J. Fixed Point Theory Appl. 11 (2012) 159–183
DOI 10.1007/s11784-012-0074-3
Published online April 3, 2012

Journal of Fixed Point Theory and Applications

Fixed point theorems and convergence theorems for generalized nonspreading mappings in Banach spaces

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Dedicated to Professor Dick Palais

Abstract. In this paper, we first prove a general fixed point theorem for nonlinear mappings in a Banach space. Then we prove a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for 2-generalized nonspreading mappings in a Banach space.

Mathematics Subject Classification (2010). Primary 47H10; Secondary 47H05.

Keywords. Banach space, nonexpansive mapping, nonspreading mapping, hybrid mapping, fixed point.

1. Introduction

Let E be a real smooth Banach space and let J be the duality mapping of E. Let C be a nonempty closed convex subset of E. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. Recently, Kohsaka and Takahashi [14] introduced the following nonlinear mapping: A mapping $T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$, where

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see also [13]. Such a mapping is deduced from the resolvent of a maximal monotone operator in a Banach space; see [14, 27, 23]. A nonspreading mapping defined by [14] is as follows in a Hilbert space: Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *nonspreading* [14] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

for all $x, y \in C$. Takahashi [22] also defined another nonlinear mapping in a Hilbert space: A mapping $T : C \to C$ is said to be *hybrid* [22] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$. Furthermore, we know that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$; see [4, 21]. The classes of nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space are deduced from the class of firmly nonexpansive mappings; see [22]. A mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see [3, 4]. Recently, Kocourek, Takahashi and Yao [10] introduced a class of nonlinear mappings called generalized hybrid containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space: A mapping $T : C \to C$ is called *generalized hybrid* [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Kocourek, Takahashi and Yao [11] also extended the class of generalized hybrid mappings in a Hilbert space to Banach spaces: Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Then a mapping $T : C \to C$ is called *generalized nonspreading* if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\big\{\phi(y, Tx) - \phi(y, x)\big\} \end{aligned} \tag{1.3}$$

for all $x, y \in C$. Very recently, Maruyama, Takahashi and Yao [16] introduced a broad class of nonlinear mappings containing the class of generalized hybrid mappings defined by [10] in a Hilbert space: A mapping $T : C \to C$ is called 2-*generalized hybrid* if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. Motivated by [16, 11], we introduced three classes of nonlinear mappings in Banach spaces [26] which contain the class of 2-generalized hybrid mappings in a Hilbert space. Then we proved fixed point theorems for these classes of nonlinear mappings in Banach spaces.

In this paper, we deal with the class of 2-generalized nonspreading mappings which is one of the three classes of nonlinear mappings defined in [26] in Banach spaces. We first prove a general fixed point theorem of nonlinear mappings in a Banach space. Using this result, we give another proof of our fixed point theorem [26] for 2-generalized nonspreading mappings in Banach spaces. Then we prove a nonlinear mean convergence theorem of Baillon's type [2] and a weak convergence theorem of Mann's type[15] for such nonlinear mappings in a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [8].

Let E be a Banach space. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [19, 20]. The following result is also in [19, 20].

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi : E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.2}$$

for $x, y \in E$, where J is the duality mapping of E; see [1, 9]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.3}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.4}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then from Lemma 2.1 we have

$$\phi(x, y) = 0 \iff x = y. \tag{2.5}$$

The following lemmas are in Xu [29] and Kamimura and Takahashi [9], respectively.

Lemma 2.2. *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3. *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to C$ is called *generalized nonexpansive* [5] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp., *sunny* *generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp., sunny generalized nonexpansive retraction) R from E onto D; see [5] for more details. The following results are in Ibaraki and Takahashi [5].

Theorem 2.4. *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Theorem 2.5. *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz \rangle \leq 0$ *for all* $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [12] proved the following results.

Theorem 2.6. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Theorem 2.7. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Very recently, Ibaraki and Takahashi [7] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Theorem 2.8. *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following theorem is proved using Theorems 2.6 and 2.8.

Theorem 2.9 (see [7]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a *Banach limit* on

l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [19].

3. Fixed point theorems

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let $n \in \mathbb{N}$. Then a mapping $T : C \to C$ is called *n-generalized nonspreading* [26] if there exist $\alpha_1, \alpha_2, \dots, \alpha_n$, $\beta_1, \beta_2, \dots, \beta_n$, $\gamma_1, \gamma_2, \dots, \gamma_n$, $\delta_1, \delta_2, \dots, \delta_n \in \mathbb{R}$ such that for all $x, y \in C$,

$$\begin{aligned} &\sum_{k=1}^{n} \alpha_k \phi(T^{n+1-k}x, Ty) + \left(1 - \sum_{k=1}^{n} \alpha_k\right) \phi(x, Ty) \\ &\quad + \sum_{k=1}^{n} \gamma_k \{\phi(Ty, T^{n+1-k}x) - \phi(Ty, x)\} \\ &\le \sum_{k=1}^{n} \beta_k \phi(T^{n+1-k}x, y) + \left(1 - \sum_{k=1}^{n} \beta_k\right) \phi(x, y) \\ &\quad + \sum_{k=1}^{n} \delta_k \{\phi(y, T^{n+1-k}x) - \phi(y, x)\}. \end{aligned} \tag{3.1}$$

Such a mapping is called $(\alpha_1, \alpha_2, \dots, \alpha_n$, $\beta_1, \beta_2, \dots, \beta_n$, $\gamma_1, \gamma_2, \dots, \gamma_n$, δ_1, $\delta_2, \dots, \delta_n)$*-generalized nonspreading.* For example, an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2,$ $\delta_1, \delta_2)$-generalized nonspreading mapping is as follows:

$$\begin{aligned} &\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\ &\quad + \gamma_1\{\phi(Ty, T^2x) - \phi(Ty, x)\} + \gamma_2\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\le \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\ &\quad + \delta_1\{\phi(y, T^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{3.2}$$

for all $x, y \in C$. This is also called a 2*-generalized nonspreading* mapping; see [26]. We know that an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading mapping is nonspreading in the sense of [14] for $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, $\alpha_2 = \beta_2 = \gamma_2 = 1$ and $\delta_2 = 0$ in (3.2).

