On the solution existence of generalized vector quasi-equilibrium problems with discontinuous multifunctions[1]

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Abstract. In this paper we deal with the following generalized vector quasi-equilibrium problem: given a closed convex set K in a normed space X, a subset D in a Hausdorff topological vector space Y, and a closed convex cone C in R^n. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. Find a point $(\hat{x}, \hat{y}) \in K \times D$ such that

$$(\hat{x}, \hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), \{f(\hat{x}, \hat{y}, z) : z \in \Gamma(\hat{x})\} \cap (-\mathrm{Int}C) = \emptyset.$$

We prove some existence theorems for the problem in which Φ may be discontinuous and K may be unbounded.

Keywords: Solution existence, generalized vector quasi-equilibrium problem, implicit generalized quasivariational inequality, lower semicontinuity, upper semicontinuity, Hausdorff lower semicontinuity, $C-$convex, C-lower semicontinuity, $C-$upper semicontinuity.

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1 Introduction

Throughout this paper, C is a closed convex cone in R^n such that $\text{Int}C \neq \emptyset$ and $C \neq R^n$, where $\text{Int}C$ denotes the interior of C. Let X and Y be a Hausdorff topological vector space, $K \subseteq X$ and $D \subseteq Y$ be nonempty sets. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. The generalized vector quasi-equilibrium is the problem of finding $(\hat{x}, \hat{y}) \in K \times D$ such that

$$(\hat{x}, \hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), \{f(\hat{x}, \hat{y}, z) : z \in \Gamma(\hat{x})\} \cap (-\text{Int}C) = \emptyset. \tag{1}$$

The problem will denoted by $\text{P}(K, \Gamma, \Phi, f)$ ((P) for short). We denote by Sol(P) the solution set of (P).

It is noted that $P(K, \Gamma, \Phi, f)$ covers several generalized quasivariational inequalities and generalized vector equilibrium problems. Here are some of them.

(A) If $n = 1$, $C = R_+$ then (P) reduces to the implicit quasivariational inequality problem: find $\hat{x} \in K$ and $\hat{y} \in \Phi(\hat{x})$ such that

$$\hat{x} \in \Gamma(\hat{x}) \text{ and } f(\hat{x}, \hat{y}, z) \geq 0 \ \forall z \in \Gamma(\hat{x}). \tag{2}$$

(B) If $\Gamma(x) = K$ for all $x \in K$ then (P) reduces to the generalized vector equilibrium problems: find $\hat{x} \in K$ and $\hat{y} \in \Phi(\hat{x})$ such that

$$\{f(\hat{x}, \hat{y}, z) : z \in K\} \cap (-\text{Int}C) = \emptyset. \tag{3}$$

(C) If $n = 1$, $C = R_+$, $Y = X^* = D$ and $f(x, y, z) = \langle y, z - x\rangle$ then (P) reduces to the generalized quasivariational inequality problem: find $\hat{x} \in K$ and $\hat{y} \in \Phi(\hat{x})$ such that

$$\hat{x} \in \Gamma(\hat{x}) \text{ and } \langle \hat{y}, z - \hat{x}\rangle \geq 0 \ \forall z \in \Gamma(\hat{x}). \tag{4}$$

The solution existence of (2), (3) and (4) has become a basic research topic which continues to attract of researchers in applied mathematics. We refer the readers to [3]-[13], [15]- [20], [26]- [34] and references given therein for recent results on the solution existence of (2), (3) and (4) with discontinuous multifunctions.

Since the generalized vector quasi-equilibrium problem covers many class of variational inequalities and vector equilibrium problems, it can be seen as an efficient model to study the solution existence of these class in a unitary form.

The aim of this paper is to derive some solution existence theorems for (P) with discontinuous multifunctions. Namely, we will establish some existence theorems in which Φ may not be continuous and K may be unbounded. Under certain conditions our results extend the results in [6], [7], [10]-[12] and some preceding results. In order to obtain the results we first reduce problem (P) to problem (1) by the scalarization method and we then use solution existence theorems in [18] to establish our results.

The rest of the paper consists of two sections. In section 2 we recall some auxiliary results and the scalariation method. Section 3 is devoted to main results.

2 Auxiliary results

Let C be a closed convex cone in R^n. A single-valued mapping $g : X \to R^n$ is called C-*upper semicontinuous* ($C-$u.s.c., for short) on X if for every $z \in Z$ the set $g^{-1}(z - \mathrm{Int}C)$ is open in X (see [27]). In [27], Tanaka proved that g is $C-$u.s.c. on X if and only if for each fixed $x \in X$ and for any $y \in \mathrm{Int}C$, there exists a neighborhood U of x such that $g(u) \in g(x) + y - \mathrm{Int}C$ for all $u \in U$. Also, g is said to be $C-$ *lower semicontinuous* ($C-$l.s.c., for short) on X if for each fixed $x \in X$ and for any $y \in \mathrm{Int}C$, there exists a neighborhood V of x such that $g(x) - y \in g(v) - \mathrm{Int}C$ for all $v \in V$.

Let K be a nonempty convex subset in X. A single-valued mapping $h : K \to Z$ is called C-*convex* if for every $x, x' \in K$ and $t \in [0, 1]$ one has

$$th(x) + (1 - t)h(x') - h(tx + (1 - t)x') \in C.$$

If $-h$ is $C-$ *convex* then h is said to be $C-$ *concave* on K.

