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Degree theory for generalized variational inequalities and applications ☆

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Abstract

In this paper, a degree theory for finite dimensional generalized variational inequalities is built and employed to prove some results on solution existence and solution stability.

Keywords: Degree theory; Variational inequality; Generalized variational inequality; Solution existence; Solution stability

1. Introduction

Many problems in analysis and in the application of analysis can be reduced to a study of the solution set of an equation $\phi(x) = p$ in an appropriate space. Degree theory has developed as means of examining the solution existence and their number of the solution.

Suppose that D is a open bounded set in R^n with the closure $\overline{D}$ and the boundary ∂D. Let $\phi : \overline{D} \to R^n$ be an continuous map and $p \in R^n$ such that $p \notin \phi(\partial D)$. The aim of degree theory is to define an integer $d(\phi, D, p)$, the degree of ϕ at p respect to D (see [7,10,20] for the definition) with the properties that $d(\phi, D, p)$ is an estimate of the number of solution of $\phi(x) = p$ in D, d is continuous in ϕ and p and d is additive in the domain D. The following list summarizes some properties most frequently used (see, for instance [7,10,18,26]).

Theorem 1.1. *Suppose that $p \notin \phi(\partial D)$. Then the following properties hold*:

(1) (*Normalization*) *If $p \in D$ then $d(I, D, p) = 1$, where I is the identity mapping.*
(2) (*Existence*) *If $d(\phi, D, p) \neq 0$ then there is $x \in D$ such that $\phi(x) = p$.*
(3) (*Additivity*) *Suppose that D_1 and D_2 are disjoint open sets of D. If $p \notin \phi(\overline{D} \setminus (D_1 \cup D_2))$ then*
$$d(D, f, p) = d(\phi, D_1, p) + d(\phi, D_2, p).$$
(4) (*Homotopy invariance*) *Suppose that $H : [0,1] \times D \to R^n$ is continuous. If $p \notin H(t, \partial D)$ for all $t \in [0,1]$ then $d(H(t,\cdot), D, p)$ is independent of t.*
(5) (*Excision*) *If D_0 is a closed set of D and $p \notin \phi(D_0)$ then $d(\phi, D, p) = d(\phi, D \setminus D_0, p)$.*

Recently, in the two-volume book [9] dedicated entirely to finite dimensional variational inequalities (VI, for brevity), Facchinei and Pang have used degree theory to obtain existence theorems for variational inequalities (see [9, Proposition 2.2.3 and Theorem 2.3.4]). These results gave a necessary and sufficient condition for a pseudomonotone VI on a general closed convex set to have a solution. In particular, Pang [19] used degree theory to obtain interesting results on sensitivity of a parametric nonsmooth equation with multivalued perturbed solution sets. This paper has been very influential for the optimization community. Also, based on degree theory, Robinson [22] provided a strong

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conclusion on the solution stability of variational conditions; Gowda [11] proved inverse and implicit function theorems for H-differentiable functions, thereby giving a unified treatment of such theorem for C^1-functions and for locally Lipschitzian function. In order to obtain these results, the authors have used degree theory as a bridge to marry nonlinear analysis and variational inequality theory under which we can study problems via nonlinear equations.

Let us assume that R^n is a finite dimensional space with the Euclidian norm and K is a closed convex set in R^n. Let $f : K \to R^n$ be a continuous mapping. The variational inequality defined by K and f denoted by $\mathrm{VI}(f, K)$, is the problem of finding a vector $x \in K$ such that it satisfies the inclusion

$$0 \in f(x) + N_K(x), \tag{1.1}$$

where $N_K(x)$ is the normal cone of K at x defined by the formula

$$N_K(x) = \begin{cases} \{x^* \in R^n : \langle x^*, y - x\rangle \leqslant 0 \ \forall y \in K\} & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We denote by $\Pi_K(x)$ the metric projection of x onto K and put

$$\Phi(x) = x - \Pi_K[x - f(x)]. \tag{1.2}$$

Φ is called the natural map. It is clear that x is a solution of (1.1) if and only if x is a solution of the equation $\Phi(x) = 0$. Let Ω be an open bounded set in R^n such that $\Omega \cap K \neq \emptyset$. We wish to investigate the number of solutions of (1.1) in Ω. Since (1.1) is equivalent to the equation $\Phi(x) = 0$, it suggests us to compute the degree $d(\Phi, \Omega, 0)$. By this way, as it mentioned above, [9,22] obtained interesting results on solution existence and solution stability of VIs.

It is natural to try to study generalized variational inequalities (GVI, for short) which is also known in the literature as set-valued variational inequalities in this direction. Namely, we consider the problem of finding $x \in K$ such that

$$0 \in F(x) + N_K(x), \tag{1.3}$$

where $F : K \to 2^{R^n}$ is a multifunction. We consider the so-called generalized natural map which defined by

$$\Phi_F(x) = x - \Pi_K(x - F(x)).$$

In this case we will meet some difficulties for applications of degree theory to our problem. Namely, we can not apply degree theory to Φ_F directly because Φ_F has no convex values and so the degree of Φ_F is undefined generally.

The aim of the present paper is to build a degree theory for GVIs via the natural map and employ the results obtained to prove some facts on the solution existence and solution stability of GVIs in finite dimensional spaces.

