r_n|L,\end{aligned} \tag{3.5}$$

where $L = \sup\{\|u_n - x_n\| : n \ge 0\}$. So, from (3.2) we have

$$\begin{aligned}&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|T_{l_{n+1}}u_n\|) + \tfrac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|T_{l_n}u_n\|) \\ &\quad + \tfrac{1}{b}|r_{n+1}-r_n|L + \|T_{l_{n+1}}u_n - T_{l_n}u_n\|.\end{aligned} \tag{3.6}$$

If $l_{n+1} > l_n$, since Γ is a semigroup, we have by (UARC)

$$\|T_{l_{n+1}}u_n - T_{l_n}u_n\| = \|T_{l_{n+1}-l_n}T_{l_n}u_n - T_{l_n}u_n\| \to 0.$$

Interchanging l_{n+1} and l_n if $l_{n+1} < l_n$. Similarly we can obtain $\|T_{l_{n+1}}u_n - T_{l_n}u_n\| \to 0$.

Thus it follows from (3.6) that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence, by Lemma 2.3, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Consequently, it follows from (3.1) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n - x_n\| = 0. \tag{3.7}$$

From (3.5) and $|r_{n+1}-r_n| \to 0$, we have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0.$$

Step 3. $\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|T_{l_n}u_n - u_n\| = 0$. Indeed, since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n}u_n$, we have

$$\begin{aligned}\|x_{n+1} - T_{l_{n+1}}u_{n+1}\| &\le \|x_{n+1} - T_{l_n}u_n\| + \|T_{l_n}u_n - T_{l_{n+1}}u_{n+1}\| \\ &\le \alpha_n\|f(x_n) - T_{l_n}u_n\| + \beta_n\|x_n - T_{l_n}u_n\| \\ &\quad + \|T_{l_n}u_n - T_{l_{n+1}}u_n\| + \|T_{l_{n+1}}u_n - T_{l_{n+1}}u_{n+1}\| \\ &\le \alpha_n\|f(x_n) - T_{l_n}u_n\| + \beta_n\|x_n - T_{l_n}u_n\| \\ &\quad + \|T_{l_n}u_n - T_{l_{n+1}}u_n\| + \|u_n - u_{n+1}\|.\end{aligned} \tag{3.8}$$

As in Step 2, we can obtain that $\|T_{l_n}u_n - T_{l_{n+1}}u_n\| \to 0$. Thus it follows from (3.8) and condition (C1) that

$$(1 - \limsup_{n\to\infty\beta_n})\limsup_{n\to\infty}\|x_n - T_{l_n}u_n\| \le 0,$$

and so $\lim_{n\to\infty}\|x_n - T_{l_n}u_n\| = 0$. For $p \in F(\Gamma)\cap EP(\Phi)$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|S_{r_n}x_n - S_{r_n}p\|^2 \\ &\le \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \tfrac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2)\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.9}$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} u_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|T_{l_n} u_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|u_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|x_n - p\|^2 - \|x_n - u_n\|^2) \\
&= \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \gamma_n \|x_n - u_n\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n \|x_n - u_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\gamma_n \|x_n - u_n\|^2 &\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

So, we have $\|x_n - u_n\| \to 0$. From

$$\|T_{l_n} u_n - u_n\| \le \|T_{l_n} u_n - x_n\| + \|x_n - u_n\|,$$

we also have $\|T_{l_n} u_n - u_n\| \to 0$.

Step 4. Fir each $s \in G$, $\lim_{n\to\infty} \|T_s u_n - u_n\| = 0$. Indeed, let $\widetilde{C}$ be any bounded subset of C which contains the sequence $\{u_n\}$. It follows that

$$\begin{aligned}
\|T_s u_n - u_n\| &\le \|T_s u_n - T_s T_{l_n} u_n\| + \|T_s T_{l_n} u_n - T_{l_n} u_n\| + \|T_{l_n} u_n - u_n\| \\
&\le 2\|T_{l_n} u_n - u_n\| + \sup_{x \in \widetilde{C}} \|T_s T_{l_n} x - T_{l_n} x\|.
\end{aligned}$$