Now we state and prove the main result in this section.

Theorem 3.1. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty closed convex subset of E. Let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of C*

and let μ be a mean on l^∞. Suppose that

$$\mu_n \phi(x_n, Ty) \le \mu_n \phi(x_n, y)$$

for all $y \in C$. Then T has a fixed point in C.

Proof. Using a mean μ and a bounded sequence $\{x_n\}$, we define a function $g : E^* \to \mathbb{R}$ as follows:

$$g(x^*) = \mu_n \langle x_n, x^* \rangle$$

for all $x^* \in E^*$. Since μ is linear, g is also linear. Furthermore, we have

$$\begin{aligned}|g(x^*)| &= |\mu_n \langle x_n, x^* \rangle| \\ &\le \|\mu\| \sup_{n \in \mathbb{N}} |\langle x_n, x^* \rangle| \\ &\le \|\mu\| \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\| \\ &= \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\|\end{aligned}$$

for all $x^* \in E^*$. Then g is a linear and continuous real-valued function on E^*. Since E is reflexive, there exists a unique element z of E such that

$$g(x^*) = \mu_n \langle x_n, x^* \rangle = \langle z, x^* \rangle$$

for all $x^* \in E^*$. Such an element z is in C. In fact, if $z \notin C$, then there exists $y^* \in E^*$ by the separation theorem [19] such that

$$\langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle.$$

So, from $\{x_n\} \subset C$ we have

$$\langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle \le \inf_{n \in \mathbb{N}} \langle x_n, y^* \rangle \le \mu_n \langle x_n, y^* \rangle = \langle z, y^* \rangle.$$

This is a contradiction. Then we have $z \in C$. From (2.3) we have that for $y \in C$ and $n \in \mathbb{N}$,

$$\phi(x_n, y) = \phi(x_n, Ty) + \phi(Ty, y) + 2\langle x_n - Ty, JTy - Jy \rangle.$$

So, we have that for $y \in C$,

$$\begin{aligned}\mu_n \phi(x_n, y) &= \mu_n \phi(x_n, Ty) + \mu_n \phi(Ty, y) + 2\mu_n \langle x_n - Ty, JTy - Jy \rangle \\ &= \mu_n \phi(x_n, Ty) + \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle.\end{aligned}$$

Since, by assumption, $\mu_n \phi(x_n, Ty) \le \mu_n \phi(x_n, y)$ for all $y \in C$, we have

$$\mu_n \phi(x_n, y) \le \mu_n \phi(x_n, y) + \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle.$$

This implies that

$$0 \le \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle.$$

We know that z is an element of C. Putting $y = z$, we have that

$$0 \le \phi(Tz, z) + 2\langle z - Tz, JTz - Jz \rangle.$$

Thus we have from (2.4) that

$$0 \le \phi(Tz, z) + \phi(z, z) + \phi(Tz, Tz) - \phi(z, Tz) - \phi(Tz, z).$$

So, we have $0 \le -\phi(z, Tz)$ and hence $0 = \phi(z, Tz)$. Since E is strictly convex, we have $Tz = z$. This completes the proof. □

Using Theorem 3.1, we prove a fixed point theorem for n-generalized nonspreading mappings in a Banach space; see [26, Remark 2].

Theorem 3.2. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to C$ be an n-generalized nonspreading mapping. Then T has a fixed point in C if and only if $\{T^m z\}$ is bounded for some $z \in C$.*

Proof. If $F(T) \ne \emptyset$, then $\{T^m z\} = \{z\}$ for $z \in F(T)$. So, $\{T^m z\}$ is bounded. Conversely, let $T : C \to C$ be n-generalized nonspreading. Then there exist

$$\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_n, \delta_1, \delta_2, \dots, \delta_n \in \mathbb{R}$$

such that for all $x, y \in C$,

$$\begin{aligned} &\sum_{k=1}^{n} \alpha_k \phi(T^{n+1-k}x, Ty) + \left(1 - \sum_{k=1}^{n} \alpha_k\right) \phi(x, Ty) \\ &\quad + \sum_{k=1}^{n} \gamma_k \{\phi(Ty, T^{n+1-k}x) - \phi(Ty, x)\} \\ &\le \sum_{k=1}^{n} \beta_k \phi(T^{n+1-k}x, y) + \left(1 - \sum_{k=1}^{n} \beta_k\right) \phi(x, y) \\ &\quad + \sum_{k=1}^{n} \delta_k \{\phi(y, T^{n+1-k}x) - \phi(y, x)\}. \end{aligned} \tag{3.3}$$

By assumption, we can take $z \in C$ such that $\{T^m z\}$ is bounded. Replacing x by $T^m z$ in (3.3), we have that for any $y \in C$ and $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\sum_{k=1}^{n} \alpha_k \phi(T^{n+1-k}T^m z, Ty) + \left(1 - \sum_{k=1}^{n} \alpha_k\right) \phi(T^m z, Ty) \\ &\quad + \sum_{k=1}^{n} \gamma_k \{\phi(Ty, T^{n+1-k}T^m z) - \phi(Ty, T^m z)\} \\ &\le \sum_{k=1}^{n} \beta_k \phi(T^{n+1-k}T^m z, y) + \left(1 - \sum_{k=1}^{n} \beta_k\right) \phi(T^m z, y) \\ &\quad + \sum_{k=1}^{n} \delta_k \{\phi(y, T^{n+1-k}T^m z) - \phi(y, T^m z)\}. \end{aligned}$$

