Asymptotic behavior for retarded parabolic equations with superlinear perturbations

Tran Dinh Ke* and Ngai-Ching Wong†

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*Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam; email: ketd@hn.vnn.vn

†Corresponding author. Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan; email: wong@math.nsysu.edu.tw

Abstract. We obtain the existence and uniqueness of solutions for a class of retarded parabolic equations with superlinear perturbations. The asymptotic behavior result is studied by using the pullback attractor framework.

Key words. Retarded equation; semilinear parabolic equation; superlinear perturbation; non-autonomous; pullback attractor.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We deal with the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = F(u, u_t) + h(x,t), \quad x \in \Omega, t > \tau, \\ \\ u(x,\tau) = u_0(x), \quad u(x, \tau+\theta) = \phi(x,\theta), \; x \in \Omega, \quad \theta \in (-\rho, 0). \end{cases} \tag{1}$$

Here A is a self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, $h \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, $\tau \in \mathbb{R}$ and $F(u, u_t)$ has the following form

$$F(u, u_t) = g(u_t) - f(u), \tag{2}$$

in which the functions $f, g \in C(\mathbb{R})$ satisfying

(F1) There exists $C_f > 0$ such that $|f(u)| \leqslant C_f(1+|u|^{p-1})$, $p > 2$, for all $u \in \mathbb{R}$,

(F2) $f(u)u \geqslant C_0|u|^p - C_1$ for some $C_0, C_1 > 0$, for all $u \in \mathbb{R}$,

(F3) $(f(u) - f(v))(u - v) \geqslant -\ell(u-v)^2$ for some $\ell > 0$ and for all $u, v \in \mathbb{R}$,

(G1) $g : L^2(-\rho, 0; L^2(\Omega)) \to L^2(\Omega)$, $g(0) = 0$ and

$$\|g(\xi) - g(\eta)\|_{L^2(\Omega)} \leqslant C_g\|\xi - \eta\|_{L^2(-\rho,0;L^2(\Omega))}$$

for $C_g > 0$ and for all $\xi, \eta \in L^2(-\rho, 0; L^2(\Omega))$,

(G2) There exist $M_0, M_1 > 0$ such that

$$|(g(\xi), \eta)_{L^2(\Omega)}| \leqslant M_0\|\xi\|_{L^2(-\rho,0;L^2(\Omega))}\|\eta\|_{L^2(\Omega)} + M_1$$

for all $(\xi, \eta) \in L^2(-\rho, 0; L^2(\Omega)) \times L^2(\Omega)$.

Throughout this work, for $t \in \mathbb{R}$, u_t stands for the function in $L^2(-\rho, 0; L^2(\Omega))$ such that $u_t(\theta) = u(t + \theta)$.

Retarded differential equations arise in many realistic models of problems in science and engineering where there is a time lag or after-effect. In particular, the parabolic case represents some issues in mathematical biology and the time lags are often seen as maturation time for population dynamics. Let us introduce some relevant literatures in J.K. Hale [1, 2], V.B. Kolmanovskii and A.D. Myshkis [3], Y. Hino et al. [4], R. P. Agarwal et al. [5], L. H. Erbe et al. [6]. In the last decade, by the growth of theory for finite and infinite dynamical systems, there are many studies for a wide class of evolution equations, for which the asymptotic behavior of solutions is considered in different frameworks. The theory of global attractor has been developed originally to deal with autonomous evolution equations (see e.g. [7, 8]). In the latter, to treat the non-autonomous equations, some approaches have been proposed such as the uniform attractor, trajectory attractor (see [9, 10, 11]), pullback attractor (see, for instance, [12, 13]). Recently, the retarded partial differential equations have become a remarkable subject due to physical and biological motivations and some natural extensions, in addition to the rich history of retarded ordinary differential equations. We refer the readers to [14, 15, 16, 17], among others. As far as we know, the existence and longtime behavior of solutions to retarded semilinear parabolic equations have been analyzed when the nonlinearity contains the retarded term only. In this work, we study a model in which the nonlinearity has both retarded term and superlinear perturbation as in (2). On the other hand, we employ the analysis of pullback attractor to investigate the long-time behavior for our problem.

The rest of the paper is organized as follows. In the next section, we study the existence and uniqueness of solution for our problem. Section 3 is devoted to the results on long-time behavior of solutions. In the last section, we discuss on some special cases when the operator A may be in degenerate forms and some classes of nonlinearities.

2 Existence and Uniqueness Results

By the assumptions of A, we see that $A : D(A) \to L^2(\Omega)$ has discrete spectrum that only contains positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\{e_k\}_{k=1}^{\infty}$ composed an orthogonal basis of Hilbert space $L^2(\Omega)$. Then one can define the operator A^α for $\alpha \in \mathbb{R}$. Now we introduce some notations which will be used in this note.

- $H = L^2(\Omega)$,
- $V = D(A^{\frac{1}{2}})$ with associated product $(u, v)_V = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)_H$,
- $C_H = C([-\rho, 0]; H),\ C_V = C([-\rho, 0]; V),\ L^2_H = L^2(-\rho, 0; H),\ L^2_V = L^2(-\rho, 0; V)$.

It's worth noting that, if $\alpha < \beta$ then $D(A^\beta) \subset D(A^\alpha)$ and this embedding is compact [8]. In particular, we have $V \subset H \equiv H' \subset V'$, and all injections are dense and compact. Here H' and V' are dual spaces of H and V respectively.

Let us mention that, though the operator A generate a analytic semigroup, we can not apply the theory of fixed points to prove the existence and uniqueness result since the nonlinearity does not satisfy the local Lipschitz property. In order to prove the existence result, we make use the iteration scheme similar to those as in [14]. We first recall the existence and uniqueness results for the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = h(x,t) - f(u), \quad x \in \Omega, t \in (\tau, T), \\ u(x,\tau) = u_0(x), \end{cases} \tag{3}$$

where the nonlinearity f satisfies (F1)-(F3).

Let $T > \tau$ and $Q_{\tau,T} = \Omega \times (\tau, T)$. By the weak solution of (3) on interval (τ, T), we mean the function $u \in L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$ satisfying the equation in (3) in the dual space $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$, where p' is the conjugate exponent of p, i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

We have the following theorem.