Since $\|T_{l_n} u_n - u_n\| \to 0$, from (UARC) we derive

$$\lim_{n\to\infty} \|T_s u_n - u_n\| = 0.$$

Step 5. $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0$, where $z = P_{F(\Gamma)\cap EP(\Phi)} f(z)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty} \langle f(z) - z, x_{n_i} - z\rangle = \limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|T_{l_n} u_n - u_n\| \to 0$, we obtain $T_{l_{n_i}} u_{n_i} \rightharpoonup w$. Let us show $w \in EP(\Phi)$. By $u_n = S_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}).$$

Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A4) we have

$$0 \geq f(y, w), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, w) \\ &\leq tf(y_t, y) \end{aligned}$$

and hence $0 \leq f(y_t, y)$. From (A3), we have

$$0 \leq f(w, y), \quad \forall y \in C$$

and hence $w \in EP(\Phi)$. We shall show $w \in F(\Gamma)$. Assume $w \notin F(\Gamma)$. Since $u_{n_i} \rightharpoonup w$ and $\lim_{n\to\infty} \|T_s u_n - u_n\| = 0$ for each $s \in G$, we deduce from Lemma 2.5 that $w \in F(\Gamma) = \bigcap_{s\in G} F(T_s)$. Since $z = P_{F(\Gamma)\cap EP(\Phi)}f(z)$, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \\ &= \langle f(z) - z, w - z\rangle \leq 0. \end{aligned}$$

Step 6. $\lim_{n\to\infty} \|x_n - z\| = \lim_{n\to\infty} \|u_n - z\| = 0$ where $z = P_{F(\Gamma)\cap EP(\Phi)}f(z)$. Indeed, since $x_{n+1} - z = \alpha_n(f(x_n) - z) + \beta_n(x_n - z) + \gamma_n(T_{l_n}u_n - z)$, by Lemma 2.3 we derive from (3.9)

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|\beta_n(x_n - z) + \gamma_n(T_{l_n}u_n - z)\|^2 + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\ &\leq (\beta_n\|x_n - z\| + \gamma_n\|x_n - z\|)^2 + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n\alpha\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \tfrac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z\|^2 + \tfrac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, x_{n+1} - z\rangle \\ &= \tfrac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z\|^2 + \tfrac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n - z\|^2 \\ &\quad + \tfrac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1 - \tfrac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha})\|x_n - z\|^2 \\ &\quad + \tfrac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\{\tfrac{\alpha_n M}{2(1-\alpha)} + \tfrac{1}{1-\alpha}\langle f(z) - z, x_{n+1} - z\rangle\}, \end{aligned}$$

where $M = \sup\{\|x_n - z\|^2 : n \geq 0\}$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$

$$\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha} \in (0,1) \quad \Leftrightarrow \quad \alpha_n(2-\alpha) \in (0,1).$$

It is clear that $\lim_{n\to\infty} \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha} = 0$. Note that condition (C2) implies that $\sum_{n=n_0}^{\infty} \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha} = \infty$. Moreover, it is obvious that

$$\limsup_{n\to\infty}\{\frac{\alpha_n M}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(z)-z, x_{n+1}-z\rangle\} \leq 0.$$

Therefore, according to Lemma 2.6, we conclude that $\lim_{n\to\infty} \|x_n - z\| = 0$, i.e., $\{x_n\}$ converges strongly to $z \in F(\Gamma) \cap EP(\Phi)$, where $z = P_{F(\Gamma)\cap EP(\Phi)} f(z)$. Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$, it follows that $\lim_{n\to\infty} \|u_n - z\| = 0$. This completes the proof of Theorem 3.1. □

Remark 3.1. Our Theorem 3.1 extends Takahashi and Takahashi Theorem 3.2 [13] to the case of nonexpansive semigroups with uniformly asymptotic regularity and to the one of the modified iterative scheme

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} u_n, \ \forall n \geq 0. \end{cases}$$

Moreover, our Theorem 3.1 removes the restrictions $\sum_n |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_n |r_{n+1} - r_n| < \infty$ in their Theorem 3.2 [13]. On the other hand, Yao and Noor's algorithm in [14, Theorem 1] is extended to develop the new one in our Theorem 3.1 for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a nonexpansive semigroup with uniformly asymptotic regularity. There is no doubt that such an extension is very interesting and quite significant.