Since $\{T^m z\}$ is bounded, we can apply a Banach limit μ to both sides of the above inequality. Then we have

$$\begin{aligned}\mu_m\Bigg(&\sum_{k=1}^{n}\alpha_k\phi(T^{m+n+1-k}z, Ty) + \left(1-\sum_{k=1}^{n}\alpha_k\right)\phi(T^m z, Ty)\\&+\sum_{k=1}^{n}\gamma_k\big\{\phi(Ty, T^{m+n+1-k}z) - \phi(Ty, T^m z)\big\}\Bigg)\\&\le \mu_m\Bigg(\sum_{k=1}^{n}\beta_k\phi(T^{m+n+1-k}z, y) + \left(1-\sum_{k=1}^{n}\beta_k\right)\phi(T^m z, y)\\&\qquad+\sum_{k=1}^{n}\delta_k\big\{\phi(y, T^{m+n+1-k}z) - \phi(y, T^m z)\big\}\Bigg).\end{aligned}$$

So, we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\alpha_k\mu_m\phi(T^{m+n+1-k}z, Ty) + \left(1-\sum_{k=1}^{n}\alpha_k\right)\mu_m\phi(T^m z, Ty)\\&\quad+\sum_{k=1}^{n}\gamma_k\big\{\mu_m\phi(Ty, T^{m+n+1-k}z) - \mu_m\phi(Ty, T^m z)\big\}\\&\le \sum_{k=1}^{n}\beta_k\mu_m\phi(T^{m+n+1-k}z, y) + \left(1-\sum_{k=1}^{n}\beta_k\right)\mu_m\phi(T^m z, y)\\&\quad+\sum_{k=1}^{n}\delta_k\big\{\mu_m\phi(y, T^{m+n+1-k}z) - \mu_m\phi(y, T^m z)\big\}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=1}^{n}\alpha_k\mu_m\phi(T^m z, Ty) + \left(1-\sum_{k=1}^{n}\alpha_k\right)\mu_m\phi(T^m z, Ty)\\&\quad+\sum_{k=1}^{n}\gamma_k\big\{\mu_m\phi(Ty, T^m z) - \mu_m\phi(Ty, T^m z)\big\}\\&\le \sum_{k=1}^{n}\beta_k\mu_m\phi(T^m z, y) + \left(1-\sum_{k=1}^{n}\beta_k\right)\mu_m\phi(T^m z, y)\\&\quad+\sum_{k=1}^{n}\delta_k\big\{\mu_m\phi(y, T^m z) - \mu_m\phi(y, T^m z)\big\}.\end{aligned}$$

This implies

$$\mu_m\phi(T^m z, Ty) \le \mu_m\phi(T^m z, y)$$

for all $y \in C$. By Theorem 3.1, T has a fixed point in C. □

4. Some properties of generalized nonspreading mappings

In this section, we obtain fundamental properties for 2-generalized nonspreading mappings in a Banach space.

Proposition 4.1. *Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$. Then a mapping $T : C \to C$ is $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading if and only if T satisfies that for any $x, y \in C$,*

$$\begin{aligned}
0 \le\ & (\beta_1 - \alpha_1)\big\{\phi(T^2x, Ty) - \phi(x, Ty)\big\} \\
& + (\beta_2 - \alpha_2)\big\{\phi(Tx, Ty) - \phi(x, Ty)\big\} + \phi(Ty, y) \\
& + 2\langle x - Ty + \beta_1(T^2x - x) + \beta_2(Tx - x), JTy - Jy\rangle \\
& - \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} - \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
& + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\}.
\end{aligned}$$

Proof. Since a mapping $T : C \to C$ is $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading, we have that for any $x, y \in C$,

$$\begin{aligned}
& \alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
& \quad + \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} + \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
& \le \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
& \quad + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\}.
\end{aligned}$$

Then we have from (2.3) that for any $x, y \in C$,

$$\begin{aligned}
0 \le\ & \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
& + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\} \\
& - \alpha_1\phi(T^2x, Ty) - \alpha_2\phi(Tx, Ty) - (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
& - \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} - \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
=\ & \beta_1\big\{\phi(T^2x, Ty) + \phi(Ty, y) + 2\langle T^2x - Ty, JTy - Jy\rangle\big\} \\
& + \beta_2\big\{\phi(Tx, Ty) + \phi(Ty, y) + 2\langle Tx - Ty, JTy - Jy\rangle\big\} \\
& + (1 - \beta_1 - \beta_2)\big\{\phi(x, Ty) + \phi(Ty, y) + 2\langle x - Ty, JTy - Jy\rangle\big\} \\
& + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\} \\
& - \alpha_1\phi(T^2x, Ty) - \alpha_2\phi(Tx, Ty) - (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
& - \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} - \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
=\ & (\beta_1 - \alpha_1)\big\{\phi(T^2x, Ty) - \phi(Tx, Ty)\big\} \\
& + (\beta_2 - \alpha_2)\big\{\phi(Tx, Ty) - \phi(x, Ty)\big\} + \phi(Ty, y) \\
& + 2\langle \beta_1T^2x + \beta_2Tx + (1 - \beta_1 - \beta_2)x - Ty, JTy - Jy\rangle \\
& - \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} - \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
& + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\}.
\end{aligned}$$

Hence we have that for any $x, y \in C$,

$$
\begin{aligned}
0 \le\ & (\beta_1 - \alpha_1)\big\{\phi(T^2x, Ty) - \phi(x, Ty)\big\} \\
& + (\beta_2 - \alpha_2)\big\{\phi(Tx, Ty) - \phi(x, Ty)\big\} + \phi(Ty, y) \\
& + 2\langle x - Ty + \beta_1(T^2x - x) + \beta_2(Tx - x), JTy - Jy\rangle \\
& - \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} - \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
& + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\}.
\end{aligned}
$$

This completes the proof. □

Let E be a Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then $p \in C$ is an asymptotic fixed point of T (see [17]) if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. Motivated by the concept of asymptotic fixed points, we have the following result. This result is used in Section 6.