For each cone C, the set

$$C^* := \{z^* \in R^n : \langle z^*, z\rangle \geq 0 \text{ for all } z \in C\}$$

is said to be the polar cone of C. Note that C^* has a compact base B^*, that is, $C^* = \bigcup_{t>0} tB^*$ where $B^* \subset C^*$ is convex and compact with $0 \notin B^*$ (see [21]). When $\mathrm{Int}C \neq \emptyset$ and $\overline{z} \in \mathrm{Int}C$, $\overline{z} \neq 0$ the the set

$$B^* = \{z^* \in C^* : \langle z^*, \overline{z}\rangle = 1\}$$

is a compact convex base for C^*. Put $C^*_+ = C^* \setminus \{0\}$. ¿From the bipolar theorem (see, e.g., [15]), we have

$$z \in C \Longleftrightarrow [\langle z^*, z\rangle \geq 0 \ \forall z^* \in C^*] \Longleftrightarrow [\langle z^*, z\rangle \geq 0 \ \forall z^* \in B] \qquad (5)$$

and

$$z \in \mathrm{Int}C \Longleftrightarrow [\langle z^*, z\rangle > 0 \ \forall z^* \in C^*_+] \Longleftrightarrow [\langle z^*, z\rangle > 0 \ \forall z^* \in B]. \qquad (6)$$

The following lemma gives us an useful tool of the scalarization procedure.

Lemma 2.1 *Let g be a single-valued mapping from K into Z and $u^* \in C^*_+$. Let $\phi : K \to R$ be a mapping defined by $\phi(x) = \langle u^*, g(x)\rangle$ for all $x \in K$. Then the following assertions are valid:*

(a) *If g is $C-$convex then ϕ is convex ;*

(b) *If g is $C-$concave then ϕ is concave;*

(c) *If g is $C-$u.s.c. then ϕ u.s.c;*

(d) *If g is $C-$l.s.c. then ϕ is l.s.c.*

Proof. Since g is $C-$convex, then for all $x, x' \in K$ and $t \in [0, 1]$ one has

$$tg(x) + (1 - t)g(x') - g(tx + (1 - t)x') \in C.$$

By (5) we have $\langle u^*, tg(x) + (1 - t)g(x') - g(tx + (1 - t)x')\rangle \geq 0$. Hence

$$t\langle u^*, g(x)\rangle + (1 - t)\langle u^*, g(x')\rangle \geq \langle u^* g(tx + (1 - t)x')\rangle.$$

This implies that

$$t\phi(x) + (1 - t)\phi(x') \geq \phi(tx + (1 - t)x').$$

Hence we obtain (a). The proof of (b) is similar to the proof of (a).

For the assertion (c) we assume that $x_n \to x$. We shall prove that $\limsup_{n\to\infty} \phi(x_n) \leq \phi(x)$. Choose $y_j \in \text{Int}C$ such that $y_j \to 0$. Then for each $j > 0$ there exists a neighborhood U_j of x such that

$$g(u) \in g(x) + y_j - \text{Int}C \ \forall u \in U_j.$$

Therefore for each j there exists $n_j > 0$ such that

$$g(x_n) \in g(x) + y_j - \text{Int}C \ \forall n > n_j.$$

By (6) it follows that $\langle u^*, g(x_n) - g(x) - y_j\rangle < 0$. Hence

$$\begin{aligned}\phi(x_n) &= \langle u^*, (g(x_n) - g(x) - y_j + g(x) + y_j\rangle \\ &= \langle u^* g(x_n) - g(x) - y_j\rangle + \langle u^*, (g(x) + y_j\rangle \\ &< \langle u^*, g(x)\rangle + \ \langle u^*, y_j\rangle\end{aligned}$$

for all $n > n_j$. This implies that $\limsup_{n\to\infty}\langle \phi(x_n) \leq \langle u^*, g(x)\rangle + \langle u^* y_j\rangle$. By letting $j \to \infty$ and noting that $\langle u^*, y_j\rangle \to 0$ we obtain

$$\limsup_{n\to\infty} \phi(x_n) \leq \langle u^*, g(x)\rangle = \phi(x).$$

The proof of assertion (d) is similar to (c). □

Recall that a multifunction $\Gamma : X \to 2^E$ from a normed space X into a normed space E is said to be lower semicontinuous (l.s.c., for short) at $\overline{x} \in X$ if for any open set V in E satisfying $V \cap \Gamma(\overline{x}) \neq \emptyset$, there exists a neighborhood U of $\overline{x}$ such that $V \cap \Gamma(x) \neq \emptyset$ for all $x \in U$. Γ is said to be Hausdorff l.s.c., at $\overline{x} \in K$ if for any $\epsilon > 0$, there exists a neighborhood W of $\overline{x}$ such that

$$\Gamma(\overline{x}) \subset \Gamma(x) + \epsilon B \text{ for all } x \in W.$$

Here B is the unit open ball in E.

We now return to problem (2). By using the Michael continuous selection theorem, in [18] we obtained the following result.

Lemma 2.2 (C.f. [18, Theorem 3.1]) *Let $X = R^m$, K be a convex compact set in X and D be a nonempty set in Y. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R$ be a single-valued mapping. Assume the following conditions are fulfilled:*

(i) *Γ is l.s.c. with nonempty convex values on K and the set $M = \{x \in K : x \in \Gamma(x)\}$ is closed;*

(ii) *the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in M$;*

(iii) *for each $z \in K$, the set $\{x \in M \mid \sup_{y \in \Phi(x)} f(x, y, z) \geq 0\}$ is closed;*

(iv) *for each $x \in M$, the set $\{z \in K \mid \sup_{y \in \Phi(x)} f(x, y, z) \geq 0\}$ is closed;*

(v) *for each $x \in M$ there exists $y \in \Phi(x)$ such that $f(x, y, x) = 0$;*

(vi) *for each $x \in M$ and $y \in \Phi(x)$, the function $f(x, y, .)$ is convex and l.s.c.;*

(vii) *for each $x \in M$ and $z \in \Gamma(x)$, the function $f(x, ., z)$ is concave and u.s.c.*

Then there exists $(\hat{x}, \hat{y}) \in K \times D$ such that

$$(\hat{x}, \hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), f(\hat{x}, y, z) \geq 0 \ \forall z \in \Gamma(\hat{x}). \tag{7}$$

3 Existence results

In this section we keep all notations in preceding sections and assume that $f : K \times D \times K \to R^n$ defined by

$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), \dots, f_n(x, y, z)),$$

where $f_i : K \times D \times K \to R,\ i = 1, 2, ..., n$ are scalar functions. For each $\xi \in C_+^*$ we consider the following problem.