As direct consequences of Theorem 3.1, we obtain two corollaries.

Corollary 3.1. Let C be a nonempty closed convex subset of H. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{l_n\}$ be a sequence in G. Let $\{\alpha_n\}$ satisfy the control conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, and (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} l_n = \infty$;

(iv) Γ is a semigroup (i.e., $T_r T_s = T_{r+s}$ for $r, s \in G$) with $F(\Gamma) \neq \emptyset$ and satisfies the uniformly asymptotic regularity condition

$$\lim_{r\in G, r\to\infty} \sup_{x\in\widetilde{C}} \|T_s T_r x - T_r x\| = 0, \quad \text{uniformly in } s \in G, \tag{UARC}$$

where $\widetilde{C}$ is any bounded subset of C. Let $f : C \to C$ be a contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} P_C x_n, \quad \forall n \geq 0.$$

Then, $\{x_n\}$ converges strongly to $z \in F(\Gamma)$, where $z = P_{F(\Gamma)}f(z)$.

Proof. Put $\Phi(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 0$ in Theorem 3.1. Then, we have $u_n = P_C x_n$. So, according to Theorem 3.1, the sequence $\{x_n\}$ generated by $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{l_n} P_C x_n, \quad \forall n \geq 0,$$

converges strongly to $z \in F(\Gamma)$, where $z = P_{F(\Gamma)}f(z)$. □

Corollary 3.2. Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4) such that $EP(\Phi) \neq \emptyset$ and let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0, 1)$. Let $\{\alpha_n\}$ satisfy the control conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, and (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Assume that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $f : C \to C$ be a contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n, \ \forall n \geq 0, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies

$$\liminf_{n\to\infty} r_n > 0 \ \text{ and } \ \lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in EP(\Phi)$, where $z = P_{EP(\Phi)}f(z)$.

Proof. Put $T_s x = x, \ \forall x \in C, s \in G$ in Theorem 3.1. Then, in terms of Theorem 3.1, the sequences $\{x_n\}$ and $\{u_n\}$ generated in Corollary 3.2 converge strongly to $z \in EP(\Phi)$, where $z = P_{EP(\Phi)}f(z)$. □

Remark 3.2. Takahashi and Takahashi derived Wittmann's theorem [11] in the case when $f(y) = x_1 \in C$ for all $y \in H$ and S is a nonexpansive mapping of C into itself in their Corollary 3.3 [13]. Our Corollary 3.1 extends their Corollary 3.3 [13] to the case of nonexpansive semigroups with uniformly asymptotic regularity. Takahashi and Takahashi also derived Combettes and Hirstoaga theorem [2] in the case when $f(y) = x_1 \in H$ for all $y \in H$ in their Corollary 3.4 [13]. Our Corollary 3.2 extends their Corollary 3.4 [13] to the case of the modified iterative scheme

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n u_n, \ \forall n \geq 0. \end{cases}$$

Furthermore, our Corollary 3.1 removes the restriction $\sum_n |\alpha_{n+1} - \alpha_n| < \infty$ in their Corollary 3.3 [13], and Corollary 3.2 removes the restrictions $\sum_n |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_n |r_{n+1} - r_n| < \infty$ in their Corollary 3.4 [13].

References

[1] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student, 63 (1994), 123-145.

[2] P.L. Combettes and S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal., 6 (2005), 117-136.

[3] S.D. Flam and A.S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Program., 78 (1997), 29-41.

[4] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl., 241 (2000), 46-55.

[5] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc., 73 (1967), 561-597.

[6] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc., 125 (1997), 3641-3645.

[7] A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, 2006, in press.

[8] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., 298 (2004), 279-291.

[9] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl., 305 (2005), 227-239.

[10] L.C. Ceng and J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math., (2007), doi:10.1016/j.cam.2007.02.022.

[11] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math., 58 (1992), 486-491.

[12] K. Geobel and W.A. Kirk, Topics on Metric Fixed-Point Theory, Cambridge University Press, Cambridge, England, 1990.

[13] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl., (2006), doi:10.1016/j.jmaa.2006.08.036.

[14] Y. Yao and M.A. Noor, On viscosity iterative methods for variational inequalities, J. Math. Anal. Appl., 325 (2007), 776-787.