ATTRACTIVE POINTS AND HALPERN'S TYPE STRONG CONVERGENCE THEOREMS IN HILBERT SPACES

WATARU TAKAHASHI, NGAI-CHING WONG, AND JEN-CHIH YAO

Abstract. In this paper, using the concept of attractive points of a nonlinear mapping, we obtain a strong convergence theorem of Halpern's type [6] for a wide class of nonlinear mappings which contains nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. Using this result, we obtain well-known and new strong convergence theorems of Halpern's typ in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi [16].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $T : C \to H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$-*generalized hybrid* mapping. The class of generalized hybrid mappings covers many well-known mappings. For example, a (1, 0)-generalized hybrid mapping T is nonexpansive, i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [14, 15] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Furthermore, it is hybrid [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. In 1992, Wittmann [27] proved the following strong convergence theorem of Halpern's type [6] in a Hilbert space; see also [20].

Theorem 1.1. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

2000 *Mathematics Subject Classification.* 47H10, 47H05, 47H09.

Key words and phrases. Attractive point, fixed point, generalized hybrid mapping, Halpern's type iteration, nonexpansive mapping.

Then $\{x_n\}$ *converges strongly to* $P_{F(T)}x$*, where* $P_{F(T)}$ *is the metric projection of* H *onto* $F(T)$*.*

Kurokawa and Takahashi [16] also proved the following strong convergence theorem for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [7] for generalized hybrid mappings.

Theorem 1.2. *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H*. Let* T *be a nonspreading mapping of* C *into itself. Let* $u \in C$ *and define two sequences* $\{x_n\}$ *and* $\{z_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots,$ *where* $\{\alpha_n\} \subset (0,1)$*,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. If* $F(T)$ *is nonempty, then* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* Pu*, where* P *is the metric projection of* H *onto* $F(T)$*.*

We do not know whether a strnog convergence theorem of Halpern's type (Theorem 1.1) for nonspreading mappings holds or not; see [16] and [7]. Very recently, Takahashi and Takeuchi [23] introduced the concept of *attractive points* of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type [3] without convexity for generalized hybrid mappings. Akashi and Takahash [1] also proved a strong convergence theorem of Halpern's type for nonexpansive mappings on star-shaped sets in a Hilbert space. However, they used essentially the properties of nonexpansiveness in the proof.

In this paper, motivated by [27], [16], [7], [23] and [1], we obtain a strong convergence theorem of Halpern's type for finding attractive points of generalized hybrid mappings in a Hilbert space. Using this result, we obtain well-known and new strong convergence theorems of Halpern's type in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi [16].

2. Preliminaries and Lemmas

Let H be a real Hilbert space. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We know that for $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore, we know that for all $x, y, z, w \in H$,

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2. \tag{2.3}$$

Let D be a closed and convex subset of H. For every $x \in H$, there exists a unique nearest point in D denoted by $P_D x$, that is, $\|x - P_D x\| \le \|x-y\|$ for every $y \in D$. This mapping P_D is called the *metric projection* of H onto D. It is known that P_D is firmly nonexpansive, that is, the following hold:

$$0 \le \langle x - P_D x, P_D x - y\rangle \quad \text{and} \quad \|x - P_D x\|^2 + \|P_D x - y\|^2 \le \|x-y\|^2$$

for any $x \in H$ and $y \in D$; see [19, 20, 21]. Let C be a nonempty subset of H. For a mapping T of C into H, we denote by $F(T)$ the set of all *fixed points* of T and by $A(T)$ the set of all *attractive points* of T, i.e.,

(1) $F(T) = \{z \in C : z = Tz\}$;
(2) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

Takahashi and Takeuchi [23] proved the following useful lemma.

Lemma 2.1. *Let C be a nonempty subset of H and let T be a mapping from C into H. Then, $A(T)$ is a closed and convex subset of H.*

We note that a mapping $T : C \to H$ in Lemma 2.1 can not be nonexpansive. The following lemma was also proved by Takahashi and Takeuchi [23].

Lemma 2.2. *Let C be a nonempty subset of H and let T be a generalized hybrid mapping from C into itself. Suppose that there exists an $x \in C$ such that $\{T^n x\}$ is bounded. Then $A(T) \neq \emptyset$.*

To prove our main result, we need two lemmas.

Lemma 2.3 ([17]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \dots$ and $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}, \ \forall n \geq n_0$.

Lemma 2.4 ([2]; see also [26]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.