Proposition 4.2. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let $T : C \to C$ be a 2-generalized nonspreading mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup p$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T^2x_n\| = 0$. Then $p \in F(T)$.*

Proof. Since $T : C \to C$ is a 2-generalized nonspreading mapping, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that for any $x, y \in C$,

$$
\begin{aligned}
& \alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\
& \quad + \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} + \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\
& \le \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\
& \quad + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\}.
\end{aligned} \tag{4.1}
$$

Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T^2x_n\| = 0$. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping J on E is uniformly norm-to-weak* continuous on each bounded subset of E; see [20]. Using $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T^2x_n\| = 0$, we have $\lim_{n\to\infty}\langle w, JTx_n - Jx_n\rangle = 0$ and $\lim_{n\to\infty}\langle w, JT^2x_n - Jx_n\rangle = 0$ for all $w \in E$. On the other hand, replacing x by x_n and y by p in (4.1), we obtain that

$$
\begin{aligned}
& \alpha_1\phi(T^2x_n, Tp) + \alpha_2\phi(Tx_n, Tp) + (1 - \alpha_1 - \alpha_2)\phi(x_n, Tp) \\
& \quad + \gamma_1\big\{\phi(Tp, T^2x_n) - \phi(Tp, x_n)\big\} + \gamma_2\big\{\phi(Tp, Tx_n) - \phi(Tp, x_n)\big\} \\
& \le \beta_1\phi(T^2x_n, p) + \beta_2\phi(Tx_n, p) + (1 - \beta_1 - \beta_2)\phi(x_n, p) \\
& \quad + \delta_1\big\{\phi(p, T^2x_n) - \phi(p, x_n)\big\} + \delta_2\big\{\phi(p, Tx_n) - \phi(p, x_n)\big\}.
\end{aligned} \tag{4.2}
$$

We have from Proposition 4.1 and (4.2) that

$$\begin{aligned}0 &\le (\beta_1-\alpha_1)\big\{\phi(T^2x_n,Tp)-\phi(x_n,Tp)\big\}\\&\quad+(\beta_2-\alpha_2)\big\{\phi(Tx_n,Tp)-\phi(x_n,Tp)\big\}+\phi(Tp,p)\\&\quad+2\langle x_n-Tp+\beta_1(T^2x_n-x_n)+\beta_2(Tx_n-x_n),JTp-Jp\rangle\\&\quad-\gamma_1\big\{\phi(Tp,T^2x_n)-\phi(Tp,x_n)\big\}-\gamma_2\big\{\phi(Tp,Tx_n)-\phi(Tp,x_n)\big\}\\&\quad+\delta_1\big\{\phi(p,T^2x_n)-\phi(p,x_n)\big\}+\delta_2\big\{\phi(p,Tx_n)-\phi(p,x_n)\big\}\\&=(\beta_1-\alpha_1)\big\{\|T^2x_n\|^2-\|x_n\|^2-2\langle T^2x_n-x_n,JTp\rangle\big\}\\&\quad+(\beta_2-\alpha_2)\big\{\|Tx_n\|^2-\|x_n\|^2-2\langle Tx_n-x_n,JTp\rangle\big\}+\phi(Tp,p)\\&\quad+2\langle x_n-Tp+\beta_1(T^2x_n-x_n)+\beta_2(Tx_n-x_n),JTp-Jp\rangle\\&\quad-\gamma_1\big\{\|T^2x_n\|^2-\|x_n\|^2-2\langle Tp,JT^2x_n-Jx_n\big\}\\&\quad-\gamma_2\big\{\|Tx_n\|^2-\|x_n\|^2-2\langle Tp,JTx_n-Jx_n\big\}\\&\quad+\delta_1\big\{\|T^2x_n\|^2-\|x_n\|^2-2\langle p,JT^2x_n-Jx_n\big\}\\&\quad+\delta_2\big\{\|Tx_n\|^2-\|x_n\|^2-2\langle p,JTx_n-Jx_n\big\}.\end{aligned}\tag{4.3}$$

From

$$\begin{aligned}\big|\|T^2x_n\|^2-\|x_n\|^2\big| &= (\|T^2x_n\|+\|x_n\|)\big|\|T^2x_n\|-\|x_n\|\big|\\&\le(\|T^2x_n\|+\|x_n\|)\|T^2x_n-x_n\|\end{aligned}$$

and

$$\begin{aligned}\big|\|Tx_n\|^2-\|x_n\|^2\big| &= (\|Tx_n\|+\|x_n\|)\big|\|Tx_n\|-\|x_n\|\big|\\&\le(\|Tx_n\|+\|x_n\|)\|Tx_n-x_n\|,\end{aligned}$$

we have $\|T^2x_n\|^2-\|x_n\|^2\to 0$ and $\|Tx_n\|^2-\|x_n\|^2\to 0$ as $n\to\infty$. So, letting $n\to\infty$ in (4.3), we have that

$$\begin{aligned}0&\le\phi(Tp,p)+2\langle p-Tp,JTp-Jp\rangle\\&=\phi(Tp,p)+\phi(p,p)+\phi(Tp,Tp)-\phi(p,Tp)-\phi(Tp,p)\\&=-\phi(p,Tp).\end{aligned}$$

Thus $\phi(p,Tp)\le 0$ and then $\phi(p,Tp)=0$. Since E is strictly convex, we obtain $p=Tp$. This completes the proof. □

5. Nonlinear ergodic theorem

Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let J be the duality mapping from E into E^*. We know that a mapping $T: C\to C$ is called n-generalized nonspreading if T satisfies (3.1). Observe that if $T: C\to C$ is an n-generalized nonspreading mapping and $F(T)\ne\emptyset$, then

$$\phi(u,Ty)\le\phi(u,y)$$

for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (3.1), we obtain

$$\begin{aligned}\sum_{k=1}^{n}\alpha_k\phi(u,Ty) + \left(1-\sum_{k=1}^{n}\alpha_k\right)\phi(u,Ty)& \\ +\sum_{k=1}^{n}\gamma_k\{\phi(Ty,u)-\phi(Ty,u)\}& \\ \le \sum_{k=1}^{n}\beta_k\phi(u,y) + \left(1-\sum_{k=1}^{n}\beta_k\right)\phi(u,y)& \\ +\sum_{k=1}^{n}\delta_k\{\phi(y,u)-\phi(y,u)\}&.\end{aligned}$$

