On the solution existence of implicit quasivariational inequalities with discontinuous multifunctions *

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In this paper we established some solution existence theorems for implicit quasivariational inequalities. We first established some results in finite-dimensional spaces and then a solution existence result in infinite-dimensional spaces was derived. Our theorems are proved for discontinuous mappings and sets which may be unbounded. The results presented in this paper are improvements of results in [6] and [7].

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1 INTRODUCTION

Let X be a real normed space and Y be a Hausdorff topological vector space. Let K be a nonempty closed convex set in X and C be a nonempty subset in Y. Let $\Phi : K \to 2^C$, $\Gamma : K \to 2^K$ be two multifunctions and $\psi : K \times C \times K \to R$ be a single-valued map. The implicit quasivariational inequality defined by Γ, Φ, ψ is the problem of finding a pair $(\hat{x}, \hat{z}) \in K \times C$ such that

$$\hat{x} \in \Gamma(\hat{x}), \hat{z} \in \Phi(\hat{x}), \psi(\hat{x}, \hat{z}, y) \leq 0 \ \forall y \in \Gamma(\hat{x}). \tag{1}$$

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For short, the problem will be denoted by $IQVI(K, \Gamma, \Phi, \psi)$. If Γ is a constant mapping and $\psi(x, z, y) = \langle z, x - y\rangle$ then the problem is called a generalized variational inequality. If, in addition, Φ is a single-valued map, then one has a variational inequality. In the finite-dimensional space, such problem was considered first by Chang and Pang in [3]. We refer to [3]-[11], [13]-[18] for various sufficient conditions of solution existence of variational inequalities and generalized quasivariational inequalities.

Problem (1) were first introduced by [6] with applications to fuzzy mappings. In [6], the authors have obtained some existence results for (1) without assuming continuity of data mappings and applied to generalized quasivariational inequalities. Recently, the results of [6] has been extended to the case of infinite dimensional spaces by [7]. The results in [7] were obtained by using the preceding results in the finite-dimensional spaces and building a section multifunction which is lower semicontinuous on finite-dimensional subspaces. In order to build the continuous section mapping, the authors had to assume the Hausdorff lower semicontinuity of Γ and condition $\mathrm{int}_{\mathrm{aff}(K)}\Gamma(x) \neq \emptyset$. However, this scheme might not be applicable when Γ is not Hausdorff lower semicontinuous and condition $\mathrm{int}_{\mathrm{aff}(K)}\Gamma(x) \neq \emptyset$ is violated.

The aim of this paper is to derive some existence theorems for $IQVI(K, \Gamma, \Phi, \psi)$ in which Φ may not be continuous, K may be unbounded and Γ is not necessarily Hausdorff lower semicontinuous. To do this, we will build a new scheme by approximating Γ with multifunctions H_j which are lower semicontinuous on finite-dimensional subspaces. Our results are improvements of results in [6] and [7].

The organization of the paper is as follows: Section 2 recalls some notions and auxiliary results. In Section 3 we state and prove main results.

2 PRELIMINARIES

For each $\rho > 0$, we denote by

$$B(x_0, \rho) := \{x \in E : \|x - x_0\| < \rho\},$$

$$\overline{B}(x_0, \rho) := \{x \in E : \|x - x_0\| \leq \rho\}$$

the open ball and closed ball with radius ρ and center at x_0, respectively. For each set $A \subset E$ and $x \in E$, $d(x, A) := \inf\{\|x - y\| : y \in A\}$ is the distance from x to A. Let $\Gamma : K \to 2^Z$ be a multifunction from $K \subset X$ into a normed space Z. The set $\mathrm{Gr}\Gamma := \{(x, y) \in K \times Z : y \in \Gamma(x)\}$ is called a graph of Γ. The multifunction Γ is said to have closed (open) graph if $\mathrm{Gr}\Gamma$ is closed (open) in $X \times Z$. The multifunction Γ is said to be lower semicontinuous at $\overline{x} \in K$ if for any open set V in Z such that $V \cap \Gamma(\overline{x}) \neq \emptyset$, there exists a neighborhood $U(\overline{x})$ in X such that $V \cap \Gamma(x) \neq \emptyset$ for all $x \in U(\overline{x}) \cap K$. Γ is said to be upper semicontinuous at $\overline{x} \in K$ if for any open set V in Z such that $\Gamma(\overline{x}) \subset V$, there exists a neighborhood $W(\overline{x})$ of $\overline{x}$ with the property that $\Gamma(x) \subset V$ for all $x \in W(\overline{x}) \cap K$.

