HAHN-BANACH-KANTOROVICH TYPE THEOREMS WITH THE RANGE SPACE NOT NECESSARILY (O)-COMPLETE

RODICA-MIHAELA DĂNEŢ AND NGAI-CHING WONG

Abstract. In the classical Hahn-Banach-Kantorovich theorem, the range space Y is Dedekind complete. In this paper, by extending the arguments of the original Hahn-Banach-Kantorovich theorem and using an idea of Y. A. Abramovich and A. W. Wickstead, we can weaken the order theoretic assumption on Y and obtain more general results in the settings of Banach lattices as well as ordered linear spaces.

1. Introduction

In the operator version of the Hahn-Banach-Kantorovich theorem, the range space Y is assumed to be Dedekind complete. This assumption can be considerably relaxed by using a weaker interpolation property, the so-called Cantor property on Y. Some generalizations of this type were given by H. B. Cohen [3], J. Lindenstrauss [9] and G. Buskes [2]. In particular, Y. A. Abramovich and A. W. Wickstead [1] provided us the following

Theorem 1 ([1]). *Let X and Y be Banach lattices such that X is separable and Y has the Cantor property. Let $P : X \to Y_+$ be a continuous seminorm. If G is a linear subspace of X and $T : G \to Y$ is a continuous linear operator satisfying $T(v) \leq P(v)$ for all v in G then there exists a continuous extension S of T to the whole of X such that $S(x) \leq P(x)$ for all x in X.*

In this paper, we obtain two new results along the line. The first one states that any positive linear operator from a majorizing subspace of a separable Banach lattice into a Banach lattice with the Cantor property can be extended. The second one states that any (o)-continuous linear operator from a subspace of an ordered linear space with (os)-property into an ordered linear space with the strong (σ)-interpolation property dominated by an (o)-continuous seminorm can also be extended.

1991 *Mathematics Subject Classification.* 47B60, 46A22.

Key words and phrases. Ordered linear space, Banach lattice, Hahn-Banach-Kantorovich type theorems, Cantor property, strong (σ)-interpolation property.

2. Preliminaries

As far as the linear-order-theoretical terminology is concerned, we mostly follow Cristescu's book [4]. In particular, an ordered linear space X is said to have the (os)-*property* if there exists a countable subset D of X such that for each x in X there is a sequence $\{x_n\}_n$ in D with $x_n \stackrel{o}{\to} x$. A linear subspace G of X is a *majorizing subspace* if for every x in X there exists a v in G with $x \leq v$. Consequently, there also exists a u in G such that $u \leq x$.

Definition. Let Y be an ordered linear space. Y is said to have the *Cantor property* (or the (σ)-*interpolation property* or the *countable property*) if for every increasing sequence $\{x_n\}_n$ and every decreasing sequence $\{z_m\}_m$ in Y with $x_n \leq z_m$, $\forall\ n,m \in \mathbb{N}$, there is a y in Y such that $x_n \leq y \leq z_m$, $\forall\ n,m \in \mathbb{N}$. Y is said to have the *strong* (σ)-*interpolation property* if for every pair of sequences $\{x_n\}_n$ and $\{z_m\}_m$ in Y with $x_n \leq z_m$, $\forall\ n,m \in \mathbb{N}$, there is a y in Y such that $x_n \leq y \leq z_m, \forall\, n,m \in \mathbb{N}$. In case Y is a vector lattice, these two notions coincide.

G. Seever [10] showed that for a completely regular space K, $C(K)$ has the Cantor property if and only if K is an F-space, i.e. every pair of disjoint open (F_σ)-sets in K has disjoint closures. C. B. Huijsmans and B. De Pagter [8] showed that an Archimedean vector lattice Y has the Cantor property if and only if Y is uniformly complete and normal. In general, for a vector lattice we have: Dedekind completeness implies Dedekind (σ)-completeness implies Cantor property implies order completeness implies uniform completeness (see e.g. [12, p. 696]).

In case Y is a Banach lattice, A. W. Wickstead [11] proved that the following are all equivalent: (1) Y has the Cantor property; (2) The space of all regular operators from convergent sequences into Y has the strong (σ)-interpolation property; (3) The space of all regular operators from convergent sequences into Y has the Riesz decomposition property. More recently, N. Dăneţ [6] showed that they are also equivalent to: $(3')$ The space of all regular operators from any separable Banach lattice into Y has the Riesz decomposition property.

3. Main results

We start with a Kantorovich-type theorem concerning the extension of a positive linear operator. Note that every positive linear operator from a majorizing subspace of a Banach lattice into a Banach lattice is continuous.