(P_ξ) Find $(\hat{x}, \hat{y}) \in K \times D$ such that

$$(\hat{x}, \hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), \langle \xi, f(\hat{x}, \hat{y}, z) \rangle \geq 0 \ \forall z \in \Gamma(\hat{x}). \tag{8}$$

We denote by $\mathrm{Sol}(P_\xi)$ the solution set of problem P_ξ.

The following result gives a relation between Sol(P) and $\mathrm{Sol}(P_\xi)$.

Lemma 3.1 (a)

$$\bigcup_{\xi \in C_+^*} \mathrm{Sol}(P_\xi) \subset \mathrm{Sol}(P). \tag{9}$$

(b) *If* Γ *has convex values and* $f(x, y, \cdot)$ *is* $C-$*strongly convex for each* $(x, y) \in M \times \Phi(x)$*, i.e.,*

$$tf(x, y, z_1) + (1-t)f(x, y, z_2) \in f(x, y, tz_1 + (1-t)z_2) + \mathrm{Int}C \cup \{0\}$$

for all $z_1, z_2 \in K$ *and* $t \in [0, 1]$*, then*

$$\bigcup_{\xi \in C_+^*} \mathrm{Sol}(P_\xi) = \mathrm{Sol}(P).$$

Proof. (a) Suppose that $(\hat{x}, \hat{y})$ belongs to the left hand side of (9). Then there exists $\xi \in C_+^*$ such that (8) holds. By (6) we have

$$f(\hat{x}, \hat{y}, z) \notin -\mathrm{Int}C \ \forall z \in \Gamma(\hat{x}).$$

This means that

$$\{f(\hat{x}, \hat{y}, z) : z \in \Gamma(\hat{x})\} \cap (-\mathrm{Int}C) = \emptyset.$$

Hence$(\hat{x}, \hat{y}) \in \mathrm{Sol}(P)$ and so $\bigcup_{\xi \in C_+^*} \mathrm{Sol}(P_\xi) \subset \mathrm{Sol}(P)$.

(b) Taking any $(\hat{x}, \hat{y}) \in \mathrm{Sol}(P)$, we have $(\hat{x}, \hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x})$ and

$$\{f(\hat{x}, \hat{y}, z) : z \in \Gamma(\hat{x})\} \cap (-\mathrm{Int}C) = \emptyset.$$

This implies that

$$\{f(\hat{x}, \hat{y}, z) + c : (z, c) \in \Gamma(\hat{x}) \times \mathrm{Int}C\} \cap (-\mathrm{Int}C) = \emptyset.$$

We want to check that the set

$$Q := \{f(\hat{x}, \hat{y}, z) + c : (z, c) \in \Gamma(\hat{x}) \times \mathrm{Int}C\}$$

is convex. Indeed, taking any $u, v \in Q$, we have $u = f(\hat{x}, \hat{y}, z_1) + c_1$ and $v = f(\hat{x}, \hat{y}, z_2) + c_2$ for some $(z_1, c_1), (z_2, c_2) \in \Gamma(\hat{x}) \times \text{Int}C$. Hence for each $t \in [0, 1]$, $tu + (1-t)v = tf(\hat{x}, \hat{y}, z_1) + (1-t)f(\hat{x}, \hat{y}, z_2) + tc_1 + (1-t)c_2$. Since $f(\hat{x}, \hat{y}, \cdot)$ is $C-$strongly convex, $tf(\hat{x}, \hat{y}, z_1) + (1-t)f(\hat{x}, \hat{y}, z_2) = f(\hat{x}, \hat{y}, tz_1 + (1-t)z_2) + c_3$ for some $c_3 \in \text{Int}C \cup \{0\}$. Consequently,

$$tu + (1-t)v = f(\hat{x}, \hat{y}, tz_1 + (1-t)z_2) + c,$$

where $c := tc_1 + (1-t)c_2 + c_3 \in \text{Int}C$. This implies that $tu + (1-t)v \in Q$. Thus Q is a convex set. According to the separation theorem of convex sets (see [14, Theorem 1, p. 163]), there exists a nonzero functional ξ such that

$$\langle \xi, f(\hat{x}, \hat{y}, z) + c \rangle \geq \langle \xi, u \rangle$$

for all $(z, c) \in \Gamma(\hat{x}) \times \text{Int}C$ and $u \in -\text{Int}C$. This implies that $\xi \in C^*_+$ and

$$\langle \xi, f(\hat{x}, \hat{y}, z) \rangle \geq 0 \ \forall z \in \Gamma(\hat{x}).$$

Hence $(\hat{x}, \hat{y}) \in \text{Sol}(P_\xi)$ and so $\text{Sol}(P) \subseteq \bigcup_{\xi \in C^*_+} \text{Sol}(P_\xi)$. Combining this with (9), we obtain the desired conclusion. The proof is complete. □

Lemma 3.1 suggests us that in order to prove the solution existence of problem (P) it is necessary to prove the solution existence of (P_ξ) for some $\xi \in C^*_+$. By this way we obtain the following existence result for the case of finite dimensional spaces.