Theorem 2.1. *Suppose that $u_0 \in H$, $h \in L^2_{loc}(\mathbb{R}, H)$ and f satisfies (F1)-(F3). Then problem* (3) *has a unique weak solution $u \in C([\tau, T]; H) \cap L^p(Q_{\tau,T}) \cap L^2(\tau, T; V)$.*

Proof. The existence statement can be proved as in [11, Theorem 2.1 and Lemma 2.2], while the uniqueness was proved in [11, Theorem 3.1]. □

For more information about regularities of weak solution, see [18, 19].

We now give the definition of weak solution for problem (1). Assume that $u_0 \in H$ and $\phi \in L^2_H$.

Definition 2.1. The function $u \in L^2(-\rho, T; H) \cap L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$ is said to be a weak solution to problem (1) on interval $(-\rho, T)$ if and only if $u(x,\tau) = u_0(x)$ a.e. in Ω, $u(\tau + .) = \phi$ in L^2_H and u satisfies the equation in (1) in the dual space $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$.

The following statement is the main result in this section.

Theorem 2.2. *Suppose* $u_0 \in H,\ \phi \in L^2_H,\ h \in L^2_{loc}(\mathbb{R}, H),\ f$ *satisfies (F1)-(F3) and* g *satisfies (G1). If* $\ell < \lambda_1$ *then the problem* (1) *has a unique weak solution* u *on* $(-\rho, T)$ *such that* $u \in C([\tau, T]; H) \cap L^2(-\rho, T; H) \cap L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$.

Proof. Let us start the proof for the uniqueness. Suppose u and v are weak solutions to (1). Denoting $w = u - v$, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2 + \|A^{\frac{1}{2}}w(t)\|_H^2 + \int_\Omega (f(u) - f(v))(u - v) = \int_\Omega (g(u_t) - g(v_t))w.$$

Then, using (F3), (G1) and Cauchy inequality, one has

$$\frac{d}{dt}\|w\|_H^2 \leqslant (2\ell + 1)\|w\|_H^2 + C_g\|u_t - v_t\|_{L^2_H}.$$

Integrating the last inequality over $(\tau, t),\ t > \tau$, we obtain

$$\|w(t)\|_H^2 \leqslant (2\ell + 1)\int_\tau^t \|w(s)\|_H^2 ds + C_g \int_\tau^t ds \int_{-\rho}^0 \|w(s+z)\|_H^2 dz. \tag{4}$$

Noting that $w(r) = 0$ for $r \in (\tau - \rho, \tau)$, one has

$$\begin{aligned}
\int_\tau^t ds \int_{-\rho}^0 \|w(s+z)\|_H^2 dz &= \int_\tau^t ds \int_{s-\rho}^s \|w(r)\|_H^2 dr, \\
&\leqslant \int_\tau^t ds \int_{\tau-\rho}^t \|w(r)\|_H^2 dr, \\
&= \int_\tau^t ds \int_\tau^t \|w(r)\|_H^2 dr, \\
&\leqslant (T - \tau)\int_\tau^t \|w(s)\|_H^2 ds.
\end{aligned} \tag{5}$$

Combining this with (4), we arrive at

$$\|w(t)\|_H^2 \leqslant (2\ell + 1 + C_g(T - \tau))\int_\tau^t \|w(s)\|_H^2 ds.$$

Then the Gronwall inequality ensures the uniqueness as desired.

We now construct a sequence of functions $\{u^n\}$ as follows. Let u_1 be the solution of the problem

$$\begin{cases} \dfrac{\partial u^1}{\partial t} + Au^1 = h(x,t) - f(u^1), \quad x \in \Omega, t \in (\tau, T), \\ u^1(x,\tau) = u_0(x), \quad x \in \Omega, \end{cases} \tag{6}$$

such that $u^1(x,\tau+\theta) = \phi(x,\theta), \quad x \in \Omega,\ \theta \in (-\rho, 0)$ for $\phi \in L^2_H$.

By Theorem 2.1, problem (6) has a unique weak solution. For $n \geqslant 2$, we consider the problem

$$\begin{cases} \dfrac{\partial u^n}{\partial t} + Au^n = h(x,t) + g(u_t^{n-1}) - f(u^n), \quad x \in \Omega, t \in (\tau, T), \\ u^n(x,\tau) = u_0(x), \quad u^n(x,\tau+\theta) = \phi(x,\theta) \quad x \in \Omega,\ \theta \in (-\rho, 0). \end{cases} \tag{7}$$

Since $g(u_t^{n-1}) \in H$, using Theorem 2.1 again, we have the existence of u^n. It remains to prove that $\{u^n\}$ is a Cauchy sequence in $L^2(\tau-\rho, T; H)$ and converges to the weak solution of (1).

Putting $w^n = u^{n+1} - u^n$ for $n \geqslant 2$, we observe that

$$\|w^n(t)\|_H^2 + 2\int_\tau^t \|A^{\frac{1}{2}} w^n(s)\|_H^2 ds + 2\int_\tau^t ds \int_\Omega (f(u^{n+1}) - f(u^n))(u^{n+1} - u^n)$$
$$= 2\int_\tau^t ds \int_\Omega (g(u_s^n) - g(u_s^{n-1}))w^n,$$

for $t > \tau$. It follows from (F3), Cauchy inequality and (G1) that

$$\|w^n(t)\|_H^2 + 2\int_\tau^t \|A^{\frac{1}{2}} w^n(s)\|_H^2 ds \leqslant (2\ell + \epsilon)\int_\tau^t \|w^n(s)\|_H^2 ds + C\int_\tau^t \|w^{n-1}(s)\|_{L_H^2}^2 ds,$$

where $C = C(\epsilon, C_g) > 0$.