Theorem 2 *Let X be a separable Banach lattice, G a majorizing subspace of X, and Y a Banach lattice with the Cantor property. If $T : G \to Y$ is a positive linear operator then there exists a positive linear operator $S : X \to Y$ such that $S(v) = T(v), \forall v \in G$.*

Proof. Let $x_0 \in X \setminus G$ and G_1 the linear hull of $G \cup \{x_0\}$. We will extend T to G_1. Because G is a majorizing subspace of X we can choose u, v from G such that $u \leq x_0 \leq v$. Since the operator T is positive we have

$$T(u) \leq T(v). \tag{1}$$

Let W be the nonempty set of all such u, v in G. Since X is separable, there exists a countable dense subset D of W. In particular, the inequality (1) holds for any u, v in D with $u \leq x_0 \leq v$. By the Cantor property of Y we can find a y_0 in Y satisfying

$$T(u) \leq y_0 \leq T(v), \text{ for all } u, v \in D, \ u \leq x_0 \leq v.$$

Since T is continuous, the last double inequality remains true for all u, v in G with $u \leq x_0 \leq v$. Now, letting $T_1(x_0) = y_0$ we obtain a desired extension of T, namely $T_1 : G_1 \to Y$, defined by

$$T_1(v + \lambda x_0) = T(v) + \lambda y_0.$$

Obviously G_1 is again a majorizing subspace of X. Moreover, $T_1 : G_1 \to Y$ is positive. Indeed, let $v + \lambda x_0 \geq 0$ with $\lambda \neq 0$. If $\lambda > 0$ then $x_0 \geq -\frac{1}{\lambda}v$ and this implies $y_0 \geq T(-\frac{1}{\lambda}v) = -\frac{1}{\lambda}T(v)$. Therefore, $T_1(x_0) \geq -\frac{1}{\lambda}T(v)$, and thus $T_1(v + \lambda x_0) \geq 0$. If $\lambda < 0$ we get the same result.

Finally, a routine application of Zorn's lemma will finish the proof. □

Recall that an *axial element* is an e in X_+ such that for each x in X there exists $\lambda > 0$ satisfying $x \leq \lambda e$.

Corollary 3 *Let X and Y be Banach lattices such that X is separable and contains an axial element e and Y has the Cantor property. Then for each y_0 in Y_+ there exists a positive linear operator $U : X \to Y$ with $U(e) = y_0$.*

Proof. Because e is an axial element of X, the linear hull $G = Sp(e)$ is a majorizing subspace of X. We define $T : G \to Y$ by $T(\lambda e) = \lambda y_0$ and then apply Theorem 2. □

Before stating another corollary of Theorem 2, we remark that any linear subspace G of an ordered linear space X containing an element in the interior $\operatorname{Int} X_+$ of the positive cone X_+ of X is majorizing. Moreover, any positive linear operator from X into an ordered linear space Y vanishing in a majorizing subspace is necessarily zero.

Corollary 4 *Let X be a separable Banach lattice with $\operatorname{Int} X_+ \neq \emptyset$, and Y a Banach lattice with the Cantor property. Then for any linear subspace G of X disjoint from $\operatorname{Int} X_+$, there exists a non-zero positive linear operator $U : X \to Y$ with $U|_G = 0$.*

Proof. We choose an element x_0 from $\operatorname{Int} X_+$ and denote by G_0 the linear hull of $G \cup \{x_0\}$. It follows that G_0 is a majorizing subspace of X. Define $T_0 : G_0 \to Y$ by $T_0(v + \lambda x_0) = \lambda y_0$ for some fixed element y_0 in Y_+.

Let us prove that T_0 is positive. Let $v \in G$ and $\lambda \neq 0$ such that $v + \lambda x_0 \geq 0$. Suppose that $\lambda < 0$. Then $-\lambda x_0 \in \operatorname{Int} X_+$ and hence $v = v + \lambda x_0 + (-\lambda x_0) \in \operatorname{Int} X_+$. This conflicts with the hypothesis that $G \cap \operatorname{Int} X_+ = \emptyset$. So $\lambda > 0$ and hence $T_0(v + \lambda x_0) = \lambda y_0 \geq 0$. By Theorem 2 we can extend T_0 to a positive linear operator $U : X \to Y$. Obviously $U|_G = 0$. □

The following results supplement Theorem 1. The first appears without proof in [7].