It notices that there have been many papers on degree theory for multifunctions in the infinite dimensional setting so far (see [3–5,12,13]). We emphasize that degree theory for GVIs in the present paper is somewhat different from degree theory for upper semicontinuous multifunctions with convex and compact values. It is built via the map Φ_F which does not necessarily have convex values.

The rest of the paper contains two sections. In Section 2 we build a degree theory for GVIs. Section 3 is devote to applications of obtained results. In this section we shall prove some facts on the solution existence and solution stability of GVIs.

2. Degree theory for GVIs

Throughout the paper, K is a closed convex set in R^n, Ω is an open bounded set in R^n such that $\Omega \cap K \neq \emptyset$. Let $F : K \to 2^{R^n}$ be a multifunction which is upper semicontinuous with compact convex values.

Recall that a map $F : K \to 2^{R^n}$ is upper semicontinuous (u.s.c., for brevity) if for all $x \in K$ and for any open set $W \subset R^n$ satisfying $F(x) \subset W$ there exists an open neighborhood U of x such that $F(y) \subset W$ for all $y \in U \cap K$. If $F(x) \neq \emptyset$ for all $x \in K$ and for any open set $W \subset R^n$ satisfying $F(x) \cap W \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap W \neq \emptyset$ for all $y \in U \cap K$ then F is said to be lower semicontinuous (l.s.c., for brevity).

The following lemma plays an essential role for building a degree theory of GVIs.

Lemma 2.1. *Suppose that $F : K \to 2^{R^n}$ is u.s.c. with closed convex values. Then for any $\epsilon > 0$ there exists a continuous map $f_\epsilon : R^n \to R^n$ such that for every $x \in K$ it holds*

$$f_\epsilon(x) \in F((x + \epsilon B) \cap K) + \epsilon B, \tag{2.1}$$

where B is the unit ball in R^n.

Proof. By our assumptions and the approximate selection theorem due to Cellina (see [1, p. 84]), for every $\varepsilon > 0$ there exists a continuous map $g_\epsilon : K \to \mathbb{R}^n$ such that

$$g_\varepsilon(x) \in F(((x + \epsilon B) \cap K) + \epsilon B \quad \forall x \in K.$$

By Tietze-Urysohn's theorem (see [8, Theorem 5.1, p. 149]), for each $\epsilon > 0$, there exists a continuous extension $f_\epsilon : R^n \to R^n$ of g_ϵ. As f_ϵ and g_ϵ agree on K, f_ϵ satisfies the conclusion of the theorem. The proof is complete. □

We now consider $\mathrm{GVI}(F, K)$. For each $\epsilon > 0$ we define a map $\Phi_\epsilon : R^n \to R^n$ by the formula

$$\Phi_\epsilon(x) = x - \Pi_K(x - f_\epsilon(x)), \tag{2.2}$$

where f_ϵ is a approximate continuous selection of F which satisfies (2.1). By the continuity of the metric projection and Lemma 2.1, Φ_ϵ is continuous on R^n and hence on $\overline{\Omega}$.

We have the following lemma on properties of Φ_ϵ.

Lemma 2.2. *Suppose that F is u.s.c. with compact convex values and $0 \notin (F + N_K)(\partial\Omega)$. Then the following assertions hold*:

(a) *there exists $\epsilon_1 > 0$ such that $0 \notin \Phi_\epsilon(\partial\Omega)$ for all* $\epsilon \in (0, \epsilon_1]$
(b) *there exists $\epsilon_2 > 0$ such that*

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$$d(\Phi_\epsilon, \Omega, 0) = d(\Phi_{\epsilon'}, \Omega, 0) \quad \textit{for all } \epsilon, \epsilon' \in (0, \epsilon_2]. \tag{2.3}$$

Proof. (a) Suppose the assertion is false. Then there exists a sequence $\epsilon_k \to 0^+$ and a sequence $x_k \in \partial\Omega$ such that $\Phi_{\epsilon_k}(x_k) = 0$. This means that

$$x_k = \Pi_K(x_k - f_{\epsilon_k}(x_k)). \tag{2.4}$$

By compactness of $\partial\Omega$ we can assume that $x_k \to x_0 \in \partial\Omega$. Since $x_k \in K \cap \partial\Omega$, by Lemma 2.1, there exist $y_k \in K$ and $z_k \in F(x_k)$ such that

$$\|y_k - x_k\| < \epsilon_k; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k.$$

Hence, $y_k \to x_0$. As $F(x_0)$ is a compact set and F is upper semicontinuous at x_0, by taking a subsequence (if necessary) we can suppose furthermore that $z_k \to z_0 \in F(x_0)$. Hence, $f_{\epsilon_k}(x_k) \to z_0$. Letting $k \to \infty$, from (2.4) we obtain $x_0 = \Pi_K(x_0 - z_0)$ with $z_0 \in F(x_0)$. By the property of the metric projection we have

$$0 \in z_0 + N_K(x_0) \subset F(x_0) + N_K(x_0),$$

which contradicts our assumptions. We obtain the proof part (a).