So, we have that

$$\phi(u,Ty) \le \phi(u,y) \tag{5.1}$$

for all $u \in F(T)$ and $y \in C$. Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for $x \in C$,

$$\begin{aligned}\sum_{k=1}^{n}\alpha_k\phi(T^{n+1-k}x,u) + \left(1-\sum_{k=1}^{n}\alpha_k\right)\phi(x,u)& \\ +\sum_{k=1}^{n}\gamma_k\{\phi(u,T^{n+1-k}x)-\phi(u,x)\}& \\ \le \sum_{k=1}^{n}\beta_k\phi(T^{n+1-k}x,u) + \left(1-\sum_{k=1}^{n}\beta_k\right)\phi(x,u)& \\ +\sum_{k=1}^{n}\delta_k\{\phi(u,T^{n+1-k}x)-\phi(u,x)\}&\end{aligned}$$

and hence

$$\begin{aligned}\sum_{k=1}^{n}(\alpha_k-\beta_k)\{\phi(T^{n+1-k}x,u)-\phi(x,u)\}& \\ +\sum_{k=1}^{n}(\gamma_k-\delta_k)\{\phi(u,T^{n+1-k}x)-\phi(u,x)\} \le 0&.\end{aligned}$$

If $\alpha_k - \beta_k = 0$ for all $k = 1, 2, \dots, n-1$, $\gamma_k \le \delta_k$ for all $k = 1, 2, \dots, n$ and $\alpha_n > \beta_n$, then we have from (5.1) that

$$\begin{aligned}(\alpha_n-\beta_n)\{\phi(Tx,u)-\phi(x,u)\}& \\ \le \sum_{k=1}^{n}(\delta_k-\gamma_k)\{\phi(u,T^{n+1-k}x)-\phi(u,x)\}& \\ \le 0&.\end{aligned}$$

So, we have that

$$\phi(Tx,u) \le \phi(x,u) \tag{5.2}$$

for all $x \in C$ and $u \in F(T)$. For example, let us consider a 2-generalized nonspreading mapping $T : C \to C$ which satisfies (3.2); i.e.,

$$\begin{aligned} &\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\ &\quad + \gamma_1\{\phi(Ty, T^2x) - \phi(Ty, x)\} + \gamma_2\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\le \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\ &\quad + \delta_1\{\phi(y, T^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{5.3}$$

for all $x, y \in C$. Then we have that $\alpha_1 = \beta_1$, $\alpha_2 > \beta_2$, $\gamma_1 \le \delta_1$ and $\gamma_2 \le \delta_2$ imply that

$$\phi(Tx, u) \le \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. Now using the technique developed by [18, 25], we can prove the following nonlinear ergodic theorem for 2-generalized nonspreading mappings in a Banach space. For proving this result, we need the following lemma.

Lemma 5.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a generalized nonexpansive mapping, that is, $F(T) \neq \emptyset$ and $\phi(Tx, u) \le \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Define, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

If each weak cluster point of $\{S_n x\}$ belongs to $F(T)$, then $\{S_n x\}$ converges weakly to the strong limit of $\{RT^n x\}$.

Proof. We know that since C is a sunny generalized nonexpansive retract of E, there exists the sunny generalized nonexpansive retraction P of E onto C. On the other hand, since a mapping $T : C \to C$ satisfies that $F(T) \neq \emptyset$ and

$$\phi(Tx, u) \le \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$, T is generalized nonexpansive. So, putting $S = TP$, we have that S is a generalized nonexpansive mapping of E into itself such that $F(S) = F(T)$. Indeed, we have

$$z \in F(T) \Longleftrightarrow z = Tz \Longleftrightarrow z = Pz = TPz \Longleftrightarrow z = Sz.$$

So, it follows that $F(S) = F(T)$. We also have that for any $x \in E$ and $u \in F(S) = F(T)$,

$$\phi(Sx, u) = \phi(TPx, u) \le \phi(Px, u) \le \phi(x, u).$$

So, S is a generalized nonexpansive mapping of E into itself such that $F(S) =$

$F(T)$. From Theorems 2.9 and 2.4, there exists the sunny generalized nonexpansive retraction R of E onto $F(T)$. From Theorem 2.7, this retraction R is characterized by

$$Rx = \operatorname{argmin}_{u\in F(T)}\phi(x,u).$$

We also know from Theorem 2.5 that

$$0 \le \langle v - Rv, JRv - Ju\rangle \quad \forall u \in F(T),\ v \in C.$$

Adding up $\phi(Rv,u)$ to both sides of this inequality, we have

$$\begin{aligned}\phi(Rv,u) &\le \phi(Rv,u) + 2\langle v - Rv, JRv - Ju\rangle \\ &= \phi(Rv,u) + \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv).\end{aligned} \tag{5.4}$$

Since $\phi(Tz,u) \le \phi(z,u)$ for any $u \in F(T)$ and $z \in C$, it follows that

$$\begin{aligned}\phi(T^nx, RT^nx) &\le \phi(T^nx, RT^{n-1}x) \\ &\le \phi(T^{n-1}x, RT^{n-1}x).\end{aligned}$$

Hence the sequence $\phi(T^nx, RT^nx)$ is nonincreasing. Putting $u = RT^nx$ and $v = T^mx$ with $n \le m$ in (5.4), we have from Theorem 2.3 that

$$\begin{aligned}g(\|RT^mx - RT^nx\|) &\le \phi(RT^mx, RT^nx) \\ &\le \phi(T^mx, RT^nx) - \phi(T^mx, RT^mx) \\ &\le \phi(T^nx, RT^nx) - \phi(T^mx, RT^mx),\end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RT^nx\}$ is a Cauchy sequence. Therefore, $\{RT^nx\}$ converges strongly to a point $q \in F(T)$ since $F(T)$ is closed from Theorem 2.8. Next consider a fixed $x \in C$ and an arbitrary subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$ such that $S_{n_i}x \rightharpoonup v$. By assumption, we know that $v \in F(T)$. Rewriting the characterization of the retraction R, we have that