Choosing $\epsilon > 0$ such that $\lambda_1 > \ell + \frac{\epsilon}{2}$ and using the fact that $\|A^{\frac{1}{2}}u\|_H^2 \geqslant \lambda_1\|u\|_H^2$ for all $u \in V$, we get

$$\|w^n(t)\|_H^2 + (2\lambda_1 - 2\ell - \epsilon)\int_\tau^t \|w^n(s)\|_H^2 ds \leqslant C\int_\tau^t \|w^{n-1}(s)\|_{L_H^2}^2 ds.$$

Taking (5) into account, we have

$$\|w^n(t)\|_H^2 \leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds. \tag{8}$$

We see that, for $r \in (\tau, t]$

$$\|w^n(r)\|_H^2 \leqslant (T-\tau)C\int_\tau^r \|w^{n-1}(s)\|_H^2 ds \leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds,$$

therefore

$$\begin{aligned}\sup_{\tau\leqslant r\leqslant t} \|w^n(r)\|_H^2 &\leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds,\\ &\leqslant (T-\tau)C\int_\tau^t \sup_{\tau\leqslant r\leqslant s} \|w^{n-1}(r)\|_H^2 ds.\end{aligned}$$

Denoting $\eta^n(t) = \sup_{\tau\leqslant r\leqslant t} \|w^n(r)\|_H^2$, for each $n \geqslant 2$, it follows that $\eta^n(.)$ is an increasing function on τ, T and

$$\eta^n(t) \leqslant (T-\tau)C\int_\tau^t \eta^{n-1}(s)ds.$$

By iterative estimates, we have

$$\eta^n(t) \leqslant \frac{(T-\tau)^n C^{n-1}}{(n-1)!}\eta^1(T), \;\; n \geqslant 2, \;\; t \in (\tau, T]. \tag{9}$$

Using the last inequality, one easily to gets the conclusion that $\{u^n\}$ is a Cauchy sequence in $C(\tau, T; H)$ and $L^2(\tau, T; H)$. It is evident that $\{u^n\}$ is a Cauchy sequence in $L^2([\tau - \rho, T], H)$ also.

We are in a position to show that, $\{u^n\}$ converges to the solution of (1). We have proved that there exists a function $u \in L^2(\tau-\rho, T; H)$ such that

$$u^n \to u \ \text{ strongly in } \ C([\tau,T];H), \tag{10}$$

$$u^n \to u \ \text{ strongly in } \ L^2(\tau-\rho,T;H), \tag{11}$$

$$u^n \to u \ \text{ a.e. in } \ Q_{\tau,T}. \tag{12}$$

On the other hand, in considering of (7)

$$\begin{aligned}
&\|u^n(t)\|_H^2 + 2\int_\tau^t \|A^{\frac{1}{2}}u^n(s)\|_H^2 ds + 2\int_\tau^t ds \int_\Omega f(u^n)u^n \\
&\quad = \|u^n(\tau)\|_H^2 + 2\int_\tau^t (h + g(u_t^{n-1}), u^n)_H ds \\
&\quad \leqslant \|u_0\|_H^2 + \int_\tau^t (\|h(.,s)\|_H^2 + 2\|u^n(s)\|_H^2 + \|g(u_s^{n-1}\|_H^2))ds \\
&\quad \leqslant \|u_0\|_H^2 + \int_\tau^t (\|h(.,s)\|_H^2 + 2\|u^n(s)\|_H^2 + C_g\|u_s^{n-1}\|_{L_H^2}^2))ds \\
&\quad \leqslant \|u_0\|_H^2 + C_g(T-\tau)\|\phi\|_{L_H^2}^2 \\
&\qquad + \int_\tau^t (\|h(.,s)\|_H^2 + 2\|u^n(s)\|_H^2 + C_g(T-\tau)\|u^{n-1}(s)\|_H^2)ds.
\end{aligned} \tag{13}$$

Here we have used the following estimate

$$\begin{aligned}
\|u_s^{n-1}\|_{L_H^2}^2 &= \int_{-\rho}^0 \|u^{n-1}(s+z)\|_H^2 dz = \int_{s-\rho}^s \|u^{n-1}(r)\|_H^2 dr \\
&\leqslant \int_{\tau-\rho}^s \|u^{n-1}(r)\|_H^2 dr \\
&= \int_{\tau-\rho}^\tau \|u^{n-1}(r)\|_H^2 dr + \int_\tau^s \|u^{n-1}(r)\|_H^2 dr \\
&= \|\phi\|_{L_H^2}^2 + \int_\tau^s \|u^{n-1}(r)\|_H^2 dr.
\end{aligned} \tag{14}$$

In view of the fact that $\{u^n\}$ is a bounded sequence in $L^2(\tau,T;H)$, from (F2) and (13) we deduce

- $\{u^n\}$ is bounded in $L^2(\tau, T; V)$ and

- $\{u^n\}$ is bounded in $L^p(Q_{\tau,T})$.

This allows us to state that

- $\{Au^n\}$ is bounded in $L^2(\tau, T; V')$ and

- $\{f(u^n)\}$ is bounded in $L^{p'}(Q_{\tau,T})$ in view of (F1).

Rewriting equation in (7) as

$$\frac{du^n}{dt} = h + g(u_t^{n-1}) - Au^n - f(u^n) \tag{15}$$

one can see that the sequence $\{\frac{du^n}{dt}\}$ is bounded in $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$. Consequently, we obtain

$$u^n \rightharpoonup u \ \text{ in } \ L^2(\tau, T; V), \tag{16}$$

$$f(u^n) \rightharpoonup \chi \ \text{ in } \ L^{p'}(Q_{\tau,T}), \tag{17}$$

$$\frac{du^n}{dt} \rightharpoonup \frac{du}{dt} \ \text{ in } \ L^2(\tau, T; V') + L^{p'}(Q_{\tau,T}), \tag{18}$$

$$g(u_t^{n-1}) \to g(u_t) \ \text{ in } \ L^2(\tau, T; H) \ \text{ by using (G1) and (11).} \tag{19}$$

Finally, using (12) and (17) it follows that $\chi = f(u)$ (see [20]) and we obtain the weak solution of (1) by passing (15) to the limit in $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$. At last, (10) ensures that $u \in C([\tau, T]; H)$ and therefore the initial condition is meaningful. The proof completes. □

3 Existence of Pullback Attractors

The definition of pullback attractor was proposed to deal with non-autonomous evolution equations since one observes that, the initial time is as just important as the final time and the trajectories of a dynamics may be unbounded as the time goes to infinity. For more detail about the discussion of this notion, see [15]. Let us now recall some related definitions and results.