(b) On the contrary, suppose there exist sequences $0 < \epsilon_k < \epsilon'_k \to 0$ such that

$$d(\Phi_{\epsilon_k}, \Omega, 0) \neq d(\Phi_{\epsilon_k'}, \Omega, 0). \tag{2.5}$$

Put

$$H(t,x) = x - \Pi_K(x - tf_{\epsilon_k}(x) - (1-t)f_{\epsilon'_k}(x)), \quad (t,x) \in [0,1] \times \overline{\Omega}.$$

We have $H(0,x) = \Phi_{\epsilon'_k}(x)$ and $H(1,x) = \Phi_{\epsilon_k}(x)$. If $0 \notin H(t, \partial\Omega)$ for all $t \in [0,1]$ then

$$d(H(0,\cdot), \Omega, 0) = d(H(1,\cdot), \Omega, 0),$$

because of (4) in Theorem 1.1. But the latter contradicts (2.5). Hence, for each k, there exists $t_k \in [0,1]$ such that $0 \in H(t_k, \partial\Omega)$. This implies that, for each k there exists $x_k \in \partial\Omega$ such that

$$x_k = \Pi_K(x_k - t_k f_{\epsilon_k}(x_k) - (1-t_k)f_{\epsilon'_k}(x_k)). \tag{2.6}$$

Since $x_k \in K \cap \partial\Omega$, by Lemma 2.1, there exist $y_k, y'_k \in K$; $z_k \in F(y_k)$ and $z'_k \in F(y'_k)$ such that

$$\|y_k - x_k\| < \epsilon'_k; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon'_k;$$

and

$$\|y'_k - x_k\| < \epsilon'_k; \quad \|z'_k - f_{\epsilon'_k}(x_k)\| < \epsilon'_k.$$

By compactness of $[0,1] \times \partial\Omega$ we can assume that $(t_k, x_k) \to (\bar{t}, \bar{x}) \in [0,1] \times \partial\Omega$. Hence, $y_k \to \bar{x}$ and $y'_k \to \bar{x}$. By standard arguments as in the proof of (a) we get $f_{\epsilon_k}(x_k) \to z_1$ and $f_{\epsilon'_k}(x_k) \to z_2$ for some $z_1, z_2 \in F(\bar{x})$. By letting $k \to \infty$, from (2.6) we get $\bar{x} = \Pi_K(\bar{x} - \bar{t}z_1 - (1-\bar{t})z_2)$. Put $\bar{z} = \bar{t}z_1 + (1-\bar{t})z_2$ then $\bar{z} \in F(\bar{x})$ and $\bar{x} = \Pi_K(\bar{x} - \bar{z})$. By the property of the metric projection we have

$$0 \in \bar{z} + N_K(\bar{x}) \subset F(\bar{x}) + N_K(\bar{x}).$$

Since $\bar{x} \in \partial\Omega$, we get a contradiction. The proof of the lemma is complete. □

From Lemma 2.2 it follows that there exists $\bar{\epsilon} > 0$ such that $0 \notin \Phi_\epsilon(\partial\Omega)$ and $d(\Phi_\epsilon, \Omega, 0) = d(\Phi_{\epsilon'}, \Omega, 0)$ for all $\epsilon, \epsilon' \in (0, \bar{\epsilon}]$. It is a basis for the following definition.

Definition 2.1. Let $F : K \to 2^{R^n}$ be an u.s.c. multifunction with compact convex values and $0 \notin (F + N_K)(\partial\Omega)$. The degree of generalized variational inequality defined by F and K respect to Ω at 0 is the common value $d(\Phi_\epsilon, \Omega, 0)$ for $\epsilon > 0$ sufficiently small and denoted by $d(F + N_K, \Omega, 0)$.

Example 2.1. Let

$$F(x) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1,1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0, \end{cases}$$

$K = [-1,1]$ and $\Omega = (-1/2, 2)$. Then $d(F + N_K, \Omega, 0) = 1$.

Indeed, for each $\epsilon > 0$ we consider the following function

$$f_\epsilon(x) = \begin{cases} 1 & \text{if } x \geqslant \epsilon, \\ x/\epsilon & \text{if } x \in (-\epsilon, \epsilon), \\ -1 & \text{if } x \leqslant -\epsilon. \end{cases}$$

If $x \in K \setminus (-\epsilon, \epsilon)$ then $(x, f_\epsilon(x)) = (x, 1) \in \operatorname{Graph} F$ and so $\operatorname{dist}((x,1), \operatorname{Graph} F) = 0$.

If $x \in K \cap (-\epsilon, \epsilon)$ then $(x, f_\epsilon(x)) = (x, x/\epsilon)$. Sine $(0, x/\epsilon) \in \operatorname{Graph} F$,

$$\operatorname{dist}((x, x/\epsilon), \operatorname{Graph} F)) \leqslant \operatorname{dist}((x, x/\epsilon), (0, x/\epsilon)) = |x| < \epsilon.$$

Thus, f_ϵ are approximate selections of F. We will compute $\Phi_\epsilon = x - \Pi_K(x - f_\epsilon(x))$. Choose $\bar{\epsilon} = 1$ and take any $\epsilon \in (0, \bar{\epsilon}]$. We have the following cases:

If $x \in (-\epsilon, \epsilon)$ then $|x - f_\epsilon(x)| \leqslant 1$. So $\Pi_K(x - f_\epsilon(x)) = 0$.
If $\epsilon \leqslant x \leqslant 2$ then $\Pi_K(x - f_\epsilon(x)) = 0$.
If $x > 2$ then $\Pi_K(x - f_\epsilon(x)) = x - 2$.
If $-2 \leqslant x \leqslant -\epsilon$ then $\Pi_K(x - f_\epsilon(x)) = 0$.
If $x < -2$ then $\Pi_K(x - f_\epsilon(x)) = -x - 2$.