Γ is said to be lower semicontinuous on X (u.s.c) if it is l.s.c (u.s.c) at every point $x \in X$. Γ is said to be Hausdorff lower semicontinuous (resp., Hausdorff upper semicontinuous) at $\overline{x} \in K$ if for any $\epsilon > 0$, there exist a neighborhood $W(\overline{x})$ such that

$$\Gamma(\overline{x}) \subset \Gamma(x) + \epsilon B \quad \text{for all } x \in W(\overline{x}) \cap K$$

$$(\text{resp.}, \Gamma(x) \subset \Gamma(\overline{x}) + \epsilon B \quad \text{for all } x \in W(\overline{x}) \cap K).$$

Here B is unit open ball in Z.

It is easy to check that Hausdorff lower semicontinuity implies lower semicontinuity and upper semicontinuity implies Hausdorff upper semicontinuity. The converse implications are true if each set $\Gamma(x)$ is nonempty and compact.

Let $\Gamma : K \to 2^K$ be a multifunction. For each $\epsilon > 0$ we define the multifunction $H_\epsilon : K \to 2^K$ by putting $H_\epsilon(x) = \{w \in K : d(w, \Gamma(x)) < \epsilon\}$ and $\overline{H}_\epsilon : K \to 2^K$ by the formula $\overline{H}_\epsilon(x) = \overline{H_\epsilon(x)}$. It is easy to prove that if the set K is convex then the sets $H_\epsilon(x)$ and $\overline{H}_\epsilon(x)$ are convex and $\overline{H_\epsilon(x)} = \{w \in K : d(w, \Gamma(x)) \leq \epsilon\}$ (see also [5, Proposition 2.3]).

The multifunction H_ϵ has some interesting properties. The following properties of H_ϵ will be needed in the sequel.

PROPOSITION 2.1 *Let X be a normed space and K be a nonempty closed convex set in X. Let $\Gamma : K \to 2^K$ be a lower semicontiuous multifunction with closed convex values. Then the following properties are valid:*
(a) if $\overline{y} \in H_\epsilon(\overline{x})$ for some $\overline{x} \in K$, then there exists a neighborhood U of $\overline{x}$ such that $\overline{y} \in H_\epsilon(x)$ for all $x \in U$;
(b) if E is a linear subspace of X such that $H_\epsilon(x) \cap E \neq \emptyset$ for all $x \in E$, then the multifunction $L_\epsilon : K \cap E \to 2^{K \cap E}$ defined by setting $L_\epsilon(x) = H_\epsilon(x) \cap E$, is lower semicontinuous on $K \cap E$ in the relative topology of E.

Proof (a) Assume that the assertion is false. Then we can find a sequence $\{x_n\}$ such that $x_n \to \overline{x}$ and $d(\overline{y}, \Gamma(x_n)) \geq \epsilon$. Take any $z \in \Gamma(\overline{x})$. By the lower semicontinuity of Γ, there exists a sequence $\{z_n\}$ such that $z_n \in \Gamma(x_n)$ and $z_n \to z$. Since $d(\overline{y}, \Gamma(x_n)) \geq \epsilon$, $\|\overline{y} - z_n\| \geq \epsilon$. By letting $n \to \infty$, we have$\|\overline{y} - z\| \geq \epsilon$. Hence $d(\overline{y}, \Gamma(\overline{x})) \geq \epsilon$. This contradicts the fact that $\overline{y} \in H_\epsilon(\overline{x})$.
(b) Let V be an open set in E and x_0 be a point in E such that $L_\epsilon(x_0) \cap V \neq \emptyset$. Since V is open in E, $V = E \cap \Omega$ for some open set Ω in X. Let $y_0 \in L_\epsilon(x_0) \cap V$ be arbitrary. We get $y_0 \in H_\epsilon(x_0) \cap \Omega$. By ($a$), there exists a neighborhood U of x_0 such that $y_0 \in H_\epsilon(x) \cap \Omega$ for all $x \in U$. Putting $W = U \cap E$, we see that W is a neighborhood of x_0 in E and $y_0 \in H_\epsilon(x) \cap \Omega \cap E \neq \emptyset$ for all $x \in W$. This implies that $L_\epsilon(x) \cap V \neq \emptyset$ for all $x \in W$. Hence L_ϵ is lower semicontinuous on $K \cap E$ in the relative topology of E. $\Box$

The following assertion was established in [5]. However, for the convenience of the reader we will give another proof by a simple argument.