Let X be a complete metric space (which in our case will be $H \times L^2_H$) and $B_X(a, r)$ be the ball in X centered at a with radius r. Instead of semigroup, we will use a two parameters process on X denoted by $U(t, \tau)$ which has the following properties

- $U(\tau, \tau) = \mathrm{Id}$,
- $U(t, \tau)U(\tau, r) = U(t, r)$ for all $t \geqslant \tau \geqslant r$.

Definition 3.1. Let U be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a X-pullback attractor for U if

(i) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geqslant \tau$ and

(ii) $\lim_{s\to+\infty} \mathrm{dist}(U(t, t - s)B, \mathcal{A}(t)) = 0$, for all bounded set B of X.

In this definition, $\mathrm{dist}(B_1, B_2)$ is Hausdorff semi-distance between two subsets $B_1, B_2 \subset X$, i.e.

$$\mathrm{dist}(B_1, B_2) = \sup_{b_1\in B_1} \inf_{b_2\in B_2} d(b_1, b_2).$$

The pullback attracting property (ii) considers the state of the system at time t when the initial time $t - s$ goes to $-\infty$.

Definition 3.2. A family of sets $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ is said to be X-pullback absorbing with respect to the process $U(t,\tau)$ if for any bounded subset B of X and any $t \in \mathbb{R}$, there exists $\tau(t,B) \leqslant t$ such that $U(t,\tau)B \subset \mathcal{B}(t)$ for all $\tau \leqslant \tau(t,B)$.

The following theorem [15, 21] shows the sufficient conditions for existence of X-pullback attractor.

Theorem 3.1. *Let $U(t,\tau)$ be a continuous process on X. If there exists a compact X-pullback absorbing $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ with respect to the process $U(t,\tau)$, then there exists a X-pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, and $\mathcal{A}(t) \subset \mathcal{B}(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{B\subset X \\ bounded}} \Lambda_B(t)},$$

where

$$\Lambda_B(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{s\geqslant n} U(t,t-s)B}.$$

We will apply this result to our work to study the existence of pullback attractor for the process generated by (1). A solution u of (1) with initial data (u_0,ϕ) will be denoted by $u = u(t,\tau,u_0,\phi)$. We define the process with respect to (1) as a mapping on $H \times L^2_H$:

$$U(t,\tau)(u_0,\phi) = (u(t,\tau,u_0,\phi), u_t(.,\tau,u_0,\phi)). \tag{20}$$

This is a natural way since the phase space is designated at the step of setting for (1). However, there is more than a way to define the process for our problem as we change the phase space. For more details, see [15].

Lemma 3.2. *Suppose that the assumptions of the Theorem 2.1 take place. Let $u = u(t,\tau,u_0,\phi)$ and $v = v(t,\tau,v_0,\psi)$ be the weak solutions of (1). Then there exists a* *number $M > 0$ such that*

$$\|u(t) - v(t)\|_H^2 \leqslant (\|u_0 - v_0\|_H^2 + M\|\phi - \psi\|_{L_H^2}^2)e^{M(t-\tau)},$$

for all $t \geqslant \tau$.

Proof. It follows from (1) that

$$\frac{1}{2}\frac{d}{dt}\|u(t) - v(t)\|_H^2 + \|A^{\frac{1}{2}}(u(t) - v(t))\|_H^2 + \int_\Omega (f(u(t)) - f(v(t)))(u(t) - v(t))$$
$$= \int_\Omega (g(u_t) - g(v_t))(u(t) - v(t)).$$

Using (F3), (G1) and Cauchy inequality with ϵ chosen such that $\lambda_1 - \ell - \frac{1}{2}\epsilon > 0$, we have

$$\frac{d}{dt}\|u(t) - v(t)\|_H^2 + 2\lambda_1\|u(t) - v(t)\|_H^2 - 2\ell\|u(t) - v(t)\|_H^2$$
$$\leqslant C_\epsilon C_g\|u_t - v_t\|_{L_H^2}^2 + \epsilon\|u(t) - v(t)\|_H^2.$$

Then

$$\frac{d}{dt}\|u(t) - v(t)\|_H^2 \leqslant C\int_{-\rho}^0 \|u(t+s) - v(t+s)\|_H^2 ds$$

where $C = C_\epsilon C_g$. Integrating from τ to t, we obtain

$$\begin{aligned}\|u(t) - v(t)\|_H^2 &\leqslant \|u_0 - v_0\|_H^2 + C\int_\tau^t ds\int_{-\rho}^0 \|u(s+z) - v(s+z)\|_H^2 dz \\ &= \|u_0 - v_0\|_H^2 + C\int_{-\rho}^0 dz\int_\tau^t \|u(s+z) - v(s+z)\|_H^2 ds \\ &\leqslant \|u_0 - v_0\|_H^2 + C\int_{-\rho}^0 dz\int_{\tau-\rho}^t \|u(s) - v(s)\|_H^2 ds \\ &= \|u_0 - v_0\|_H^2 + C\rho\|\phi - \psi\|_{L_H^2}^2 + C\rho\int_\tau^t \|u(s) - v(s)\|_H^2 ds.\end{aligned}$$

Hence, the Gronwall inequality gives

$$\|u(t)-v(t)\|_H^2 \leqslant (\|u_0-v_0\|_H^2 + C\rho\|\phi-\psi\|_{L_H^2}^2)e^{C\rho(t-\tau)}$$

for any $t \geqslant \tau$. □

The aim of the next lemma is to construct an $H \times L_H^2$-pullback absorbing for the process $U(t,\tau)$.

Lemma 3.3. *Under assumptions of Theorem 2.2 and that g satisfies (G2), the solution of* (1) *satisfies*

$$\|u(t)\|_H^2 \leqslant e^{-\alpha(t-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L_H^2} + \int_\tau^t e^{-\alpha(t-s)}\|h(s)\|_H^2 ds + l_2(1-e^{-\alpha(t-\tau)})$$

where α, l_1 and l_2 are positive constants.