From the above we obtain

$$\Pi_K(x - f_\epsilon(x)) = \begin{cases} 0 & \text{if } x \in [-2,2], \\ x - 2 & \text{if } x > 2, \\ -x - 2 & \text{if } x < -2. \end{cases}$$

Hence,

$$\Phi_\epsilon(x) = \begin{cases} x & \text{if } x \in [-2,2], \\ 2 & \text{if } x > 2, \\ 2x + 2 & \text{if } x < -2. \end{cases}$$

Note that $\partial\Omega = \{-1/2; 2\}$; $F(-1/2) + N_K(-1/2) = \{1\}$ and $F(2) + N_K(2) = \emptyset$. Hence, $0 \notin (F + N_K)(\partial\Omega)$. We now compute $d(\Phi_\epsilon, \Omega, 0)$. As Φ_ϵ is differentiable in Ω we get

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$$d(\Phi_\epsilon, \Omega, 0) = \sum_{x \in \Phi_\epsilon^{-1}(0)} \operatorname{sign} \Phi'_\epsilon(x) = 1.$$

Since the latter equality is true for all $\epsilon \in (0, \bar{\epsilon}]$, we obtain $d(F + N_K, \Omega, 0) = 1$.

We have the following theorem on existence.

Theorem 2.1. *Suppose that* $0 \notin (F + N_K)(\partial\Omega)$. *Then the following assertions hold*:

(a) (*Existence*) *if* $d(F + N_K, \Omega, 0) \neq 0$ *then there exists* $x \in \Omega \cap K$ *such that*

$$0 \in F(x) + N_K(x).$$

(b) *if* $f : R^n \to R^n$ *is a continuous map such that* $f(x) \in F(x)$ *for all* $x \in K$ *then* $d(F + N_K, \Omega, 0) = d(\Phi, \Omega, 0)$, *where* $\Phi(x) = x - \Pi_K(x - f(x))$.

Proof. (a) By definition, there exists $\bar{\epsilon} > 0$ such that $d(F + N_K, \Omega, 0) = d(\Phi_\epsilon, \Omega, 0)$ for all $\epsilon \in (0, \bar{\epsilon}]$. Let $\{\epsilon_k\}$ be a sequence such that $\epsilon_k \to 0^+$. Then $d(\Phi_{\epsilon_k}, \Omega, 0) \neq 0$ for k sufficiently large. By (2) in Theorem 1.1, there exists $x_k \in \Omega$ such that $\Phi_{\epsilon_k}(x_k) = 0$. This is equivalent to

$$x_k = \Pi_K(x_k - f_{\epsilon_k}(x_k)). \tag{2.7}$$

Since $x_k \in K \cap \Omega$, by Lemma 2.1, there exists $y_k \in K$ and $z_k \in F(y_k)$ such that

$$\|y_k - x_k\| < \epsilon_k; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k.$$

By compactness of $K \cap \overline{\Omega}$ we can assume that $x_k \to x_0 \in K \cap \overline{\Omega}$. Hence, $y_k \to x_0$. By standard arguments we get $z_k \to z_0$ and $f_{\epsilon_k}(x_k) \to z_0$ for some $z_0 \in F(x_0)$. Letting $k \to \infty$, from (2.7) we obtain $x_0 = \Pi_K(x_0 - z_0)$. The property of the metric projection yields

$$0 \in z_0 + N_K(x_0) \subset F(x_0) + N_K(x_0).$$

Since $0 \notin (F + N_K)(\partial\Omega)$ we have $x_0 \in K \cap \Omega$.

(b) By putting $f_\epsilon = f$ for all $\epsilon > 0$ we get the desired property. The proof of the theorem is complete. □

Example 2.2. Consider Example 2.1 we have $d(F + N_K, \Omega, 0) = 1$. By the above theorem, $GVI(F, K)$ has a solution $x \in \Omega \cap K$. In this case, $x = 0$ is a solution.

The following theorem contains most usual properties of degree theory.

Theorem 2.2. *Assume that* $0 \notin (F + N_K)(\partial\Omega)$. *The following assertions hold*:

(*a*) (*Homotopy invariance*) *If* $F_1, F_2 : K \to 2^{R^n}$ *are u.s.c. multifunctions with compact convex values and* $0 \notin (tF_1 + (1-t)F_2 + N_K)(\partial\Omega)$ *for all* $t \in [0, 1]$ *then*

$$d(F_1 + N_K, \Omega, 0) = d(F_2 + N_K, \Omega, 0).$$

(*b*) (*Additivity*) *If* Ω_1, Ω_2 *are disjoint open subsets of* Ω *such that* $0 \notin (F + N_K)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ *then*

$$d(F + N_K, \Omega, 0) = d(F + N_K, \Omega_1, 0) + d(F + N_K, \Omega_2, 0).$$

(*c*) (*Excision*) *If* $D \subset \Omega$ *is a closed set such that* $0 \notin (F + N_K)(D)$ *then*

$$d(F + N_K, \Omega, 0) = d(F + N_K, \Omega \setminus D, 0).$$

Proof. (a) Let $f_\epsilon, g_\epsilon : R \times R^n \to R^n$ be approximate selections of F_1 and F_2, respectively satisfying the conclusion of Lemma 2.1. Put

$$\Phi_\epsilon^t(x) = x - \Pi_K(x - tf_\epsilon(x) - (1-t)g_\epsilon(x)).$$

We claim that there is $\bar{\epsilon} > 0$ such that $0 \notin \Phi_\epsilon^t(\partial\Omega)$ for all $\epsilon \in [0, \bar{\epsilon}]$ and $t \in [0, 1]$. In fact, if the claim is false, then there exist a sequence $t_k \in [0, 1]$ and a sequence $\epsilon_k \to 0^+$ such that $0 \in \Phi_{\epsilon_k}^{t_k}(\partial\Omega)$. Hence, for each k, there exists $x_k \in \partial\Omega$ such that

$$x_k = \Pi_K(x_k - t_k f_{\epsilon_k}(x_k) - (1 - t_k) g_{\epsilon_k}(x_k)).$$

By compactness of $[0, 1] \times \partial\Omega$ we can assume that $(t_k, x_k) \to (t_0, x_0) \in [0, 1] \times \partial\Omega$. By standard arguments we can show that $f_{\epsilon_k}(x_k) \to z_1$ for some $z_1 \in F_1(x_0)$ and $g_{\epsilon_k}(x_k) \to z_2$ for some $z_2 \in F_2(x_0)$. Letting $k \to \infty$ from the above we obtain

$$x_0 = \Pi_K(x_0 - t_0 z_1 - (1 - t_0) z_2).$$

By the property of the metric projection we get

$$\begin{aligned} 0 &\in t_0 z_1 + (1 - t_0) z_2 + N_K(x_0) \\ &\subset t_0 F_1(x_0) + (1 - t_0) F_2(x_0) + N_K(x_0). \end{aligned}$$

This contradicts the assumption and so our claim is proved. We now can apply (4) of Theorem 1.1 to Φ_ϵ^t to get $d(\Phi_\epsilon^0, \Omega, 0) = d(\Phi_\epsilon^1, \Omega, 0)$ for all $\epsilon \in (0, \bar{\epsilon}]$. Hence, $d(F_1 + N_K, \Omega, 0) = d(F_2 + N_K, \Omega, 0)$.

(b) We will show that there exists $\bar{\epsilon} > 0$ such that $0 \notin \Phi_\epsilon(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ for all $\epsilon \in (0, \bar{\epsilon}]$. Indeed, if the assertion is false then there exists a sequence $\epsilon_k \to 0^+$ and $x_k \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$ such that $x_k = \Pi_K(x_k - f_{\epsilon_k}(x_k))$. By compactness of $\overline{\Omega}$ we can assume that $x_k \to x_0 \in \overline{\Omega}$. If $x_0 \in \Omega_1 \cup \Omega_2$ then $x_k \in \Omega_1 \cup \Omega_2$ for k sufficiently large. This contradicts the fact that $x_k \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$. Hence, $x_0 \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$. By standard arguments we have $f_{\epsilon_k}(x_k) \to z_0$ for some $z_0 \in F(x_0)$. Letting $k \to \infty$ from the above we obtain $x_0 = \Pi_K(x_0 - z_0)$. This implies that $0 \in F(x_0) + N_K(x_0)$ for some $x_0 \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$, which is a contradiction.

Thus, we have $0 \notin \Phi_\epsilon(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ for all $\epsilon \in (0, \bar{\epsilon}]$. By (3) of Theorem 1.1, we get

$$d(\Phi_\epsilon, \Omega, 0) = d(\Phi_\epsilon, \Omega_1, 0) + d(\Phi_\epsilon, \Omega_2, 0).$$

It follows that

$$d(F + N_K, \Omega, 0) = d(F + N_K, \Omega_1, 0) + d(F + N_K, \Omega_2, 0).$$

(c) By standard arguments we show that $0 \notin \Phi_\epsilon(D)$ for all $\epsilon \in (0, \bar{\epsilon}]$. Applying (5) of Theorem 1.1 to Φ_ϵ we obtain the desired conclusion. The proof of the theorem is complete. □

Definition 2.2. A vector $x_0 \in K$ is called an isolated solution of $\mathrm{GVI}(F,K)$ if there exists a neighborhood V of x_0 such that x_0 is the unique solution of $\mathrm{GVI}(F,K)$ in $\overline{V}$.

Theorem 2.3. *Suppose that x_0 is an isolated solution of $\mathrm{GVI}(F,K)$ and $\mathscr{U}$ is the collection of all open bounded neighborhoods V of x_0 such that $\overline{V}$ does not contain another solution of $\mathrm{GVI}(F,K)$. Then*

$$d(F+N_K,V_1,0)=d(F+N_K,V_2,0)$$

for all $V_1,V_2\in\mathscr{U}$. The common value $d(F+N_K,V,0)$ for $V\in\mathscr{U}$ is called the index of $F+N_K$ and denoted by $i(F+N_K,x_0,0)$.