PROPOSITION 2.2 *Let X be a normed space and $\Gamma : K \to 2^K$ be Hausdorff upper semicontinuous. Then $\overline{H}_\epsilon$ has closed graph in $K \times K$.*

Proof. We take sequences $\{x_n\}$ and $\{y_n\}$ in K such that $d(y_n, \Gamma(x_n)) \leq \epsilon$ and $x_n \to x_0$, $y_n \to y_0$ and show that $d(y_0, \Gamma(x_0)) \leq \epsilon$. In fact, for each $n > 0$, there exists $z_n \in \Gamma(x_n)$ such that $\|y_n - z_n\| < \epsilon + 1/n$. We take any $\mu > 0$, by the Hausdorff upper semicontinuity of Γ there exists $n_0 > 0$ such that

$$\Gamma(x_n) \subset \Gamma(x_0) + \mu B \text{ for all } n > n_0,$$

from which it follows that

$$d(z_n, \Gamma(x_0)) < \mu \text{ for all } n > n_0.$$

Therefore for all $n > n_0$ we have

$$\begin{aligned} d(y_0, \Gamma(x_0)) &\leq \|y_0 - y_n\| + d(y_n, \Gamma(x_0)) \\ &\leq \|y_0 - y_n\| + \|y_n - z_n\| + d(z_n, \Gamma(x_0)) \\ &< \|y_0 - y_n\| + \epsilon + 1/n + \mu. \end{aligned}$$

By letting $n \to \infty$ we obtain

$$d(y_0, \Gamma(x_0)) \leq \epsilon + \mu \text{ for all } \mu > 0.$$

This means that $d(y_0, \Gamma(x_0)) \leq \epsilon$. The proof is complete. $\square$

3 EXISTENCE RESULTS

First, we have the following existence result in finite-dimensional spaces.

THEOREM 3.1 *Let K be a nonempty convex compact set in R^m, C be a nonempty subset in Y. Assume that the following conditions are fulfilled:*
(i) the multifunction Γ is lower semicontinuous with nonempty convex values and the set $M := \{x \in K : x \in \Gamma(x)\}$ is closed;
(ii) the set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in M$;
(iii) for each $y \in K$ the set $\{x \in M : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;
(iv) for each $x \in M$ the set $\{y \in K : \inf_{z\in\Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;
(v) for each $x \in M$ and each $z \in \Phi(x)$ one has $\psi(x, z, x) = 0$;
(vi) for each $x \in M$ and $z \in \Phi(x)$, the function $\psi(x, z, .)$ is concave on $\Gamma(x)$;
(vii) for each $x \in M$ and each $y \in \Gamma(x)$, the function $\psi(x, ., y)$ is lower semicontinuous (in the sense of single-valued maps) and convex on $\Phi(x)$.
Then $IQVI(K, \Gamma, \Phi, \psi)$ has a solution in $K \times C$.

Proof We will complete the proof of the theorem by proving some lemmas and using some techniques in [6].

We first define the multifunction $G : M \to 2^K$ by putting

$$G(x) = \{y \in K : \inf_{z\in\Phi(x)} \psi(x, z, y) > 0\}.$$

If $G(x^*) = \emptyset$ for some $x^* \in M$ then

$$\inf_{z\in\Phi(x^*)} \psi(x^*, z, y) \leq 0 \ \forall \ y \in \Gamma(x^*),$$

and hence

$$\sup_{y\in\Gamma(x^*)} \inf_{z\in\Phi(x^*)} \psi(x^*, z, y) \leq 0.$$

By Theorem 5 at p. 216 of [1], taking into account assumptions (i), (ii), (vi) and (vii) we then get

$$\inf_{z\in\Phi(x^*)} \sup_{y\in\Gamma(x^*)} \psi(x^*, z, y) \leq 0. \tag{2}$$

By assumption (vii) the function $z \to \sup_{y\in\Gamma(x^*)} \psi(x^*, z, y)$ is l.s.c. on the compact set $\Phi(x^*)$. From this and (2) we can find a point $z^* \in \Phi(x^*)$ such that

$$\sup_{y\in\Gamma(x^*)} \psi(x^*, z^*, y) \leq 0.$$

Consequently, (x^*, z^*) is a solution. We now assume that $G(x) \neq \emptyset$ for all $x \in M$.

LEMMA 3.1 *The multifunction $G : M \to 2^K$ has the following properties:*
(a) G is lower semicontinuous on M and has convex values;
(b) G has a open graph in $M \times K$.