Theorem 3.1 *Let $X = R^m$, K be a closed convex set in X, K_0 be a nonempty bounded set in K and D be a nonempty set in Y. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. Assume that there exists $\xi \in C^*_+$ such that the following conditions are fulfilled:*

(i) *Γ is l.s.c. with nonempty convex values on K and the set $M = \{x \in K : x \in \Gamma(x)\}$ is closed;*

(ii) *the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in M$;*

(iii) *for each $z \in K$, the set $\{x \in M \mid \sup_{y \in \Phi(x)} \langle \xi, f(x, y, z) \rangle \geq 0\}$ is closed;*

(iv) *for each $x \in M$, the set $\{z \in K \mid \sup_{y \in \Phi(x)} \langle \xi, f(x, y, z) \rangle \geq 0\}$ is closed;*

(v) *for each $x \in M$ and for each $y \in \Phi(x)$ such that $f(x, y, x) = 0$;*

(vi) *for each $x \in M$ and $y \in \Phi(x)$, the function $f(x, y, .)$ is $C-$convex and l.s.c.;*

(vii) *for each* $x \in M$ *and* $z \in \Gamma(x)$*, the function* $f(x,.,z)$ *is* $C-$*concave and u.s.c.;*

(viii) $\Gamma(x) \cap K_0 \neq \emptyset$ *for all* $x \in K$*, for each* $x \in M \backslash K_0$ *there exists* $z \in \Gamma(x) \cap K_0$ *such that* $f(x,y,z) \in -\mathrm{Int}C$ *for all* $y \in \Phi(x)$*.*

Then there exists $\hat{x} \in \Gamma(\hat{x})$ *such that*

$$\max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x},y,z) \rangle \geq 0 \ \forall z \in \Gamma(\hat{x}) \qquad (10).$$

Moreover there exists $\hat{y} \in \Phi(\hat{x})$ *such that* $(\hat{x},\hat{y})$ *is a solution of* $\mathrm{P}(K,\Gamma,f,\Phi)$*.*

Proof. Take $r > 0$ such that $K_0 \subset \mathrm{int}B_r$ where B_r is the closed ball in R^m with radius r and center at 0. We put $\Omega_r = K \cap B_r$ and define the multifunction $\Gamma_r : \Omega_r \to 2^{\Omega_r}$ by $\Gamma_r(x) = \Gamma(x) \cap B_r$ and $\phi : K \times D \times K \to R$ by $\phi(x,y,z) = \langle \xi, f(x,y,z) \rangle$. According to Lemma 3.1 in [34], Γ_r is l.s.c. on Ω_r. Put

$$\Phi_r = \Phi \mid_{\Omega_r}, \phi_r = \phi \mid_{\Omega_r \times D \times \Omega_r} .$$

By (vi) and Lemma 2.1, $\phi(x,y,\cdot)$ is convex and l.s.c. Also, $\phi(x,\cdot,z)$ is concave and u.s.c. Hence the components $\Omega_r, \Gamma_r, \Phi_r$ and ϕ_r meet all conditions of Lemma 2.2. By this lemma, there exists $(\hat{x},\hat{y}) \in \Gamma_r(\hat{x}) \times \Phi_r(\hat{x})$ such that

$$\phi_r(\hat{x},\hat{y},z) \geq 0 \ \forall z \in \Gamma_r(\hat{x}).$$

Since $\Phi_r(\hat{x}) = \Phi(\hat{x})$ and $\phi_r(\hat{x},\hat{y},z) = \phi(\hat{x},\hat{y},z)$ we get

$$(\hat{x},\hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), \phi(\hat{x},\hat{y},z) \geq 0 \ \forall z \in \Gamma_r(\hat{x}). \qquad (11)$$

We now claim that

$$\phi(\hat{x},\hat{y},z) \geq 0 \ \forall z \in \Gamma(\hat{x}). \qquad (12)$$

In fact, from (viii) we get $\hat{x} \in K_0$. Take any $z \in \Gamma(\hat{x})$ then $(1-t)\hat{x} + tz \in \Gamma(\hat{x}) \cap B_r$ for a sufficiently small $t \in (0,1)$. Hence (11) implies

$$\phi(\hat{x},\hat{y},(1-t)\hat{x} + tz) \geq 0.$$

By (vi) and Lemma 2.1 we have

$$\begin{aligned} 0 \leq \phi(\hat{x},\hat{y},t\hat{x} + (1-t)z) &\leq t\phi(\hat{x},\hat{y},\hat{x}) + (1-t)\phi(\hat{x},\hat{y},z) \\ &= 0 + (1-t)\phi(\hat{x},\hat{y},z). \end{aligned}$$

This implies (12). It is obvious that (12) implies (10). ¿From (12) and Lemma 3.1, we have

$$\{f(\hat{x},\hat{y},z) : z \in \Gamma(\hat{x})\} \cap (-\mathrm{Int}C) = \emptyset.$$

Consequently, $(\hat{x}, \hat{y})$ is a solution of the problem. The proof is complete. $\square$

When $C = R^n_+ := \{(x_1, x_2, \ldots, x_n) \in R^n : x_i \geq 0, i = 1, 2, \ldots, n\}$ then $C^* = C$ and $\text{Int}C = \{(x_1, x_2, \ldots, x_n) \in R^n : x_i > 0, i = 1, 2, \ldots, n\}$. In this case we have

Corollary 3.1 *Let $X = R^m$, K be a closed convex set in X, K_0 be a nonempty bounded set in K and D be a nonempty set in Y. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. Assume that there exists an index i, $1 \leq i \leq n$ such that the following conditions are fulfilled:*

(i) *Γ is l.s.c. with nonempty convex values on K and the set $M = \{x \in K : x \in \Gamma(x)\}$ is closed;*

(ii) *the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in M$;*

(iii) *for each $z \in K$, the set $\{x \in M \mid \sup_{y \in \Phi(x)} f_i(x, y, z) \geq 0\}$ is closed;*

(iv) *for each $x \in M$, the set $\{z \in K \mid \sup_{y \in \Phi(x)} f_i(x, y, z) \geq 0\}$ is closed;*

(v) *for each $x \in M$ and for each $y \in \Phi(x)$ such that $f(x, y, x) = 0$;*

(vi) *for each $x \in M$ and $y \in \Phi(x)$, the function $f(x, y, .)$ is $C-$convex and l.s.c.;*

(vii) *for each $x \in M$ and $z \in \Gamma(x)$, the function $f(x, ., z)$ is $C-$concave and u.s.c.*

(viii) *$\Gamma(x) \cap K_0 \neq \emptyset$ for all $x \in K$, for each $x \in M \backslash K_0$ there exists $z \in \Gamma(x) \cap K_0$ such that $f(x, y, z) \in -\text{Int}C$ for all $y \in \Phi(x)$.*

Then problem $\text{P}(K, \Gamma, f, \Phi)$ has a solution $(\hat{x}, \hat{y}) \in K_0 \times D$.