Proof. From equation (1), we get

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2 + 2C_0\|u(t)\|_{L^p(\Omega)}^p \leqslant 2C_1|\Omega| + 2(h(t)+g(u_t),u(t))_H.$$

In view of (G2) and Cauchy inequality, we have

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2 + 2C_0\|u(t)\|_{L^p(\Omega)}^p \leqslant 2C_1|\Omega| + 2M_1 + (C_\epsilon+1)\|u(t)\|_H^2 + \epsilon\|u_t\|_{L_H^2}^2 + \|h(t)\|_H^2$$

where ϵ is chosen later. Since $p>2$, it follows from the Young inequality that there is a number $C_p>0$ such that $(C_\epsilon+1)\|u\|_H^2 \leqslant 2C_0\|u\|_{L^p(\Omega)}^p + C_p$. Therefore

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2 \leqslant 2C_1|\Omega| + 2M_1 + C_p + \epsilon\int_{-\rho}^0 \|u(t+s)\|_H^2 ds + \|h(t)\|_H^2. \quad (21)$$

Now with $\alpha > 0$, one has

$$\frac{d}{dt}(e^{\alpha t}\|u(t)\|_H^2) = \alpha e^{\alpha t}\|u(t)\|_H^2 + e^{\alpha t}\frac{d}{dt}\|u(t)\|_H^2$$
$$\leqslant e^{\alpha t}\big[(\alpha - 2\lambda_1)\|u(t)\|_H^2 + \|h(t)\|_H^2 + 2C_1|\Omega| + 2M_1 + C_p + \epsilon\int_{-\rho}^{0}\|u(t+s)\|_H^2 ds\big].$$

Taking integration on $[\tau, t]$ for $t > \tau$ yields

$$e^{\alpha t}\|u(t)\|_H^2 \leqslant e^{\alpha\tau}\|u(\tau)\|_H^2 + (\alpha - 2\lambda_1)\int_\tau^t e^{\alpha s}\|u(s)\|_H^2 ds + \int_\tau^t e^{\alpha s}\|h(.,s)\|_H^2 ds$$
$$+ \frac{2C_1|\Omega| + 2M_1 + C_p}{\alpha}(e^{\alpha t} - e^{\alpha\tau}) + \epsilon\int_{-\rho}^{0} dz\int_\tau^t e^{\alpha s}\|u(s+z)\|_H^2 ds. \quad (22)$$

The last term in (22) can be estimated as

$$\int_{-\rho}^{0} dz\int_\tau^t e^{\alpha s}\|u(s+z)\|_H^2 ds = \int_{-\rho}^{0} dz\int_\tau^t e^{-\alpha z}e^{\alpha(s+z)}\|u(s+z)\|_H^2 ds$$
$$\leqslant \rho e^{\alpha\rho}\int_{\tau-\rho}^{t} e^{\alpha s}\|u(s)\|_H^2 ds$$
$$\leqslant \rho e^{\alpha\rho}\big[e^{\alpha\tau}\int_{\tau-\rho}^{\tau}\|u(s)\|_H^2 ds + \int_\tau^t e^{\alpha s}\|u(s)\|^2 ds\big].$$

Thus

$$e^{\alpha t}\|u(t)\|_H^2 \leqslant e^{\alpha\tau}\|u(\tau)\|_H^2 + (\alpha + \epsilon\rho e^{\alpha\rho} - 2\lambda_1)\int_\tau^t e^{\alpha s}\|u(s)\|_H^2 ds$$
$$+ \epsilon\rho e^{\alpha(\tau+\rho)}\|\phi\|_{L_H^2} + \int_\tau^t e^{\alpha s}\|h(s)\|_H^2 ds + \frac{2C_1|\Omega| + 2M_1 + C_p}{\alpha}(e^{\alpha t} - e^{\alpha\tau}). \quad (23)$$

Now choosing ϵ and α such that $\alpha + \epsilon\rho e^{\alpha\rho} \leqslant 2\lambda_1$ we obtain that

$$\|u(t)\|_H^2 \leqslant e^{-\alpha(t-\tau)}\|u_0\|_H^2 + \epsilon\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L_H^2}$$
$$+ \int_\tau^t e^{-\alpha(t-s)}\|h(s)\|_H^2 ds + \frac{2C_1|\Omega| + 2M_1 + C_p}{\alpha}(1 - e^{-\alpha(t-\tau)}). \quad (24)$$

We have the conclusion of the lemma. □

As a consequence, we have the following result.

Theorem 3.4. *Suppose that f satisfies (F1)-(F3) and g satisfies (G1)-(G2). If $\ell < \lambda_1$ and h has the following property*

$$\int_{-\infty}^{t} \|h(s)\|_H^2 ds < +\infty, \quad \textit{for each } \ t \in \mathbb{R},$$

then the process $U(t,\tau)$ associated with (1) *has an $H \times L_H^2$-pullback absorbing.*

Proof. By Lemma 3.3, we have

$$\|u(t+\theta,\tau)\|_H^2 \leqslant e^{-\alpha(t+\theta-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t+\theta-\tau-\rho)}\|\phi\|_{L_H^2} + \int_{-\infty}^{t+\theta} \|h(s)\|_H^2 ds + l_2(1-e^{-\alpha(t+\theta-\tau)}),$$

for $t \geqslant \tau + \rho$ and for all $\theta \in (-\rho, 0)$.

One can see that, there exists $\hat{\tau} = \hat{\tau}(t, u_0, \phi)$ such that, for all $\tau \leqslant \hat{\tau}$, the following inequality holds

$$e^{-\alpha(t-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L_H^2} \leqslant 1.$$

Therefore

$$\|u(t,\tau)\|_H^2 \leqslant \int_{-\infty}^{t} \|h(s)\|_H^2 ds + l_2 + 1, \tag{25}$$

$$\|u_t(\theta,\tau)\|_H^2 \leqslant \int_{-\infty}^{t} \|h(s)\|_H^2 ds + l_2 + 1 \tag{26}$$

for all $\tau \leqslant \hat{\tau} - \rho$. Hence, it is obvious that

$$\|U(t,\tau)(u_0,\phi)\|_{H\times L_H^2} = \|u(t)\|_H + \|u_t\|_{L_H^2} \leqslant (1+\sqrt{\rho})R(t) \tag{27}$$

where

$$R(t) = \Big(\int_{-\infty}^{t} \|h(s)\|_H^2 ds + l_2 + 1\Big)^{\frac{1}{2}}. \tag{28}$$

Then, for any bounded set $D \subset H \times L^2_H$, one easily deduce that $U(t,\tau)D \subset \mathcal{B}_H(t) = B_{H\times L^2_H}(0, (1+\sqrt{\rho})R(t))$, for all $\tau \leqslant \hat{\tau}(t,D)-\rho$. Thus $U(t,\tau)$ has an $H \times L^2_H$-pullback absorbing. □

Denote

$$F(u) = \int_0^u f(r)dr. \tag{29}$$

In order to prove further properties of the process $U(t,\tau)$, we need the following condition.