$$0 \le \langle T^kx - RT^kx, JRT^kx - Ju\rangle$$

and hence

$$\begin{aligned}\langle T^kx - RT^kx, Ju - Jq\rangle &\le \langle T^kx - RT^kx, JRT^kx - Jq\rangle \\ &\le \|T^kx - RT^kx\| \cdot \|JRT^kx - Jq\| \\ &\le K\|JRT^kx - Jq\|,\end{aligned}$$

where K is an upper bound for $\|T^kx - RT^kx\|$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$, we get

$$\left\langle S_nx - \frac{1}{n}\sum_{k=0}^{n-1} RT^kx, Ju - Jq\right\rangle \le K\frac{1}{n}\sum_{k=0}^{n-1}\|JRT^kx - Jq\|,$$

where $S_nx = \frac{1}{n}\sum_{k=0}^{n-1} T^kx$. Letting $n_i \to \infty$ and remembering that J is continuous, we get

$$\langle v - q, Ju - Jq \rangle \leq 0.$$

This holds for any $u \in F(T)$. Therefore $Rv = q$. But because $v \in F(T)$, we have $v = q$. Thus the sequence $\{S_n x\}$ converges weakly to the point q, where $q = \lim_{n\to\infty} RT^n x$. □

Using Lemma 5.1, we obtain the following nonlinear ergodic theorems for 2-generalized nonspreading mappings in a Banach space.

Theorem 5.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a 2-generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. Since a 2-generalized nonspreading mapping $T : C \to C$ with $F(T) \neq \emptyset$ satisfies that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$, T is generalized nonexpansive. Fix $x \in C$. To show the theorem, it is sufficient to show from Lemma 5.1 that each weak cluster point of $\{S_n x\}$ belongs to $F(T)$. Since $T : C \to C$ is a 2-generalized nonspreading mapping, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ satisfying (5.3). Since $F(T) \neq \emptyset$, we have from (5.1) that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Taking a fixed point u of T, we have that for $x \in C$, $\phi(u, T^n x) \leq \phi(u, x)$ for all $n \in \mathbb{N}$. Then $\{T^n x\}$ is bounded. Replacing x by $T^k x$ in (5.3), we have that for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\alpha_1\phi(T^{k+2}x, Ty) + \alpha_2\phi(T^{k+1}x, Ty) + (1 - \alpha_1 - \alpha_2)\phi(T^k x, Ty)\\
&\quad + \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} + \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\}\\
&\leq \beta_1\phi(T^{k+2}x, y) + \beta_2\phi(T^{k+1}x, y) + (1 - \beta_1 - \beta_2)\phi(T^k x, y)\\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}\\
&= \beta_1\{\phi(T^{k+2}x, Ty) + \phi(Ty, y) + 2\langle T^{k+2}x - Ty, JTy - Jy\rangle\}\\
&\quad + \beta_2\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle\}\\
&\quad + (1 - \beta_1 - \beta_2)\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\langle T^k x - Ty, JTy - Jy\rangle\}\\
&\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}.
\end{aligned} \tag{5.5}$$

Then we have from Proposition 4.1 that

$$\begin{aligned}
0 &\le (\beta_1-\alpha_1)\big\{\phi(T^{k+2}x,Ty)-\phi(T^kx,Ty)\big\}\\
&\quad+(\beta_2-\alpha_2)\big\{\phi(T^{k+1}x,Ty)-\phi(T^kx,Ty)\big\}+\phi(Ty,y)\\
&\quad+2\langle\beta_1T^{k+2}x+\beta_2T^{k+1}x+(1-\beta_1-\beta_2)T^kx-Ty,JTy-Jy\rangle\\
&\quad-\gamma_1\big\{\phi(Ty,T^{k+2}x)-\phi(Ty,T^kx)\big\}-\gamma_2\big\{\phi(Ty,T^{k+1}x)-\phi(Ty,T^kx)\big\}\\
&\quad+\delta_1\big\{\phi(y,T^{k+2}x)-\phi(y,T^kx)\big\}+\delta_2\big\{\phi(y,T^{k+1}x)-\phi(y,T^kx)\big\}\\
&= (\beta_1-\alpha_1)\big\{\phi(T^{k+2}x,Ty)-\phi(T^kx,Ty)\big\}\\
&\quad+(\beta_2-\alpha_2)\big\{\phi(T^{k+1}x,Ty)-\phi(T^kx,Ty)\big\}+\phi(Ty,y)\\
&\quad+2\langle T^kx-Ty+\beta_1(T^{k+2}x-T^kx)+\beta_2(T^{k+1}x-T^kx),JTy-Jy\rangle\\
&\quad-\gamma_1\big\{\phi(Ty,T^{k+2}x)-\phi(Ty,T^kx)\big\}-\gamma_2\big\{\phi(Ty,T^{k+1}x)-\phi(Ty,T^kx)\big\}\\
&\quad+\delta_1\big\{\phi(y,T^{k+2}x)-\phi(y,T^kx)\big\}+\delta_2\big\{\phi(y,T^{k+1}x)-\phi(y,T^kx)\big\}.
\end{aligned} \tag{5.6}$$

Summing up these inequalities in (5.6) with respect to $k=0,1,\dots,n-1$ and dividing by n, we have

$$\begin{aligned}
0 &\le \frac{1}{n}(\beta_1-\alpha_1)\big\{\phi(T^{n+1}x,Ty)+\phi(T^nx,Ty)-\phi(Tx,Ty)-\phi(x,Ty)\big\}\\
&\quad+\frac{1}{n}(\beta_2-\alpha_2)\big\{\phi(T^nx,Ty)-\phi(x,Ty)\big\}+\phi(Ty,y)\\
&\quad+2\langle S_nx-Ty,JTy-Jy\rangle\\
&\quad+\frac{2}{n}\langle\beta_1(T^{n+1}x+T^nx-Tx-x)+\beta_2(T^nx-x),JTy-Jy\rangle\\
&\quad-\frac{1}{n}\gamma_1\big\{\phi(Ty,T^{n+1}x)+\phi(Ty,T^nx)-\phi(Ty,Tx)-\phi(Ty,x)\big\}\\
&\quad-\frac{1}{n}\gamma_2\big\{\phi(Ty,T^nx)-\phi(Ty,x)\big\}\\
&\quad+\frac{1}{n}\delta_1\big\{\phi(y,T^{n+1}x)\phi(y,T^nx)-\phi(y,Tx)-\phi(y,x)\big\}\\
&\quad+\frac{1}{n}\delta_2\big\{\phi(y,T^nx)-\phi(y,x)\big\},
\end{aligned}$$