Proof. For the proof we put $\xi = (0, 0, \ldots, \xi_i \ldots, 0, 0)$, where ξ_i is the ith component of ξ and $\xi_i = 1$. It easy to see that $\xi \in C^*_+$ and conditions of Theorem 3.1 are satisfied. The conclusion follows directly from Theorem 3.1. $\square$

Let us give an illustrative example for Theorem 3.1.

Example 3.1 Let $X = R$, $K = [0, 1] \subset X$, $Y = R$, $D = [1, 4]$ and

$$C = R^2_+ = \{(x, y) \mid x \geq 0, y \geq 0\}.$$

Let Γ, Φ and f be defined by:

$$\Gamma(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ (0, 1] & \text{if } 0 < x \leq 1, \end{cases}$$

$$\Phi(x) = \begin{cases} [2,4] & \text{if } x = 0 \\ \{1\} & \text{if } 0 < x \leq 1, \end{cases}$$

$f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$, where $f_1(x,y,z) = y(z^2 - x^2)$, $f_2(x,y,z) = y(z^4 - x^4)$. Then the set $\{0\} \times [2,4]$ is a solution set of $\mathrm{P}(K,\Gamma,\Phi,f)$. Moreover Φ is not upper semicontinuous on [0, 1].

Indeed, by putting $\xi = (1,0)$ $(i = 1)$, we see that all conditions of Theorem 3.1. are fulfilled. Taking $\hat{x} = 0$ and $\hat{y} \in \Phi(0) = [2,4]$ we have $0 \in \Gamma(0)$ and

$$f(0,\hat{y},z) = (0,0) \notin -\mathrm{Int}C \ \forall z \in \Gamma(0).$$

Hence the set $\{0\} \times [2,4]$ is a solution set of the problem. Since $x_n = 1/n \to 0$ and $y_n = 1 \in \Phi(x_n)$ but $1 \notin \Phi(0)$, Φ is not u.s.c. at $x = 0$.

In the rest of this section we shall derive some existence results for the case of infinite dimensional spaces.

Theorem 3.2 *Let X be a Banach space, K be a closed convex set of X and D be a nonempty set in Y. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. Let K_1, K_2 be two nonempty compact sets of K such that $K_1 \subset K_2$, K_1 is finite dimensional and $\xi \in C_+^*$. Assume that:*

(i) *Γ is Hausdorff l.s.c. with nonempty closed graph and convex values;*

(ii) *the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in \Gamma(x)$;*

(iii) *for each $z \in K$, the set $\{x \in K \mid \sup_{y\in\Phi(x)}\langle\xi, f(x,y,z)\rangle \geq 0\}$ is compactly closed;*

(iv) *for each $x \in K$, the set $\{z \in K \mid \sup_{y\in\Phi(x)}\langle\xi, f(x,y,z)\rangle \geq 0\}$ is finitely closed;*

(v) *for each $x \in K$ and for each $y \in \Phi(x)$ such that $f(x,y,x) = 0$;*

(vi) *for each $x \in K$ and $y \in \Phi(x)$, the function $f(x,y,.)$ is $C-$convex and l.s.c.;*

(vii) *for each $x \in K$ and $z \in \Gamma(x)$, the function $f(x,.,z)$ is $C-$concave and u.s.c.*

(viii) $\mathrm{Int}_{\mathrm{aff(K)}}\Gamma(x) \neq \emptyset$;

(iv) *$\Gamma(x) \cap K_1 \neq \emptyset$ for all $x \in K$. Moreover for each $x \in K \backslash K_2$ with $x \in \Gamma(x)$ there exists $z \in \Gamma(x) \cap K_1$ such that $f(x,y,z) \in -\mathrm{Int}C$ for all $y \in \Phi(x)$.*

Then there exists a pair $(\hat{x},\hat{y}) \in K_2 \times D$ which solves $\mathrm{P}(K,\Gamma,\Phi,f)$.

Proof. The proof is based on the scheme given by [10].

Let $L = \text{aff}(K)$ and L_0 be the linear subspace corresponding to L. For each $x \in \overline{\text{co}}K_2$, there exists $z_x \in \text{Int}_L\Gamma(x)$, the interior of $\Gamma(x)$ in L which is nonempty by (viii).

The following lemma plays an important role in our arguments.

Lemma 3.2 ([9], Proposition 2.5) *Let T be a topological space, X be a nomerd space, L be an affine manifold of X, $\Gamma : T \to 2^L$ a Hausdorff lower semicontinuous multifunction with nonempty closed convex values, and $\overline{x} \in X$, $\overline{y} \in Int_L(\Gamma(\overline{x}))$. Then there exists a neighborhood U of $\overline{x}$ such that $\overline{y} \in Int_L(\Gamma(x))$ for all $x \in U$.*

By Lemma 3.2, there exists a neighborhood U_x of x in X such that $z_x \in \text{Int}_L\Gamma(u)$ for all $u \in U_x \cap K$. Since $\overline{\text{co}}K_2$ is a compact set and

$$\overline{\text{co}}K_2 \subset \bigcup_{x \in \overline{\text{co}}K_2} (U_x \cap L),$$

then there exists $x_1, x_2, ..., x_m \in \overline{\text{co}}K_2$ such that

$$\overline{\text{co}}K_2 \subset \bigcup_{i=1}^{m} [U_{x_i} \cap L].$$

Putting

$$P_0 = \bigcup_{i=1}^{m} (U_{x_i} \cap L)$$

then $P_0 \subset L$. Since $L \backslash P_0 \neq \emptyset$ and closed in L, then

$$\xi := \inf\{d(a, L\backslash P_0) : a \in \overline{\text{co}}K_2\} > 0.$$

Putting

$$P = \overline{\text{co}}K_2 + (\overline{B}(0, \xi/2) \cap L_0)$$

we have that P is a closed convex set in L and $P \subset P_0$.