3. Strong Convergence Theorem of Halpern's Type

In this section, we prove a strong convergence theorem of Halpern's type [6] for finding attractive points of generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $T : C \to H$ be a generalized hybrid mapping. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A(T)$.*

Proof. Since $T : C \to H$ is generalized hybrid, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. Suppose that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Replacing x by x_n in (3.1), we have that

$$\alpha\|Tx_n - Ty\|^2 + (1 - \alpha)\|x_n - Ty\|^2 \leq \beta\|Tx_n - y\|^2 + (1 - \beta)\|x_n - y\|^2.$$

From this inequality,

$$\begin{aligned}&\alpha(\|Tx_n - x_n\|^2 + \|x_n - Ty\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + (1-\alpha)\|x_n - Ty\|^2 \\ &\quad \le \beta(\|Tx_n - x_n\|^2 + \|x_n - y\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + (1-\beta)\|x_n - y\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\alpha(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + \|x_n - Ty\|^2 \\ &\quad \le \beta(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + \|x_n - y\|^2.\end{aligned}$$

Since $\{x_n\}$ is bounded and $x_n - Tx_n \to 0$, we have that

$$\limsup_{n\to\infty} \|x_n - Ty\|^2 \le \limsup_{n\to\infty} \|x_n - y\|^2.$$

Since $\|x_n - Ty\|^2 = \|x_n - y\|^2 + \|y - Ty\|^2 + 2\langle x_n - y, y - Ty\rangle$ and $x_n \rightharpoonup z$, we also have that

$$\limsup_{n\to\infty} \|x_n - y\|^2 + \|y - Ty\|^2 + 2\langle z - y, y - Ty\rangle \le \limsup_{n\to\infty} \|x_n - y\|^2$$

and hence

$$\|y - Ty\|^2 + 2\langle z - y, y - Ty\rangle \le 0.$$

Using (2.3), we have that

$$\|y - Ty\|^2 + \|z - Ty\|^2 - \|z - y\|^2 - \|y - Ty\|^2 \le 0$$

and hence

$$\|z - Ty\|^2 - \|z - y\|^2 \le 0$$

for all $y \in C$. This implies $z \in A(T)$. This completes the proof. □

Now we prove a strong convergence theorem of Halpern's type for finding attractive points of generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let H be a Hilbert space and let C be a convex subset of H. Let T be a generalized hybrid mapping from C into itself with $A(T) \ne \emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $z \in C$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{A(T)}z$.

Proof. Let $x_1 \in C$ and $u \in A(T)$. Put $M = \|x_1 - u\| + \|z - u\|$. Define $z_n = \beta_n x_n + (1-\beta_n)Tx_n$. Then we have from (2.2) that

$$\begin{aligned}\|z_n - u\|^2 &= \beta_n\|x_n - u\|^2 + (1-\beta_n)\|Tx_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\le \beta_n\|x_n - u\|^2 + (1-\beta_n)\|x_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\le \|x_n - u\|^2.\end{aligned} \tag{3.2}$$

It is obvious that $\|x_1 - u\| \leq M$. Suppose that $\|x_k - u\| \leq M$ for some $k \in \mathbb{N}$. Then we have from (3.2) that

$$\begin{aligned}\|x_{k+1} - u\| &= \|\alpha_k z + (1-\alpha_k)z_k - u\| \\ &\leq \alpha_k\|z-u\| + (1-\alpha_k)\|z_k - u\| \\ &\leq \alpha_k\|z-u\| + (1-\alpha_k)\|x_k - u\| \\ &\leq \alpha_k M + (1-\alpha_k)M \\ &= M.\end{aligned}$$

By mathematical induction, we have that $\|x_n - u\| \leq M$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. We have from $u \in A(T)$ that $\|Tx_n - u\| \leq \|x_n - u\|$ and hence $\{Tx_n\}$ is also bounded. Take $\bar{x} = P_{A(T)}z$. We have from (3.2) that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\|z-\bar{x}\|^2 + (1-\alpha_n)\|z_n - \bar{x}\|^2 \\ &\leq \alpha_n\|z-\bar{x}\|^2 + (1-\alpha_n)(\|x_n-\bar{x}\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2) \\ &\leq \alpha_n\|z-\bar{x}\|^2 + \|x_n-\bar{x}\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2.\end{aligned} \tag{3.3}$$

We have from (3.3) that

$$\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \leq \alpha_n\|z-\bar{x}\|^2 + \|x_n-\bar{x}\|^2 - \|x_{n+1}-\bar{x}\|^2. \tag{3.4}$$

We also have that

$$\|x_{n+1} - x_n\| \leq \alpha_n\|z - x_n\| + (1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|. \tag{3.5}$$

Case A: Put $\Gamma_n = \|x_n - \bar{x}\|^2$ for all $n \in \mathbb{N}$. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. It follows from $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and (3.4) that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.6}$$