(F4) There exist the positive constants C_2, C_3 and C_4 such that

$$C_2|u|^p - C_4 \leqslant F(u) \leqslant C_3|u|^p + C_4.$$

Theorem 3.5. *Under assumptions of the Theorem 3.4 and the condition (F4), the process $U(t,\tau)$ associated with* (1) *has a $V \times L^2_V$-pullback absorbing.*

Proof. Let $u(t) = U(t,\tau)(u_0,\phi)$. We first need an estimate for $\int_\zeta^{\zeta+1}\big(\|u(s)\|^2_V + F(u(s))\big)ds$. One can proceed as in the proof of Lemma 3.3 to get that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_H + 2\|u(t)\|^2_V + 2C_0\|u(t)\|^p_{L^p(\Omega)} &\leqslant 2C_1|\Omega| + 2M_1 \\ &\quad + (1+M_0^2)\|u(t)\|^2_H + \|u_t\|^2_{L^2_H} + \|h(t)\|^2_H.\end{aligned}$$

Since $p > 2$, it follows from the Young inequality that there is a number $C_p > 0$ such that $(1+M_0^2)\|u\|^2_H \leqslant C_0\|u\|^p_{L^p(\Omega)} + C_p$. Therefore

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_H + 2\|u(t)\|^2_V + C_0\|u(s)\|^p_{L^p(\Omega)} &\leqslant 2C_1|\Omega| + 2M_1 + C_p \\ &\quad + \int_{-\rho}^0 \|u(t+s)\|^2_H ds + \|h(t)\|^2_H. \end{aligned} \tag{30}$$

Now for a given time $t > \tau$ and for all ζ such that $\tau < \zeta \leqslant t$, we have

$$\|u(\zeta+1)\|_H^2 - \|u(\zeta)\|_H^2 + 2\int_\zeta^{\zeta+1} \big(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\big)ds \leqslant 2C_1|\Omega| + 2M_1 + C_p + \int_\zeta^{\zeta+1} ds \int_{-\rho}^0 \|u(s+z)\|_H^2 dz + \int_{-\infty}^{t+1} \|h(s)\|_H^2 ds.$$

This implies that

$$\int_\zeta^{\zeta+1} \big(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\big)ds \leqslant C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{1}{2}\|u(\zeta)\|_H^2 + \frac{\rho}{2}\int_{\zeta-\rho}^{\zeta+1} \|u(s)\|_H^2 ds + \frac{1}{2}\int_{-\infty}^{t+1} \|h(s)\|_H^2 ds.$$

Invoking (25) and (28), we have

$$\int_\zeta^{\zeta+1} \big(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\big)ds \leqslant C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{1}{2}R^2(\zeta) + \frac{1}{2}\rho(\rho+1)R^2(\zeta+1) + \frac{1}{2}\int_{-\infty}^{t+1} \|h(s)\|_H^2 ds, \quad (31)$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$. In view of (F4), we can write

$$\int_\zeta^{\zeta+1} \big(\|u(s)\|_V^2 + \int_\Omega F(u(s))\big)ds \leqslant \bar{R}^2(t) \quad (32)$$

where

$$\min(1, \frac{C_0}{2C_3})\bar{R}^2(t) = C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{C_0 C_4}{2C_3}|\Omega| + \frac{1}{2}R^2(t) + \frac{1}{2}\rho(\rho+1)R^2(t+1) + \frac{1}{2}\int_{-\infty}^{t+1} \|h(s)\|_H^2 ds.$$

Our next step is to get an estimate for $\frac{d}{d\zeta}\big(\|u(\zeta)\|^2 + \int_\Omega F(u(s))\big)$. Multiplying the equation in (1) by $\dot{u} = \frac{du}{d\zeta}$, using (G2) and Cauchy inequality, we have

$$\|\dot{u}(\zeta)\|_H^2 + \frac{d}{d\zeta}\big(\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta))\big) = (h + g(u_\zeta), \dot{u}(\zeta)) \leqslant \|\dot{u}(\zeta)\|_H^2 + M_1 + \frac{M_0^2}{2}\|u_\zeta\|_{L_H^2}^2 + \frac{1}{2}\|h(\zeta)\|_H^2.$$

Taking (26) into account, one gets

$$\frac{d}{d\zeta}\big(\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta))\big) \leqslant M_1 + \frac{\rho M_0^2}{2}R^2(t) + \frac{1}{2}\|h(\zeta)\|_H^2 \tag{33}$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$. On the other hand, it is evident that

$$\int_\zeta^{\zeta+1} \|h(s)\|_H^2 ds \leqslant \int_{-\infty}^{t+1} \|h(s)\|_H^2 ds < +\infty. \tag{34}$$

Putting (32)-(34) into the uniform Gronwall inequality, we deduce that, there exists $\hat{R} = \hat{R}(t, u_0, \phi, h) > 0$ such that

$$\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta)) \leqslant \hat{R}^2(t)$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$.

Using (F4) again, we obtain

$$\|u(t)\|_V^2 + C_2\|u(t)\|_{L^p(\Omega)}^p \leqslant \hat{R}^2(t) + C_4|\Omega|, \tag{35}$$

$$\|u_t(\theta)\|_V^2 + C_2\|u_t(\theta)\|_{L^p(\Omega)}^p \leqslant \hat{R}^2(t) + C_4|\Omega|, \tag{36}$$

$$\|u_t\|_{L_V^2}^2 \leqslant \rho\hat{R}^2(t) + \rho C_4|\Omega|, \tag{37}$$

for all $\tau \leqslant \hat{\tau} - \rho$ and $\theta \in (-\rho, 0)$.