Theorem 5 *Suppose X and Y are ordered linear spaces, G is a linear subspace of X with the (os)-property, and Y has the strong (σ)-interpolation property. Let $T : G \to Y$ be an (o)-continuous linear operator and $P : X \to Y_+$ an (o)-continuous seminorm such that $T(v) \leq P(v)$ for all v in G. Then for any x_0 in $X \setminus G$ we can extend T to an (o)-continuous linear operator $T_1 : G_1 = Sp(G \cup \{x_0\}) \to Y$ such that $T_1(z) \leq P(z)$ for all z in G_1.*

Proof. Because G has the (os)-property, there exists a countable subset D of G such that, for each v in G, there is a sequence $(v_n)_{n\in\mathbb{N}}$ in D with $v_n \xrightarrow{o} v$. If $u, v \in G$ then

$$T(u) - T(v) = T(u-v) \leq P(u-v) =$$
$$= P((u+x_0) - (v+x_0)) \leq P(u+x_0) + P(v+x_0).$$

So

$$-P(v+x_0) - T(v) \leq P(u+x_0) - T(u), \text{ for all } u, v \in G. \tag{2}$$

In particular, the inequality holds for all u, v in D. Using the strong (σ)-interpolation property of Y we find a y_0 in Y such that

$$-P(v+x_0) - T(v) \leq y_0 \leq P(u+x_0) - T(u), \text{ for all } u, v \in D. \tag{3}$$

But T and P are (o)-continuous and hence the inequalities (3) hold for all u, v in G. Now, by letting

$$T_1(v + \lambda x_0) = T(v) + \lambda y_0$$

we obtain a linear extension of T to G_1.

It remains to show that $T_1(v + \lambda x_0) \leq P(v + \lambda x_0)$ for all v in G and λ in $\mathbb{R}$, or equivalently,

$$T(v) + \lambda y_0 \leq P(v + \lambda x_0) \text{ for all } v \in G \text{ and } \lambda \in \mathbb{R}. \tag{4}$$

If $\lambda = 0$, the inequality (4) is valid because $T_1 = T \leq P$ on G. If $\lambda > 0$, using the right inequality in (3), for $\frac{1}{\lambda}v$ instead of u, we obtain

$$y_0 \leq P(\frac{1}{\lambda}v + x_0) - T(\frac{1}{\lambda}v) = \frac{1}{\lambda}\left[P(v+\lambda x_0) - T(v)\right].$$

Therefore,

$$T(v) + \lambda y_0 \leq P(v + \lambda x_0).$$

If $\lambda < 0$, we use the left inequality in (3) to establish (4) instead.

Being dominated by the (o)-continuous seminorm P, the extension T_1 of T is (o)-continuous as well. □

Corollary 6 *Suppose in Theorem 5, in addition, every linear subspace of X has the (os)-property. Then there exists an (o)-continuous linear operator $S : X \to Y$ such that $S(v) = T(v)$ for all v in G, and $S(x) \leq P(x)$ for all x in X.*

Proof. It follows from Theorem 5 and an application of Zorn's lemma. □

References

[1] Y. A. Abramovich and A. W. Wickstead, The regularity of order bounded operators into C(K), II, *Quart J. Math. Oxford (2)* **44** (1993), 257-270.

[2] G. Buskes, Extension of Riesz homomorphisms, Thesis, *Univ. of Nijmegen*, 1983.

[3] H. B. Cohen, The K-norm extension property for Banach spaces, *Proc. Amer. Math. Soc.* **15** (1964), 797-802.

[4] R. Cristescu, Ordered Vector Spaces and linear operators, Ed. Acad. Abacus Press Kent, England, 1976.

[5] R. Cristescu, Order structures in normed vector spaces (in Romanian), Ed. Şt. şi enciclopedică, Bucuresti, 1983.

[6] N. Dăneţ, The Riesz decomposition property for the space of regular operators, *Proc. Amer. Math. Soc.* (to appear).

[7] R. Dăneţ, Extension theorems with the range having an interpolation property, in *Seminaire d'espaces linéaires ordonnés topologiques*, Univ. of Bucharest, **17** (1998), 97-98.

[8] C. B. Huljsmans and B. de Pagter, On z-ideals and d-ideals in Riesz Spaces II, *Indag. Math.* **42** (1980), 391-408.

[9] J. Lindenstrauss, On the extension of operators with range in a C(K)-space, *Proc. Amer. Math. Soc.* **15** (1964), 218-224.

[10] G. L. Seever, Measures of F-spaces, *Trans. Amer. Math. Soc.* **193** (1968), 267-280.

[11] A. W. Wickstead, Spaces of operators with Riesz separation property, *Indag. Math. N.S. (2)* **6** (1995), 235-245.

[12] A. C. Zaanen, Riesz spaces II, North-Holland Mathematical Library **30**, North-Holland Publishing Co., Amsterdam-New York, 1983.

Department of Mathematics, Technical University of Civil Engineering of Bucharest, 122-124, Lacul Tei Blvd., 72302 Bucharest 38, Romania.

E-mail address: `ndanet@fx.ro`

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan, R.O.C.

E-mail address: `wong@math.nsysu.edu.tw`