From $\lim_{n\to\infty}\alpha_n = 0$, (3.5) and (3.6), we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle z-\bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle z-\bar{x}, x_{n_i} - \bar{x}\rangle. \tag{3.8}$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup v$. By (3.6) and Lemma 3.1, we have that $v \in A(T)$. We have from (3.8) that

$$\limsup_{n\to\infty}\langle z-\bar{x}, x_n - \bar{x}\rangle = \langle z-\bar{x}, v-\bar{x}\rangle \leq 0. \tag{3.9}$$

On the other hand, since $x_{n+1} - \bar{x} = \alpha_n(z-\bar{x}) + (1-\alpha_n)(z_n - \bar{x})$, we have from (2.1) and (3.2) that

$$\begin{aligned}&\|x_{n+1} - \bar{x}\|^2 \leq (1-\alpha_n)\|z_n-\bar{x}\|^2 + 2\alpha_n\langle z-\bar{x}, x_{n+1}-\bar{x}\rangle \\ &\leq (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle z-\bar{x}, x_{n+1}-\bar{x}\rangle \\ &= (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle z-\bar{x}, x_{n+1}-x_n\rangle + 2\alpha_n\langle z-\bar{x}, x_n-\bar{x}\rangle.\end{aligned} \tag{3.10}$$

By $\sum_{n=1}^{\infty}\alpha_n = \infty$, (3.7), (3.9), (3.10) and Lemma 2.4, we have $\lim_{n\to\infty}x_n = \bar{x}$.

Case B: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then it follows from Lemma 2.3 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. We have from (3.4) that

$$\begin{aligned} &\beta_{\tau(n)}(1-\beta_{\tau(n)})\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\ &\qquad \leq \alpha_{\tau(n)}\|z-\bar{x}\|^2 + \|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \\ &\qquad \leq \alpha_{\tau(n)}\|z-\bar{x}\|^2. \end{aligned} \tag{3.11}$$

By $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and (3.11), we have that

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{3.12}$$

We have from (3.10) that

$$\|x_{\tau(n)+1} - \bar{x}\|^2 \leq (1-\alpha_{\tau(n)})\|x_{\tau(n)} - \bar{x}\|^2 + 2\alpha_{\tau(n)}\langle z-\bar{x}, x_{\tau(n)+1} - \bar{x}\rangle. \tag{3.13}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, (3.13) and $\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned} \|x_{\tau(n)} - \bar{x}\|^2 &\leq 2\langle z-\bar{x}, x_{\tau(n)+1} - \bar{x}\rangle \\ &= 2\langle z-\bar{x}, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\langle z-\bar{x}, x_{\tau(n)} - \bar{x}\rangle. \end{aligned} \tag{3.14}$$

By $\lim_{n\to\infty} \alpha_n = 0$, (3.5) and (3.12), we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.15}$$

Since $\{x_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{x_{\tau(n_i)}\}$ such that

$$\limsup_{n\to\infty} \langle z-\bar{x}, x_{\tau(n)} - \bar{x}\rangle = \lim_{i\to\infty} \langle z-\bar{x}, x_{\tau(n_i)} - \bar{x}\rangle. \tag{3.16}$$

Following the same argument as the proof of Case A for $\{x_{\tau(n_i)}\}$, we have that

$$\limsup_{n\to\infty} \langle z-\bar{x}, x_{\tau(n)} - \bar{x}\rangle \leq 0. \tag{3.17}$$

Using (3.14), (3.15) and (3.17), we have that

$$\lim_{n\to\infty} \|x_{\tau(n)} - \bar{x}\| = 0. \tag{3.18}$$

By (3.15) we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - \bar{x}\| = 0. \tag{3.19}$$

Using Lemma 2.3 for (3.19) again, we have that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

This completes the proof. ∎

4. Applications

In this section, using Theorem 3.2, we establish well-known and new strong convergence theorems of Halpern's type in a Hilbert space. We first prove a strong convergence theorem of Halpern's type for finding fixed points of generalized hybrid mappings, which is related to Wittmann's theorem (Theorem 1.1).

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$. Let $z \in C$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\bar{x} \in F(T)$, where $\bar{x} = P_{F(T)}z$.