Let $\mathcal{F}$ be the family of all finite-dimensional linear subspaces of X containing $K_1 \cup \{z_{x_1}, z_{x_2}, ..., z_{x_n}\}$. Fix $S \in \mathcal{F}$ and put

$$\Omega = \overline{K \cap P \cap S}, \;\; K_0 = K_2 \cap \Omega.$$

Note that $K_1 \subset K \cap P \cap S \subset \Omega \subset K \cap S$.

We next define the multifunction $\Gamma_S : \Omega \to 2^{\Omega}$ by setting

$$\Gamma_S(x) := \Gamma(x) \cap \Omega = G(x) \cap \overline{K \cap P \cap S}.$$

Put

$$\Phi_S = \Phi\mid_{\Omega},\ f_S = f\mid_{\Omega\times D\times\Omega},\ M_S = \{x\in\Omega : x\in\Gamma_S(x)\}.$$

The task is now to check that Theorem 3.1 can be applied to the problem $\mathrm{P}(\Omega, \Gamma_S, \Phi_S, f_S)$ where Ω plays a role as K in Theorem 3.1. To do this we need

Lemma 3.3 ([8], Lemma 3.3) *The multifunction $\Gamma_S : \Omega \to 2^{\Omega}$ is lower semicontinuous on Ω in the relative topology of S.*

(a_1) It is easy to see that Γ_S has a closed graph. Since

$$M_S = \{x\in\Omega : x\in\Gamma_S(x)\} = \Omega\cap\{x\in K : x\in\Gamma(x)\},$$

M_S is closed in S. Therefore condition (i) of Theorem 3.1 is satisfied.

(a_2) Condition (ii) is automatically satisfied.

(a_3) For each $z\in\Omega$ we get

$$\{x\in M_S \mid \sup_{y\in\Phi_S(x)}\langle\xi, f_S(x,y,z)\rangle \geq 0\} =$$
$$\{x\in K \mid \sup_{y\in\Phi(x)}\langle\xi, f(x,y,z)\rangle \geq 0\}\cap M_S$$

which is closed by (iii)(taking into account M_S is closed, $M_S\subset S$, S is finite-dimensional). Hence condition (iii) of Theorem 3.1 is satisfied.

(a_4) For each $x\in M_S$, we have

$$\{x\in\Omega \mid \sup_{y\in\Phi_S(x)}\langle\xi, f_S(x,y,z)\rangle \geq 0\} =$$
$$\{x\in K \mid \sup_{y\in\Phi(x)}\langle\xi, f(x,y,z)\rangle \geq 0\}\cap\Omega$$

This implies that condition (iv) of Theorem 3.1 is also satisfied.

(a_5) The conditions (v), (vi), (vii) of Theorem 3.2 are automatically fulfilled.

(a_6) Finally for each $x\in M_S\backslash K_0$, then $x\in K\backslash K_2$ and $x\in\Gamma(x)$. By condition (iv) there exists $z\in\Gamma(x)\cap K_1\subset\Gamma_S(x)$ such that $f(x,y,z) = f_S(x,y,z)\in -\mathrm{Int}C$ for all $y\in\Phi_S(x)$. Therefore condition (viii) of Theorem 3.1 is valid.

Thus all conditions of Theorem 3.1 are fulfilled. By Theorem 3.1, there exists $\hat{x}_S\in\Gamma_S(\hat{x}_S)$ such that

$$\max_{y\in\Phi_S(\hat{x}_S)}\langle\xi, f_S(\hat{x}_S, y, z)\rangle \geq 0\ \forall z\in\Gamma_S(\hat{x}_S).$$

Since $f_S(\hat{x}_S, y, z) = f(\hat{x}_S, y, z), \Phi_S(\hat{x}_S) = \Phi(\hat{x}_S)$ we get

$$\max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, z) \rangle \geq 0 \ \forall z \in \Gamma \hat{x}_S) \cap \Omega. \tag{13}$$

We now show that

$$\max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, z) \rangle \geq 0 \ \forall z \in \Gamma(\hat{x}_S) \cap S. \tag{14}$$

In fact, we fix any $z \in \Gamma(\hat{x}_S) \cap S$. Since

$$\begin{aligned} &\hat{x}_S \in K_2 \subset \overline{\text{co}} K_2 \subset K \subset L, \\ &z \in \Gamma(\hat{x}_S) \subset K \subset L, \\ &L - L \subset L_0, \end{aligned}$$

we have

$$\hat{x}_S + t(z - \hat{x}_S) \in K \cap [\overline{\text{co}} K_2 + \overline{B}(0, \xi/2) \cap L_0] = K \cap P$$

for a sufficiently small $t \in (0, 1)$. By the convexity of $\Gamma(\hat{x}_S) \cap S$ we get

$$\hat{x}_S + t(z - \hat{x}_S) \in K \cap P \cap S \cap \Gamma(\hat{x}_S) \subset \Omega \cap \Gamma(\hat{x}_S).$$

Hence (13) implies

$$\max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, \hat{x}_S + t(z - \hat{x}_S)) \rangle \geq 0. \tag{15}$$

By (iv) and using the similar argument as in the proof of Theorem 3.1, (15) implies

$$\max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, z) \rangle \geq 0.$$

Hence we obtained (14). We now consider the net $\{\hat{x}_S\}_{s \in \mathcal{F}}$, where $\mathcal{F}$ is ordered by the ordinary set inclusion $\supseteq$. By the compactness of K_2 we can assume that $\hat{x}_S \to \hat{x} \in K_2$. Since Γ has a closed graph, $\hat{x} \in \Gamma(\hat{x})$.