We arrive at the conclusion that, for any bounded set $D \subset V \times L_V^2$, $U(t,\tau)D \subset \mathcal{B}_V(t) = B_{V\times L_V^2}\Big(0, (1+\sqrt{\rho})\sqrt{\hat{R}^2(t) + C_4|\Omega|}\Big)$, for all $\tau \leqslant \hat{\tau}(t, D) - \rho$. Thus $U(t,\tau)$ has a $V \times L_V^2$-pullback absorbing. □

Remark 3.1. In our case, the external force h is not supposed to be translation bounded, i.e.

$$\|h\|_{L_b^2(H)}^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(s)\|_H^2 ds < +\infty,$$

since we need not the uniform boundedness of the trajectories of (1) under the analysis of pullback attractor. However, this class of external force can be used for our problem. Indeed, if h is a translation bounded then for $\alpha > 0$, we have

$$\begin{aligned}\int_{\tau}^{t} e^{-\alpha(t-s)}\|h(s)\|_H^2 ds &\leqslant \sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} e^{-\alpha(t-s)}\|h(s)\|_H^2 ds \\ &\leqslant \sum_{k=0}^{\infty} e^{-\alpha k}\int_{t-k-1}^{t-k} \|h(s)\|_H^2 ds \\ &\leqslant \frac{1}{1-e^{-\alpha}}\|h\|^2_{L^2_b(H)}.\end{aligned}$$

Thus, using Lemma 3.3, we obtain the same results as in Theorem 3.4 and Theorem 3.5. It is worth mentioning that, there is no relations between our class of external force and the set of translation bounded functions.

Remark 3.2. By priori estimates, we can prove the existence of C_H-pullback absorbing and C_V-pullback absorbing with respect to the process $U(t,\tau)$ if we change the phase space from $H \times L^2_H$ to C_H. In this case, we have a simple definition for $U(t,\tau)$ as follows

$$U(t,\tau)(\phi) = u_t(.,\tau,\phi(0),\phi),$$

where $u \in C([-\rho,T];H) \cap L^2(\tau,T;V) \cap L^p(Q_{\tau,T})$ is the solution of (1) on interval $[-\rho,T]$, for any $T \in \mathbb{R},\ \ T > \tau$. The readers are referred to [15] for this approach.

We now can state the main theorem of this section.

Theorem 3.6. *Under assumptions of Theorem 3.5, the process $U(t,\tau)$ associated with* (1) *has an $H \times L^2_H$-pullback attractor.*

Proof. By Lemma 3.2, we see that $U(t,\tau)$ is continuous mapping on $H \times L^2_H$. In order to apply Theorem 3.1, we prove that, there exists a compact $H \times L^2_H$-pullback absorbing with respect to $U(t,\tau)$. From Theorem 3.5, $U(t,\tau)$ has a $V \times L^2_V$-pullback absorbing $\{\mathcal{B}_V(t)\}$. Let

$$\mathcal{B}(t) = \bigcup_{\tau \leqslant \hat{\tau}(t,\mathcal{B}_V)-\rho} U(t,\tau)\mathcal{B}_V(t).$$

It is easy to see that, $\{\mathcal{B}(t)\}$ is a $V \times L^2_V$-pullback absorbing of $U(t,\tau)$. We now show that $\mathcal{B}(t)$ is precompact in $H \times L^2_H$. Let Π_1 and Π_2 are projectors on $H \times L^2_H$, i.e. $\Pi_1 : (u_0, \phi) \mapsto u_0,\ \ \Pi_2 : (u_0, \phi) \mapsto \phi$. One observes that $\Pi_1\mathcal{B}(t)$ is bounded in V and then it is precompact in H. It remains to prove that $\Pi_2\mathcal{B}(t)$ is precompact in L^2_H.

Let $u_t \in \Pi_2\mathcal{B}(t)$. For a given $t > \tau + \rho$, (36) ensures that $u(t+\theta),\ \theta \in (-\rho, 0)$, belong to a bounded set in $V \cap L^p(\Omega)$. Denoting $\Theta = [t-\rho, t]$ and $Q_\Theta = \Omega \times \Theta$, it follows that u belong to a bounded set in $L^2(\Theta; V \cap L^p(\Omega))$. We rewrite equation in (1) as one in dual space $L^2(\Theta; V') + L^{p'}(Q_\Theta)$ that

$$\dot{u}(\zeta) = h(\zeta) + g(u_\zeta) - Au(\zeta) - f(u(\zeta)). \tag{38}$$

At first, for any $v \in L^2(\Theta; V) \cap L^p(Q_\Theta)$ we have

$$\begin{aligned} |\langle h, v\rangle| &\leqslant \|h\|^2_{L^2(Q_\Theta)}\|v\|^2_{L^2(Q_\Theta)} \\ &\leqslant C\|h\|^2_{L^2(Q_\Theta)}\|v\|^2_{L^2(\Theta;V)}. \end{aligned} \tag{39}$$

Using (G2), one gets

$$\begin{aligned} |\langle g(u_\zeta), v\rangle| &\leqslant M_0 \int_{t-\rho}^{t} \|u_\zeta\|_{L^2_H}\|v(\zeta)\|_H d\zeta + \rho M_1 \\ &\leqslant CM_0\Big(\int_{t-\rho}^{t} d\zeta \int_{-\rho}^{0} \|u(\zeta+z)\|^2_H dz\Big)^{\frac{1}{2}}\|v\|_{L^2(\Theta,V)} + \rho M_1 \\ &\leqslant \rho CM_0\Big(\int_{t-2\rho}^{t} \|u(\zeta)\|^2_H d\zeta\Big)^{\frac{1}{2}}\|v\|_{L^2(\Theta,V)} + \rho M_1. \end{aligned} \tag{40}$$

It is obvious that

$$|\langle Au, v\rangle| \leqslant \|u\|_{L^2(\Theta;V)}\|v\|_{L^2(\Theta;V)}. \tag{41}$$

At last, taking (F1) into account,

$$\begin{aligned}|\langle f(u), v\rangle| &\leqslant \Big(\int_{Q_\Theta} |f(u)|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\int_{Q_\Theta} |v|^p\Big)^{\frac{1}{p}} \\ &\leqslant C_f\|u\|^{p-1}_{L^p(Q_\Theta)}\|v\|_{L^p(Q_\Theta)} + C. \end{aligned}\tag{42}$$

Combining (39)-(42), we obtain that $\dot{u}$ belongs to a bounded set in $L^2(\Theta; V') + L^{p'}(Q_\Theta) \subset L^{p'}(\Theta; V' + L^{p'}(\Omega))$. Using compactness Lemma in [20], we conclude that u belongs to a compact set in $L^2(\Theta; H)$, or equivalently, $\{u_t \in \Pi_2\mathcal{B}(t)\}$ is precompact in $L^2([-\rho, 0]; H)$. The proof is complete. □

4 Further Remarks

Let us discuss on some special cases of operator A. A typical example for A is that $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. In this case, we have $V = D(A^{\frac{1}{2}}) = H_0^1(\Omega)$. Now we introduce two examples, in which A is degenerate elliptic operator.