Proof. We will use the same arguments as in [10] for the proof below.

Taking any $V\in\mathscr{U}$ we have $0\notin(F+N_K)(\partial V)$. Therefore $d(F+N_K,V,0)$ is well defined. We now assume that $V_1,V_2\in\mathscr{U}$. Put $V=V_1\cup V_2\in\mathscr{U}$ and $D=\overline{V}_1\cap V_2^c$, where $V_2^c=R^n\setminus V_2$. We have that D is a compact set in $\overline{V}$ and $0\notin(F+N_K)(D)$. By (c) in Theorem 2.2, we get

$$d(F+N_K,V,0)=d(F+N_K,V\setminus D,0)=d(F+N_K,V_2,0).$$

Using a similar argument for $D=\overline{V}_2\cap V_1^c$, we get

$$d(F+N_K,V,0)=d(F+N_K,V\setminus D,0)=d(F+N_K,V_1,0).$$

Thus, $d(F+N_K,V_1,0)=d(F+N_K,V_2,0)$. □

3. Applications

In this section we shall employ the obtained results in Section 2 to prove some facts on solution existence and solution stability of GVIs.

The following theorem is an extensions of a result in [9] (see [9, Pr. 2.2.3]).

Theorem 3.1. *Let $K\subset R^n$ be a nonempty closed convex set and $F:K\to 2^{R^n}$ be an u.s.c. multifunction with nonempty compact convex values. Assume that there exists a vector $\hat{x}\in K$ such that the set*

$$L_{\leqslant}(\hat{x}):=\{x\in K:\inf_{x^*\in F(x)}\langle x^*,x-\hat{x}\rangle\leqslant 0\}$$

is bounded (possibly empty).

Then $\mathrm{GVI}(F,K)$ has a solution.

Proof. Let Ω be an open ball containing $L_{\leqslant}(\hat{x})\cup\{\hat{x}\}$. We must have $L_{\leqslant}(\hat{x})\cap\partial\Omega=\emptyset$ and hence,

$$\inf_{x^*\in F(x)}\langle x^*,x-\hat{x}\rangle>0\quad\forall x\in K\cap\partial\Omega. \tag{3.1}$$

If $0\in(F+N_K)(\partial\Omega)$, then $\mathrm{GVI}(F,K)$ has a solution. Otherwise, the degree $d(F+N_K,\Omega,0)$ is well defined. Hence, there exists $\epsilon_1>0$ such that $0\notin\Phi_\epsilon(\partial\Omega)$ for all $\epsilon\in(0,\epsilon_1]$. Recall that $\Phi_\epsilon(x)=x-\Pi_K(x-f_\epsilon(x))$, where f_ϵ is approximate selection of F which is continuous on R^n.

We claim that there exists $\epsilon_2>0$ such that for every $\epsilon\in(0,\epsilon_2]$ it holds

$$\langle f_\epsilon(x),x-\hat{x}\rangle\geqslant 0\quad\forall x\in K\cap\partial\Omega. \tag{3.2}$$

Indeed, if the assertion is false then there exist sequences $\epsilon_k\to 0^+$ and $x_k\in K\cap\partial\Omega$ such that

$$\langle f_{\epsilon_k}(x_k),x_k-\hat{x}\rangle<0\quad\forall k\in\mathbb{N}. \tag{3.3}$$

By Lemma 2.1, there exists $(y_k,z_k)\in\mathrm{Graph}F$ such that $\|y_k-x_k\|<\varepsilon_k$ and $\|f_{\epsilon_k}(x_k)-z_k\|<\varepsilon_k$. By compactness of $K\cap\partial\Omega$ we may suppose that there exists $\bar{x}\in K\cap\partial\Omega$ such that $x_k\to\bar{x}$. Then $y_k\to\bar{x}$. As $F(\bar{x})$ is a compact set and F is upper semicontinuous at $\bar{x}$, by taking a subsequence (if necessary) we can suppose furthermore that $z_k\to\bar{z}\in F(\bar{x})$. Then $f_{\epsilon_k}(x_k)\to\bar{z}$. Letting $k\to\infty$, from (3.3) we obtain $\langle\bar{z},\bar{x}-\hat{x}\rangle\leqslant 0$, hence

$$\inf_{x^*\in F(\bar{x})}\langle x^*,\bar{x}-\hat{x}\rangle\leqslant 0.$$

This contradicts (3.1) and our claim is obtained.

Put $\bar{\epsilon}=\min\{\epsilon_1,\epsilon_2\}$. We now show that $d(\Phi_\epsilon,\Omega,0)=1$ for all $\epsilon\in(0,\bar{\epsilon}]$ and so $d(F+N_K,\Omega,0)=1$. For this we build a homotopy as in [9].