Proof It is easy to see that $G(x)$ is a convex set by (vi). Let V be an open set and $x_0 \in M$ such that $G(x_0) \cap V \neq \emptyset$. Take $y_0 \in G(x_0) \cap V$. Then $y_0 \in V$ and $\inf_{z\in\Phi(x_0)} \psi(x_0, z, y_0) > 0$. By (iii), there exists a neighborhood U of x_0 such that $\inf_{z\in\Phi(x)} \psi(x, z, y_0) > 0$ for all $x \in U \cap M$. Hence we have $y_0 \in G(x) \cap V$ for all $x \in U \cap M$. This means that $G(x)$ is l.s.c. on M.

For the proof of (b) we take $(\overline{x}, \overline{y}) \in \mathrm{Gr}G$. Then $(\overline{x}, \overline{y})$ belongs to $M \times K$ and $\inf_{z\in\Phi(\overline{x})} \psi(\overline{x}, z, \overline{y}) > 0$. By (iv), there exists a neighborhood W of $\overline{y}$ such that

$$\inf_{z\in\Phi(\overline{x})} \psi(\overline{x}, z, y) > 0 \ \forall y \in W \cap K.$$

Hence $W \cap K \subset G(\overline{x})$. By Proposition 2.1 in [4], there exist a neighborhood U of $\overline{x}$ and a neighborhood $W' \subset W$ such that $W' \cap X \subset G(x)$ for all $x \in U \cap M$. This implies that $(U \cap M) \times (W' \cap X) \subset \mathrm{Gr}G$. Consequently, G has open graph in $M \times K$.

We next define the multifunction $F : M \to 2^K$ by setting

$$F(x) := \Gamma(x) \cap G(x) = \{y \in \Gamma(x) : \inf_{z \in \Phi(x)} \psi(x, z, y) > 0\}.$$

LEMMA 3.2 *The multifunction F is lower semicontinuous on M and $F(\hat{x}) = \emptyset$ for some $\hat{x} \in M$.*

Proof We first show that F is l.s.c. on M. Let x_0 be a point in M and V be an open set such that $F(x_0) \cap V \neq \emptyset$. Take any $y_0 \in F(x_0) \cap V$. By (b) of Lemma 3.1, there exist a neighborhood W of y_0 and a neighborhood U_1 of x_0 such that $W \cap K \subset G(x)$ for all $x \in U_1 \cap M$. Since $\Gamma(x_0) \cap W \cap V \neq \emptyset$ and Γ is lower semicontinuous at x_0, there exists a neighborhood U_2 of x_0 such that $\Gamma(x) \cap W \cap V \neq \emptyset$ for all $x \in U_2 \cap M$. Putting $U = (U_1 \cap M) \cap (U_2 \cap M)$, we have

$$\begin{aligned} \Gamma(x) \cap G(x) \cap V &\supset \Gamma(x) \cap W \cap K \cap V \\ &= \Gamma(x) \cap W \cap V \neq \emptyset \end{aligned}$$

for all for all $x \in U$. Hence $F(x) \cap V \neq \emptyset$ for all $x \in U$. This implies that F is l.s.c on M.

We now suppose that $F(x) \neq \emptyset$ for all $x \in M$. Then we can build the multifuction $T : K \to 2^K$ by the following formula

$$T(x) = \begin{cases} F(x) & \text{if } x \in M \\ \Gamma(x) & \text{if } x \notin M. \end{cases}$$

It is clear that T has nonempty convex values. Moreover T is l.s.c. on K. Indeed, let V be an open set and x_0 be a point in K such that $T(x_0) \cap V \neq \emptyset$. If $x_0 \in M$ then $T(x_0) = F(x_0)$ and $F(x_0) \cap V \neq \emptyset$. By the lower semicontinuity of F, there exists a neighborhood U_1 of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U_1 \cap M$. Since $F(x_0) \subset \Gamma(x_0)$, $\Gamma(x_0) \cap V \neq \emptyset$. By the lower semicontinuity of Γ, there exists a neighborhood U_2 of x_0 such that $U_2 \subset U_1$ and $\Gamma(x) \cap V \neq \emptyset$ for all $x \in U_2 \cap X$. It is easy to see that $T(x) \cap V \neq \emptyset$ for all $x \in U_2 \cap X$. If $x_0 \in K \setminus M$, then $T(x_0) = \Gamma(x_0)$ and $\Gamma(x_0) \cap V \neq \emptyset$. By the lower semicontinuity of Γ and noting that $K \setminus M$ is an open set in K, there exist two neighborhoods U_3 and U_4 of x_0 such that $\Gamma(x) \cap V \neq \emptyset$ for all $x \in U_3 \cap X$ and $U_4 \cap K \subset K \setminus M$. Putting $U = (U_3 \cap K) \cap (U_4 \cap K)$ we have $T(x) \cap V \neq \emptyset$ for all $x \in U$. Thus there exists a neighborhood U_0 of x_0 such that $T(x) \cap V \neq \emptyset$ for all $x \in U_0 \cap K$. Sine x_0 is arbitrary, T is l.s.c. on K.