The Grushin type operator.

Let

$$A = -G_k := -\Delta_x - |x|^{2k}\Delta_y = -\sum_{i=1}^{N_1}\frac{\partial^2}{\partial x_i^2} - |x|^{2k}\sum_{i=1}^{N_2}\frac{\partial^2}{\partial y_i^2}$$

where $k \geqslant 0$. In this case, Ω is a smooth bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This operator is first introduced in [22] and one knows that it is not elliptic if $k > 0$ and Ω intersects with the hyperplane $\{x = 0\}$.

In [23], to study the boundary value problem, the authors use the natural energy space $S_0^1(\Omega)$ defined as the completion of $C_0^1(\bar{\Omega})$ in the norm

$$\|u\|_{S_0^1(\Omega)} = \Big(\int_\Omega \big(|\nabla_x u|^2 + |x|^{2k} |\nabla_y u|^2 \big) dxdy \Big)^{1/2}.$$

We have the continuous embedding $S_0^1(\Omega) \subset L^p(\Omega)$, for $2 \leqslant p \leqslant 2_k^* = \frac{2N(k)}{N(k)-2}$, where $N(k) = N_1 + (k+1)N_2$. Moreover, this embedding is compact if $2 \leqslant p < 2_k^*$ and 2_k^* is the so-called critical exponent for the embedding. In view of the compact embedding $S_0^1(\Omega) \subset L^2(\Omega)$, we see that G_k is positively definite and has compact resolvent (for more details, see [23]). Thus it can be used for our problem with $V = S_0^1(\Omega)$.

The Caldiroli-Musina type operator.

In the second example, we are interested in the case $A = -\mathrm{div}(\sigma(x)\nabla u)$. The degeneracy of A is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes at some points. In [24], where a semilinear degenerate elliptic problem was studied, the authors assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumption

$(\mathcal{H}_\alpha)$ $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,2), \liminf_{x \to z} |x - z|^{-\alpha} \sigma(x) > 0$ for every $z \in \overline{\Omega}$.

By $\mathcal{D}_0^{1,2}(\Omega, \sigma)$ we denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\| = \big(\int_\Omega \sigma(x) |\nabla v|^2 \big)^{\frac{1}{2}}.$$

According to [24], if σ satisfy $(\mathcal{H}_\alpha)$ then the following assertions hold

i) $\mathcal{D}_0^{1,2}(\Omega, \sigma) \subset L^{2_\alpha^*}(\Omega)$ continuously,

ii) $\mathcal{D}_0^{1,2}(\Omega, \sigma) \subset L^r(\Omega)$ compactly if $r \in [1, 2_\alpha^*)$

where $2^*_\alpha = \frac{2N}{N-2+\alpha}$. In particular, we have a compact embedding $\mathcal{D}_0^{1,2}(\Omega) \subset L^2(\Omega)$. This leads to the fact that A is positively definite and has compact resolvent. We can put it into (1) and the energy space $\mathcal{D}_0^{1,2}(\Omega)$ plays the role of V.

Some classes of nonlinearity.

As a final discussion, we would like to show some particular cases of nonlinearity $F(u, u_t)$ in our problem. In the cases $g(u_t) = 0$, we have the reaction-diffusion problem, which is studied by many authors (see [20, 11, 19, 18] and references therein). The particular case of $f(u)$ is that $f(u) = |u|^{p-1}u,\ p > 2$, which satisfies (F1)-(F4) obviously. Let us mention that, one can deal with the more general case, namely, $f(u) = |u|^{p-1}u + k(u)$ provided that

- $k(u) \leqslant C_k(|u|^{q-1} + 1),\ C_k > 0,\ q < p,$
- $\big(k(u) - k(v)\big)(u - v) \geqslant -\ell|u - v|^2,$
- $k_0|u|^q - k_1 \leqslant K(u) \leqslant k_0|u|^q - k_2$, where $K(u) = \int_0^u k(s)ds$ and k_0, k_1, k_2 are positive numbers.

As a special case of the retarded term $g(u_t)$, we recall the work of A.V. Rezounenko and J. Wu [17], in which a so-called *state-dependent* selective delay term was introduced

$$g(u_t) = \int_{-\rho}^{0} \Big\{ \int_\Omega b(u(x+\theta, y))k(x-y)dy \Big\} \chi(\theta, u_t)d\theta.$$

We can give some restrictions on g, which are similar to those in [17]; precisely,

- $b : \mathbb{R} \to \mathbb{R}$ has the Lipschitz property and $|b(w)| \leqslant b_0|w| + b_1,\ \ b_0, b_1 > 0,$

- $k : \Omega \to \mathbb{R}$ is bounded,
- $\chi : [-\rho, 0] \times L^2_H \to \mathbb{R}$ is Lipschitz with respect to the second coordinate and

$$\|\chi(.,v)\|_{L^2(-\rho,0)} \leqslant C_\chi \text{ for some } C_\chi > 0 \text{ and for all } v \in H.$$

By these assumptions, one can proceed as in [17] to prove that

$$|(g(\xi),\eta)_H| \leqslant M_0(\|\xi\|_{L^2_H} + 1)\|\eta\|_H,$$

for some $M_0 > 0$ and for all $(\xi, \eta) \in L^2_H \times H$. Actually, the last inequality can replace (G2), since we employ (G2) in the situation that

$$|(g(\xi),\eta)_H| \leqslant \epsilon\|\xi\|^2_{L^2_H} + C_\epsilon(\|\eta\|^2_H + 1).$$

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