Fix any $\epsilon\in(0,\bar{\epsilon}]$ and put

$$H(t,x)=x-\Pi_K(t(x-f_\epsilon(x)+(1-t)\hat{x}),\ (t,x)\in[0,1]\times\overline{\Omega}.$$

We have $H(0,x)=x-\hat{x}$ and $H(1,x)=\Phi_\epsilon(x)$. Note that $d(H(0,\cdot),\Omega,0)=1$. We now claim that $0\notin H(t,\partial\Omega)$ for all $t\in[0,1]$. In fact, it is obvious that $0\notin H(0,\partial\Omega)$ and $0\notin H(1,\partial\Omega)$. Assume that there exist $t\in(0,1)$ and $x\in\partial\Omega$ such that $0=H(t,x)$. By the property of the metric projection we have

$$\langle x-t(x-f_\epsilon(x)-(1-t)\hat{x},y-x\rangle\geqslant 0\ \forall y\in K.$$

In particular, for $y=\hat{x}$ we get

$$\langle tf_\epsilon(x)+(1-t)(x-\hat{x}),\hat{x}-x\rangle\geqslant 0.$$

This implies

$$\langle f_\epsilon(x),\hat{x}-x\rangle\geqslant\frac{1-t}{t}\|x-\hat{x}\|^2>0,$$

where the last inequality holds because $t\in(0,1)$ and $x\neq\hat{x}$. But then it follows that $\langle f_\epsilon(x),x-\hat{x}\rangle<0$ which contradicts (3.2). Thus, $0\notin H(t,\partial\Omega)$ for all $t\in[0,1]$. By the homotopy invariance ((4) in Theorem 1.1) we obtain $d(H(0,\cdot),\Omega,0)=d(H(1,\cdot),\Omega,0)=1$.

In summary, we have proved that $d(\Phi_\epsilon,\Omega,0)=1$ for all $\epsilon\in(0,\bar{\epsilon}]$. By the degree definition of GVIs we have $d(F+N_K,\Omega,0)=1$. According to Theorem 2.1, there exists $x_0\in\Omega\cap K$ such that $0\in F(x_0)+N_K(x_0)$. The proof of the theorem is complete. □

In the rest of the paper we will present a result on solution stability of GVIs. Let us assume that M and Λ are subsets of R^k and R^m, respectively; $F:M\times R^n\to 2^{R^n}$ and $K:\Lambda\to 2^{R^n}$ be multifunctions. Consider the parametric generalized variational inequality

$$0\in F(\mu,x)+N_{K(\lambda)}(x), \tag{3.4}$$

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where $N_{K(\lambda)}(x)$ is the value at x of the normal cone operator associated with the set $K(\lambda)$ and $(\mu,\lambda) \in M \times \Lambda$ are parameters. We denote by $S(\mu,\lambda)$ the solution set of the problem (3.4) corresponding to (μ,λ) and suppose that $x_0 \in S(\mu_0,\lambda_0)$ for a given $(\mu_0,\lambda_0) \in M \times \Lambda$.

Our main concern is now to investigate the behaviour of $S(\mu,\lambda)$ when (μ,λ) vary around (μ_0,λ_0). This problem has been addressed by many authors in the last two decades. For the relevant literature of the problem we refer the reader to [14–17,22–25] and several references given therein.

The following result gives a sufficient condition for the lowercontinuity of the solution map of (3.4). It is an extension of results in [14,22] for the case of GVIs.

Theorem 3.2. *Assume that X_0, Λ_0 and M_0 are neighborhoods of x_0, λ_0 and μ_0, respectively and the following conditions are satisfied*:

(i) *$F(\cdot,\cdot)$ is l.s.c. on $M_0 \times X_0$ with closed convex values and $F(\mu_0,\cdot)$ is u.s.c. with compact convex values;*
(ii) *$K : \Lambda_0 \to 2^{R^n}$ is closed convex valued and pseudo-Lipschitz continuous around (λ_0,x_0), i.e., there exist neighborhoods V of λ_0, W of x_0 and a constant $k>0$ such that*

$$K(\lambda)\cap W \subset K(\lambda') + k\|\lambda-\lambda'\|B(0,1) \quad \forall \lambda,\lambda' \in V\cap\Lambda;$$

(iii) *x_0 is an isolated solution and there exists $\bar{\sigma}>0$ such that*

$$i(\sigma F(\mu_0,\cdot)+N_{K(\lambda_0)\cap X_0},x_0,0)\neq 0 \ \forall\ \sigma\in(0,\bar{\sigma}].$$

Then there exist a neighborhood U_0 of μ_0, a neighborhood V_0 of λ_0 and an open bounded neighborhood Q_0 of x_0 such that the solution map $\hat{S} : U_0 \times V_0 \to 2^{R^n}$ of (3.4) defined by $\hat{S}(\mu,\lambda) = S(\mu,\lambda)\cap Q_0$ is nonempty valued and lower semicontinuous at (μ_0,λ_0).

Proof. By (i) and the continuous selection theorem due to Michael (see [26, Theorem 9G, p. 466]), there exists a continuous mapping $f : M_0 \times X_0 \to R^n$ such that $f(\mu,x) \in F(\mu,x)$ for all $(\mu,x)\in M_0\times X_0$. By Tietze-Urysohn's theorem (see [8, Theorem 5.1, p. 149]) we can assume that f is continuous on $R^k \times R^n$.