The following lemma will complete the proof of Theorem 3.2.

Lemma 3.4

$$\max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle \geq 0 \ \forall \ z \in Int_L \Gamma(\hat{x}). \tag{16}$$

Proof. On the contrary, suppose that that there exists $\hat{z} \in \text{Int}_L \Gamma(\hat{x})$ such that

$$\max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x}, y, \hat{z}) \rangle < 0. \tag{17}$$

By Lemma 3.2 there exists $\delta > 0$ such that

$$\hat{z} \in \mathrm{Int}_L \Gamma(x) \ \forall x \in B(\hat{x}, \delta) \cap K. \tag{18}$$

It also follows from (17) that

$$\hat{x} \in \{x \in K \mid \max_{y \in \Phi(x)} \langle \xi, f(x, y, \hat{z}) \rangle < 0.\}$$

which is open set by (iii). Therefore there exists $\alpha \in (0, \delta)$ such that

$$\max_{y \in \Phi(x)} \langle \xi, f(x, y, \hat{z}) \rangle < 0 \ \forall x \in B(\hat{x}, \alpha) \cap K. \tag{19}$$

Since $\hat{x}_S \to \hat{x}$, there exists $S_0 \in \mathcal{F}$ such that $\hat{x}_S \in B(\hat{x}, \alpha)$ for all $S \supseteq S_0$. So we can choose $S \in \mathcal{F}$ satisfying $\hat{z} \in S$ and $\hat{x}_S \in B(\hat{x}, \alpha)$. Combining this with (18), we obtain $\hat{z} \in \Gamma(\hat{x}_S) \cap S$. Hence it follows from (14) that

$$\hat{x}_S \in \Gamma(\hat{x}_S), \ \max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, \hat{z}) \rangle \geq 0. \tag{20}$$

On the other hand, (19) implies that

$$\hat{x}_S \in \Gamma(\hat{x}_S), \ \max_{y \in \Phi(\hat{x}_S)} \langle \xi, f(\hat{x}_S, y, \hat{z}) \rangle < 0$$

which contradicts to (20). The lemma is proved.

We now take any $z \in \Gamma(\hat{x}) \subset L$. Since $\Gamma(\hat{x})$ is a closed convex set in X, $\Gamma(\hat{x})$ is a closed convex set in L which is the closure of $\mathrm{Int}_L \Gamma(\hat{x})$ in L (see [2] Theorem 2, pp. 19). Hence there exists a sequence $z_n \in \mathrm{Int}_L \Gamma(\hat{x})$ such that $z_n \to z$. By Lemma 3.4 we have

$$\max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x}, y, z_n) \rangle \geq 0.$$

By letting $n \to \infty$ and using assumption (iv) yields

$$\max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle \geq 0 \ \forall \ z \in \Gamma(\hat{x}).$$

Hence

$$\inf_{z \in \Gamma(\hat{x})} \max_{y \in \Phi(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle \geq 0.$$

By the minimax theorem (see [1, Theorem 5]) we have

$$\max_{y \in \Phi(\hat{x})} \inf_{z \in \Gamma(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle \geq 0.$$

Since the function $y \mapsto \inf_{z \in \Gamma(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle$ is u.s.c., there exists a point $\hat{y} \in \Phi(\hat{x})$ such that

$$\inf_{z \in \Gamma(\hat{x})} \langle \xi, f(\hat{x}, \hat{y}, x) \rangle = \max_{y \in \Phi(\hat{x})} \inf_{z \in \Gamma(\hat{x})} \langle \xi, f(\hat{x}, y, z) \rangle \geq 0.$$

This implies that

$$\langle \xi, f(\hat{x}, \hat{y}, z\rangle \geq 0 \ \forall z \in \Gamma(\hat{x}).$$

By Lemma 3.1, $(\hat{x}, \hat{y})$ is a solution of the problem. The proof is complete. □

For the scalar case we have

Corollary 3.2 ([10], Theorem 1.2) *Let X be a real Banach space, let K be a closed convex subset of X, let $\Gamma : K \to 2^K$ and $\Phi : K \to 2^{X^*}$ be two multifunctions. Let K_1, K_2 be two nonempty compact subsets of K such that $K_1 \subset K_2$ and K_1 is finite-dimensional. Assume that:*

(i) *the set $\Phi(x)$ is nonempty, weakly-star compact for each $x \in K$, and convex for each $x \in K$, with $x \in \Gamma(x)$;*

(ii) *for each $z \in K$, the set $\{x \in K : \inf_{y\in\Phi(x)}\langle y, x - z\rangle \leq 0\}$ is compactly closed;*

(iii) *the multifunction Γ is Hausdorff l.s.c. with closed graph and convex values;*

(iv) $\Gamma(x) \cap K_1 \neq \emptyset$ *for all $x \in X$;*

(v) $int_{aff(K)}(\Gamma(x)) \neq \emptyset$ *for all $x \in K$;*

(vi) *for each $x \in K \setminus K_2$, with $x \in \Gamma(x)$, one has*

$$\sup_{z\in\Gamma(x)\cap K_1} \inf_{y\in\Phi(x)} \langle y, x - z\rangle > 0.$$

Then there exists $(\hat{x}, \hat{y}) \in K_2 \times X^*$ such that

$$\hat{x} \in \Gamma(\hat{x}),\ \hat{y} \in \Phi(\hat{x}) \text{ and } \langle \hat{y}, \hat{x} - z\rangle \leq 0 \ \forall z \in \Gamma(\hat{x}).$$