where $S_nx=\frac{1}{n}\sum_{k=0}^{n-1}T^kx$. Since $\{T^nx\}$ is bounded by assumption, $\{S_nx\}$ is bounded. Thus we have a subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$ such that $\{S_{n_i}x\}$ converges weakly to a point $u\in C$. Letting $n_i\to\infty$ in the above inequality, we obtain

$$0\le\phi(Ty,y)+2\langle u-Ty,JTy-Jy\rangle.$$

Putting $y=u$, we obtain

$$\begin{aligned}
0 &\le \phi(Tu,u)+2\langle u-Tu,JTu-Ju\rangle\\
&=\phi(Tu,u)+\phi(u,u)+\phi(Tu,Tu)-\phi(u,Tu)-\phi(Tu,u)\\
&=-\phi(u,Tu).
\end{aligned}$$

Hence we have $\phi(u, Tu) \leq 0$ and then $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. This completes the proof. □

Theorem 5.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. Since the identity mapping I is a sunny generalized nonexpansive retract of E onto E, E is a nonempty closed convex sunny generalized nonexpansive retract of E. We also know that $\alpha > \beta$, together with $\gamma \leq \delta$, implies that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. So, we have the desired result from Theorem 5.1. □

Theorem 5.4 (see [10]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since C is a nonempty closed convex subset of H, there exists the metric projection of H onto C. In a Hilbert space, the metric projection of H onto C is equivalent to the sunny generalized nonexpansive retraction of E onto C. On the other hand, a generalized hybrid mapping $T : C \to C$ with $F(T) \neq \emptyset$ is quasi-nonexpansive; i.e.,

$$\phi(Tx, u) = \|Tx - u\|^2 \leq \|x - u\|^2 = \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. So, we have the desired result from Theorem 5.1. □

Remark. We do not know whether a nonlinear ergodic theorem of Baillon's type for nonspreading mappings holds or not.

6. Weak convergence theorems

In this section, we prove a weak convergence theorem of Mann's iteration for generalized nonspreading mappings in a Banach space. For proving it, we need the following two lemmas. The following lemma was obtained by Takahashi and Yao [28].

Lemma 6.1. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = R_C(\alpha_n x_n + (1 - \alpha_n)Tx_n) \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C. Then $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$, where $R_{F(T)}$ is a sunny generalized nonexpansive retraction of C onto $F(T)$.

From Lemma 2.2, we also have the following result. For the sake of completeness, we give the proof.

Lemma 6.2. *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - abg(\|x - y\|)$$

for all $x, y \in B_r$ and $a, b, c \geq 0$ with $a + b + c = 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Proof. If $a + b = 0$, then

$$\begin{aligned}\|ax + by + cz\|^2 &= \|cz\|^2 = c^2\|z\|^2 \leq c\|z\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - abg(\|x - y\|).\end{aligned}$$

If $a + b > 0$, then we have from Lemma 2.2 that

$$\begin{aligned}\|ax + by + cz\|^2 &= \left\|(a+b)\left(\frac{a}{a+b}x + \frac{b}{a+b}y\right) + cz\right\|^2 \\ &\leq (a+b)\left\|\frac{a}{a+b}x + \frac{b}{a+b}y\right\|^2 + c\|z\|^2 \\ &\quad - (a+b)cg\left(\left\|\frac{a}{a+b}x + \frac{b}{a+b}y - z\right\|\right) \\ &\leq (a+b)\left\|\frac{a}{a+b}x + \frac{b}{a+b}y\right\|^2 + c\|z\|^2 \\ &\leq (a+b)\Bigg(\frac{a}{a+b}\|x\|^2 + \frac{b}{a+b}\|y\|^2 \\ &\qquad\qquad - \frac{a}{a+b}\frac{b}{a+b}g(\|x - y\|)\Bigg) + c\|z\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - \frac{ab}{a+b}g(\|x - y\|) \\ &\leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - abg(\|x - y\|).\end{aligned}$$

This completes the proof. □

Using Lemmas 6.1 and 6.2, and the technique developed by [6], we prove the following theorem.

Theorem 6.3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a 2-generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $0 < a \leq a_n, b_n, c_n \leq b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Proof. Let $m \in F(T)$. By the assumption, we know that T is a generalized nonexpansive mapping of C into itself. So, we have

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(a_n x_n + b_n T x_n + c_n T^2 x_n, m)\\ &\leq a_n \phi(x_n, m) + b_n \phi(T x_n, m) + c_n \phi(T^2 x_n, m)\\ &\leq a_n \phi(x_n, m) + b_n \phi(x_n, m) + c_n \phi(x_n, m)\\ &= \phi(x_n, m).\end{aligned}$$