According to Lemma 1.1 in [24], it follows from (ii) that there exist a neighborhood $\Lambda_0' \subset \Lambda_0 \cap V$ of λ_0, a neighborhood $X_0' \subset X_0 \cap W$ of x_0 and a constant $k_0>0$ such that

$$\|\Pi_{K(\lambda)\cap X_0}(z) - \Pi_{K(\lambda')\cap X_0}(z)\| \leqslant k_0\|\lambda-\lambda'\|^{1/2}$$

for all $\lambda,\lambda' \in \Lambda_0'$ and $z \in X_0'$. Hence, for any $z,z' \in X_0'$ and $\lambda,\lambda'\in\Lambda_0'$ we have

$$\begin{aligned}\|\pi(\lambda,z)-\pi(\lambda',z')\| &= \|\Pi_{K(\lambda)\cap X_0}(z)-\Pi_{K(\lambda')\cap X_0}(z')\|\\ &\leqslant \|\Pi_{K(\lambda)\cap X_0}(z)-\Pi_{K(\lambda)\cap X_0}(z')\|\\ &\quad + \|\Pi_{K(\lambda)\cap X_0}(z')-\Pi_{K(\lambda')\cap X_0}(z')\|\\ &\leqslant \|z-z'\|+k_0\|\lambda-\lambda\|^{\frac{1}{2}}.\end{aligned}$$

Consequently, $\pi : \Lambda_0' \times X_0' \to X_0$ is uniformly continuous on $\Lambda_0'\times X_0'$.

Choose $\sigma_0 \in (0,\bar{\sigma}]$ such that $x_0 - \sigma_0 f(\mu_0,x_0) \in X_0'$. By the continuity of f, there exist a neighborhood $X_1 \subset X_0'$ of x_0, a neighborhood $M\prime_0 \subset M_0$ of μ_0 such that

$$x-\sigma_0 f(\mu,x) \in X_1 \ \forall\ (x,\mu)\in X_1\times M_0'.$$

Consider the function

$$\Phi_{\sigma_0}(\mu,\lambda,x) = x - \Pi_{K(\lambda)\cap X_0}(x-\sigma_0 f(\mu,x))$$

with $(\mu,\lambda,x)\in M_0'\times\Lambda_0'\times X_1$. By the above, Φ_{σ_0} is continuous on $M_0'\times\Lambda_0'\times X_1$.

From (iii) and Theorem 2.3, there exists an open bounded neighborhood $Q_0 \subset X_1$ of x_0 such that x_0 is the unique solution in $\overline{Q}_0$ of the generalized equation

$$0\in F(\mu_0,x)+N_{K(\lambda_0)}(x).$$

This is equivalent to x_0 is the unique solution in $\overline{Q}_0$ of the generalized equation

$$0\in \sigma_0 F(\mu_0,x)+N_{K(\lambda_0)}(x).$$

Since x_0 belongs to the interior of X_0, it is also the unique solution in $\overline{Q}_0$ of the generalized equation

$$0\in \sigma_0 F(\mu_0,x)+N_{K(\lambda_0)\cap X_0}(x).$$

Moreover, we have

$$\begin{aligned}&d(\sigma_0F(\mu_0,\cdot)+N_{K(\lambda_0)\cap X_0},Q_0,0)\\ &\quad = i(\sigma_0F(\mu_0,\cdot)+N_{K(\lambda_0)\cap X_0},x_0,0)\neq 0.\end{aligned}$$

As $\sigma_0 f(\mu_0,x)\in\sigma_0F(\mu_0,x)$ for all $x\in K(\lambda_0)\cap X_0$, Theorem 2.1 implies Q2

$$d(\Phi_{\sigma_0}(\mu_0,\lambda_0,\cdot),Q_0,0) = d(\sigma_0F(\mu_0,.)+N_{K(\lambda_0)\cap X_0},Q_0,0)\neq 0. \tag{3.5}$$

Note that any solution of equation $\Phi_{\sigma_0}(\mu,\lambda,x)=0$ is also a solution of $GVI(F(\mu,\cdot),K(\lambda)\cap X_0)$. Hence, x_0 is a unique solution of the equation $\Phi_{\sigma_0}(\mu_0,\lambda_0,x)=0$ in Q_0. Taking any $w\in\partial Q_0$, we have $\Phi_{\sigma_0}(\mu_0,\lambda_0,w)\neq 0$. This implies that there exist a $\delta_w>0$ such that $0\notin B(\Phi_{\sigma_0}(\mu_0,\lambda_0,w),\delta_w) := B_w$. By the continuity of Φ_{σ_0}, there exist a neighborhood $U_w\subset M_0'$ of μ_0, a neighborhood $\Lambda_w\subset\Lambda_0^{P\prime}$ of λ_0 and a neighborhood Q_w of w such that $\Phi_{\sigma_0}(\mu,\lambda,z)\in B_w$ for all $(\mu,\lambda,z)\in U_w\times\Lambda_w\times Q_w$. Since ∂Q_0 is a compact set, there are some $w_1,w_2,\ldots,w_n$ such that $\partial Q_0\subset\cup_{i=1}^n Q_{w_i}$. Put $U_0 = \cap_{i=1}^n U_{w_i}$, $V_0=\cap_{i=1}^n\Lambda_{w_i}$.

We now use similar arguments to the proof of Theorem 2.1 in [14] (see also [22, Theorem 3.2]) to show that U_0,V_0 and Q_0 satisfy the conclusion of the theorem. The proof is complete. □

4. Uncited reference

[2,6,21]. Q1

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