Proof. For the proof we put $f(x, y, z) = \langle y, z - x\rangle$, $D = Y = X^*$, $Z = R$ and $C = \{x \in R \mid x \geq 0\}$. Then we have $C^* = C$ and $C^*_+ = \{u \in R \mid u > 0\}$. Choose $\xi = 1$. We want to verify conditions of Theorem 3.2. It easily seen that f meets all conditions of Theorem 3.2. Since $\Phi(x)$ is a compact set, for each $z \in K$ we have

$$\{x \in K \mid \inf_{y\in\Phi(x)} \langle y, x - z\rangle \leq 0\} = \{x \in K \mid \min_{y\in\Phi(x)} \langle y, x - z\rangle \leq 0\} =$$
$$=\{x \in K \mid \max_{y\in\Phi(x)} \langle y, z - x\rangle \geq 0\}$$

which is a compactly closed set. Moreover for each $x \in K$, the set

$$\{z \in K : \inf_{y\in\Phi(x)} \langle y, x - z\rangle \leq 0\}$$

is also closed and satisfies

$$\{z \in K \mid \inf_{y\in\Phi(x)} \langle y, x-z\rangle \leq 0\} = \{z \in K \mid \min_{y\in\Phi(x)} \langle y, x-z\rangle \leq 0\} =$$
$$=\{z \in K \mid \max_{y\in\Phi(x)} \langle y, z-x\rangle \geq 0\}.$$

Therefore conditions (iii) and (iv) of Theorem 3.2 are valid.

Finally, (vi) implies that for each $x \in K \setminus K_2$ there exists $z \in \Gamma(x) \cap K_1$ such that $f(x, y, z) \in -\text{Int}C$ for all $y \in \Phi(x)$. Thus all conditions of Theorem 3.2 are fulfilled. The conclusion now follows directly from Theorem 3.2. □

Remark 3.1 In the proof of Theorem 3.2 we used Lemma 3.2 as a main tool for the arguments. In the infinite-dimensional setting, in general, a lower semicontinuous multifunction has no property described in Lemma 3.2, even if X is an Hilbert space; see remark 3.1 of [9] and the references given there.

The following theorem deals with the case where Γ is not Hausdorff lower semicontinuous and condition $\text{Int}_{\text{aff(K)}}\Gamma(x) \neq \emptyset$ can be omitted.

Theorem 3.3 *Let X be a normed space, K be a closed convex set of X and D be a nonempty set in Y. Let $\Gamma : K \to 2^K$, $\Phi : K \to 2^D$ be two multifunctions and $f : K \times D \times K \to R^n$ be a single-valued mapping. Let K_1, K_2 be two nonempty compact sets of K such that $K_1 \subset K_2$, K_1 is finite dimensional. Assume that there exists $\xi \in C^*_+$ and $\eta > 0$ such that the following conditions are fulfilled:*

(i) Γ *is l.s.c. with closed convex values and Hausdorff upper semicontinous;*

(ii) *the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each x with $d(x, \Gamma(x)) < \eta$;*

(iii) *the set $\{(x, z) \in K \times K : \sup_{y\in\Phi(x)}\langle\xi, f(x, y, z)\rangle \geq 0\}$ is closed;*

(iv) *for each $x \in K$ there exists $y \in \Phi(x)$ such that $f(x, y, x) = 0$;*

(v) *for each $x \in K$ and $y \in \Phi(x)$, the function $f(x, y, .)$ is $C-$convex and l.s.c.;*

(vi) *for each $(x, z) \in K \times K$, the function $f(x, ., z)$ is $C-$concave and u.s.c.;*

(vii) *$\Gamma(x) \cap K_1 \neq \emptyset$ for all $x \in K$. Moreover for each $x \in K\setminus K_2$ with $d(x, \Gamma(x)) < \eta$ there exists $z \in \Gamma(x) \cap K_1$ such that $f(x, y, z) \in -\text{Int}C$ for all $y \in \Phi(x)$.*

Then there exists a pair $(\hat{x}, \hat{y}) \in K \times D$ which solves P(K, Γ, Φ, f).

Proof. Define a mapping $\phi : K \times D \times K \to R$ by putting

$$\phi(x, y, z) = \langle\xi, f(x, y, z)\rangle.$$

We now apply a existence result of problem (2) to $\mathrm{P}_\xi(K,\Gamma,\Phi,\phi)$. By Theorem 3.3 in [18], there exists $(\hat{x},\hat{y}) \in K \times D$ such that

$$(\hat{x},\hat{y}) \in \Gamma(\hat{x}) \times \Phi(\hat{x}), \phi(\hat{x},\hat{y},z) \geq 0 \ \forall z \in \Gamma(\hat{x}).$$

By lemma 3.1, $(\hat{x},\hat{y})$ is a solution of $\mathrm{P}(K,\Gamma,\Phi,f)$. $\Box$

Remark 3.2 In Theorem 3.1 and Theorem 3.2, conditions (iii) and (iv) are verified via a functional $\xi \in C_+^*$. One of main difficulties is to find such functionals. Under certain conditions, says, if D is compact, Φ is upper semicontinuous and the function $(x,y) \mapsto f(x,y,z)$ is $C-$ upper continuous, then we can choose any $\xi \in C_+^*$. However example 2.1 revealed that although Φ is not u.s.c., there exists $\xi \in C_+^*$ under which conditions (iii) and (iv) are fulfilled. Besides, Lemma 2.1 showed that under suitable conditions the solution existence of P_ξ is necessary for the solution existence of (P). It is natural to know if we can prove the solution existence of (P) without P_ξ. Namely, one may ask whether the conclusion of Theorem 3.1 and Theorem 3.2 are still valid if conditions (iii) and (iv) are replaced by the following conditions:

(iii)' *for each* $z \in K$, *the set* $\{x \in M \mid \exists y \in \Phi(x), f(x,y,z) \notin -\mathrm{Int}C\}$ *is closed;*

(iv)' *for each* $x \in M$, *the set* $\{z \in K \mid \exists y \in \Phi(x), f(x,y,z) \notin -\mathrm{Int}C\}$ *is closed.*

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