Proof. Since T is generalised hybrid, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x-y\|^2 \tag{4.1}$$

for all $x, y \in C$. Replacing x by a fixed point u of T, we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1-\alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1-\beta)\|u-y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an $(\alpha,\ \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Thus we have that if $u \in F(T)$, then

$$\|Ty - u\| \leq \|y - u\|$$

for all $y \in C$. This implies that $F(T) \subset A(T)$. Then we have that $A(T)$ is nonempty. From Theorem 3.2, it follows that $\{x_n\}$ converges strongly to $\bar{x} \in A(T)$. Since C is closed and $x_n \to \bar{x}$, we have $\bar{x} \in C$. From $\bar{x} \in A(T) \cap C$, we have that

$$\|T\bar{x} - \bar{x}\| \leq \|\bar{x} - \bar{x}\| = 0$$

and hence $\bar{x} \in F(T)$. Furthermore, we have that

$$\|z - \bar{x}\| = \min\{\|z-u\| : u \in A(T)\} \leq \min\{\|z-u\| : u \in F(T)\}$$

and hence $\bar{x} = P_{F(T)}z$. This completes the proof. ∎

As direct consequences of Theorems 3.2 and 4.1, we have the following results.

Theorem 4.2. *Let H be a Hilbert space and let C be a convex subset of H. Let T be a nonexpansive mapping from C into itself, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Assume $A(T) \neq \emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $z \in C$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{A(T)}z$. *Additionally, if* C *is closed and convex, then* $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(T)}z$.

Proof. Since a (1, 0)-generalized hybrid mapping T is nonexpansive, we have the desired result from Theorems 3.2 and 4.1. □

Theorem 4.3. *Let* H *be a Hilbert space and let* C *be a convex subset of* H. *Let* T *be a nonspreading mapping from* C *into itself, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Assume $A(T) \neq \emptyset$ *and let* $P_{A(T)}$ *be the metric projection of* H *onto* $A(T)$. *Let* $z \in C$ *and let* $\{x_n\}$ *be a sequence in* C *defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n z + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0, 1)$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{A(T)}z$. *Additionally, if* C *is closed and convex, then* $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(T)}z$.

Proof. Since a (2, 1)-generalized hybrid mapping T is nonspreading, we have the desired result from Theorems 3.2 and 4.1. □

Theorem 4.4. *Let* H *be a Hilbert space and let* C *be a convex subset of* H. *Let* T *be a hybrid mapping from* C *into itself, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Assume $A(T) \neq \emptyset$ *and let* $P_{A(T)}$ *be the metric projection of* H *onto* $A(T)$. *Let* $z \in C$ *and let* $\{x_n\}$ *be a sequence in* C *defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n z + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0, 1)$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

Then $\{x_n\}$ *converges strongly to* $\bar{x} = P_{A(T)}z$. *Additionally, if* C *is closed and convex, then* $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(T)}z$.

Proof. Since a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping T is nonspreading, we have the desired result from Theorems 3.2 and 4.1. □

Theorem 4.3 solves a problem posed by Kurokawa and Takahashi [16]. We know that a nonspreading mapping is not continuous in general. In fact, we can give the following example [10] of nonspreading mappings in a Hilbert space. Let H be a real Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx = \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D. \end{cases}$$

Then the mapping S is a nonspreading mapping which is not continuous.

Acknowledgements. The first author was partially supported by Grant-in-Aid for Scientific Research No. 23540188 from Japan Society for the Promotion of Science. The second and the third authors were partially supported by the grant NSC 99-2115-M-110-007-MY3 and the grant NSC 99-2115-M-037-002-MY3, respectively.

References

[1] S. Akashi and W. Takahashi, *Strong convergence theorem for nonexpansive mappings on star-shaped sets in Hilbert spaces*, Appl. Math. Comput., to appear.
[2] K. Aoyama, Y. Kimura, W.Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.
[3] J.-B. Baillon. *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511-1514.
[4] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.
[5] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
[6] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
[7] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces*, Sci. Math. Jpn. **73** (2011), 31–40.
[8] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **71** (2009), 21–32.
[9] S. Iemoto and W. Takahashi, *Approximating fixed points of nonexpansive mappings and non-spreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.
[10] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, to appear.
[11] S. Itoh and W. Takahashi, *Single-valued mappings, Multivalued mappings and fixed-point theorems*, J. Math. Anal. Appl. **59** (1977), 514–521.
[12] S. Itoh and W. Takahashi, *The common fixed points theory of singlevalued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.
[13] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
[14] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM. J. Optim. **19** (2008), 824–835.
[15] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.
[16] Y. Kurokawa and W. Takahashi, *Weak and strong convergence theorems for nonlspreading mappings in Hilbert spaces*, Nonlinear Anal. **73** (2010), 1562–1568.
[17] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Anal. **16** (2008), 899-912.
[18] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.
[19] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[20] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000.
[21] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[22] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[23] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.
[24] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[25] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.
[26] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.
[27] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.

(Wataru Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: wong@math.nsysu.edu.tw

(Jen-Chih Yao) Center for General Education, Kaohsiung Medical University, Kaohsiung 80702, Taiwan
E-mail address: yaojc@kmu.edu.tw