Thus $\lim_{n\to\infty} \phi(x_n, m)$ exists. Then we have that $\{x_n\}$ is bounded. This implies that $\{Tx_n\}$ and $\{T^2 x_n\}$ are bounded. Put

$$r = \sup_{n\in\mathbb{N}} \left\{ \|x_n\|, \|Tx_n\|, \|T^2 x_n\| \right\}.$$

Using Lemma 6.2, we have that

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(a_n x_n + b_n T x_n + c_n T^2 x_n, m)\\ &\leq \|a_n x_n + b_n T x_n + c_n T^2 x_n\|^2\\ &\quad - 2\langle a_n x_n + b_n T x_n + c_n T^2 x_n, Jm\rangle + \|m\|^2\\ &\leq a_n \|x_n\|^2 + b_n \|Tx_n\|^2 + c_n \|T^2 x_n\|^2\\ &\quad - a_n b_n g(\|Tx_n - x_n\|) - 2a_n \langle x_n, Jm\rangle\\ &\quad - 2b_n \langle Tx_n, Jm\rangle - 2c_n \langle T^2 x_n, Jm\rangle + \|m\|^2\\ &= a_n(\|x_n\|^2 - 2\langle x_n, Jm\rangle + \|m\|^2)\\ &\quad + b_n(\|Tx_n\|^2 - 2\langle Tx_n, Jm\rangle + \|m\|^2)\\ &\quad + c_n(\|T^2 x_n\|^2 - 2\langle T^2 x_n, Jm\rangle + \|m\|^2) - a_n b_n g(\|Tx_n - x_n\|)\\ &= a_n \phi(x_n, m) + b_n \phi(Tx_n, m) + c_n \phi(T^2 x_n, m)\\ &\quad - a_n b_n g(\|Tx_n - x_n\|)\\ &\leq a_n \phi(x_n, m) + b_n \phi(x_n, m) + c_n \phi(x_n, m)\\ &\quad - a_n b_n g(\|Tx_n - x_n\|)\\ &= \phi(x_n, m) - a_n b_n g(\|Tx_n - x_n\|).\end{aligned}$$

Then we obtain that

$$a_n b_n g(\|Tx_n - x_n\|) \leq \phi(x_n, m) - \phi(x_{n+1}, m).$$

From the assumptions of $\{a_n\}$ and $\{b_n\}$, we have

$$\lim_{n\to\infty} g(\|Tx_n - x_n\|) = 0. \tag{6.1}$$

Similarly, we have that

$$\begin{aligned}
\phi(x_{n+1}, m) &= \phi(a_n x_n + b_n Tx_n + c_n T^2 x_n, m) \\
&\leq \|a_n x_n + b_n Tx_n + c_n T^2 x_n\|^2 \\
&\quad - 2\langle a_n x_n + b_n Tx_n + c_n T^2 x_n, Jm\rangle + \|m\|^2 \\
&\leq a_n\|x_n\|^2 + b_n\|Tx_n\|^2 + c_n\|T^2 x_n\|^2 - a_n c_n g(\|T^2 x_n - x_n\|) \\
&\quad - 2a_n\langle x_n, Jm\rangle - 2b_n\langle Tx_n, Jm\rangle - 2c_n\langle T^2 x_n, Jm\rangle + \|m\|^2 \\
&\leq a_n\phi(x_n, m) + b_n\phi(x_n, m) + c_n\phi(x_n, m) - a_n c_n g(\|T^2 x_n - x_n\|) \\
&\leq \phi(x_n, m) - a_n c_n g(\|T^2 x_n - x_n\|).
\end{aligned}$$

Then we obtain that

$$a_n c_n g(\|T^2 x_n - x_n\|) \leq \phi(x_n, m) - \phi(x_{n+1}, m).$$

From the assumptions of $\{a_n\}$ and $\{c_n\}$, we have

$$\lim_{n\to\infty} g(\|T^2 x_n - x_n\|) = 0. \tag{6.2}$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. We have from (6.1), (6.2) and Proposition 4.2 that v is a fixed point of T. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We know that $u, v \in F(T)$. Put $a = \lim_{n\to\infty}(\phi(x_n, u) - \phi(x_n, v))$. Since

$$\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2,$$

we have $a = 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain

$$\langle u - v, Ju - Jv\rangle = 0.$$

Since E is strictly convex, it follows that $u = v$. Therefore, $\{x_n\}$ converges weakly to an element u of $F(T)$. On the other hand, we know from Lemma 6.1 that $\{R_{F(T)} x_n\}$ converges strongly to an element z of $F(T)$. From Lemma 2.5, we also have

$$\langle x_n - R_{F(T)} x_n, JR_{F(T)} x_n - Ju\rangle \geq 0.$$

So, we have $\langle u - z, Jz - Ju\rangle \geq 0$. Since J is monotone, we also have $\langle u - z, Jz - Ju\rangle \leq 0$. So, we have $\langle u - z, Jz - Ju\rangle = 0$. Since E is strictly convex, we have $z = u$. This completes the proof. □

Using Theorem 6.3, we can prove the following weak convergence theorems; see also [24].

Theorem 6.4. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading mapping such that $\alpha_1 = \beta_1$, $\alpha_2 > \beta_2$, $\gamma_1 \leq \delta_1$ and $\gamma_2 \leq \delta_2$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $0 < a \leq a_n, b_n, c_n \leq b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} R x_n$.

Proof. Since the identity mapping I is a sunny generalized nonexpansive retract of E onto E, E is a nonempty closed convex sunny generalized nonexpansive retract of E. We also know that $\alpha_1 = \beta_1$ and $\alpha_2 > \beta_2$ together with $\gamma_1 \leq \delta_1$ and $\gamma_2 \leq \delta_2$ imply that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. So, we have the desired result from Theorem 6.3. □

Theorem 6.5 (see [16]**).** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a 2-generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $0 < a \leq a_n, b_n, c_n \leq b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} P x_n$.

Proof. Since C is a nonempty closed convex subset of H, there exists the metric projection of H onto C. In a Hilbert space, the metric projection of H onto C is equivalent to the sunny generalized nonexpansive retraction of E onto C. On the other hand, a 2-generalized hybrid mapping $T : C \to C$ with $F(T) \neq \emptyset$ is quasi-nonexpansive; i.e.,

$$\phi(Tx, u) = \|Tx - u\|^2 \leq \|x - u\|^2 = \phi(x, u)$$

for all $x \in C$ and $u \in F(T)$. So, we have the desired result from Theorem 6.3. □

Remark. We do not know whether a weak convergence theorem of Mann's type for nonspreading mappings holds or not.

Acknowledgements

The first author was partially supported by Grant-in-Aid for Scientific Research No. 23540188 from Japan Society for the Promotion of Science. The second and the third authors were partially supported by the grant NSC 99-2115-M-110-007-MY3 and the grant NSC 99-2115-M-037-002-MY3, respectively.

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