According to the Michael continuous selection theorem (Theorem $3.1'''$) in [12], there exists a continuous function $f : K \to K$ such that $f(x) \in T(x)$ for all $x \in K$. By the classical Brouwer fixed-point theorem, there exists a point x^* such that $x^* = f(x^*) \in T(x^*)$. This implies that $x^* \in F(x^*)$. Therefore from (v) we obtain $0 = \inf_{z \in \Phi(x^*)} \psi(x^*, z, x^*) > 0$ which is absurd. The lemma is proved.

To finish the proof of the theorem we will use Lemma 3.2. From the above lemma, there exists $\hat{x} \in M$ such that $F(\hat{x}) = \emptyset$. By the definition of F,

$$\hat{x} \in \Gamma(\hat{x}), \quad \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0 \ \forall \ y \in \Gamma(\hat{x}).$$

Hence

$$\sup_{y \in \Gamma(\hat{x})} \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0.$$

According to Theorem 5 at p. 216 of [1], taking into account assumptions (i), (ii), (vi) and (vii) we then get

$$\inf_{z \in \Phi(\hat{x})} \sup_{y \in \Gamma(\hat{x})} \psi(\hat{x}, z, y) \leq 0. \tag{3}$$

By assumption (vii) the function $z \rightarrow \sup_{y \in \Gamma(\hat{x})} \psi(\hat{x}, z, y)$ is l.s.c. on the compact set $\Phi(\hat{x})$. Hence from (3) we can find a point $\hat{z} \in \Phi(\hat{x})$ such that

$$\sup_{y \in \Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

This means that $(\hat{x}, \hat{z})$ is a solution of $IQVI(K, \Gamma, \Phi, \psi)$. The proof is now complete. $\square$

To illustrate Theorem 3.1, we give the following example.

Example 3.1 Let $X = Y = R$, $K = [0, 1]$ and $C = [1, 4]$. Let Γ, Φ and ψ be defined by:

$$\Gamma(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ (0, 1] & \text{if } 0 < x \leq 1, \end{cases}$$

$$\Phi(x) = \begin{cases} [2, 4] & \text{if } x = 0 \\ \{1\} & \text{if } 0 < x \leq 1, \end{cases}$$

$\psi(x, z, y) = z(x^2 - y^2)$. Then the set $\{0\} \times [2, 4]$ is a solution set of $IQVI(K, \Gamma, \Phi, \psi)$. Moreover Φ is not upper semicontinuous on [0, 1].
Indeed, it easy to check that Γ is l.s.c. on [0,1] and $M = \{x \in [0, 1] : x \in \Gamma(x)\} = [0, 1]$. Hence Condition (i) of Theorem 3.1 is valid. Condition (ii) is obvious. For each $x, y \in [0, 1]$ we have

$$\{x' \in M : \inf_{z \in \Phi(x')} \psi(x', z, y) \leq 0\} = [0, y].$$

and

$$\{y' \in M : \inf_{z \in \Phi(x)} \psi(x, z, y') \leq 0\} = [x, 1].$$

Hence Conditions (iii) and (iv) are fulfilled. Conditions (v)-(vii) are straightforward to verify.

Taking $\hat{x} = 0$ and $\hat{z} \in \Phi(0) = [2,4]$ we have $0 \in \Gamma(0)$ and

$$\psi(0, \hat{z}, y) = -\hat{z}y^2 \le 0 \ \forall y \in \Gamma(0).$$

This implies that the set $\{0\} \times [2,4]$ is a solution set of the problem. Since $x_n = 1/n \to 0$ and $y_n = 1 \in \Phi(x_n)$ but $1 \notin \Phi(0) = [2,4]$, Φ is not u.s.c. on K.

When K is unbounded we have the following result.

THEOREM 3.2 *Let K be a nonempty closed convex set in R^m, K_0 be a compact set in K, and C be a nonempty subset in Y. Assume (i) - (vii) as in Theorem 3.1 and the following assumption:*
(viii) for each $x \in M \setminus K_0$ one has

$$\sup_{y \in \Gamma(x) \cap K_0} \inf_{z \in \Phi(x)} \psi(x, z, y) > 0.$$

Then $IQVI(K, \Gamma, \Phi, \psi)$ has a solution in $K_0 \times C$.

Proof We now choose $r > 0$ such that $K_0 \subset \mathrm{int}B_r$, where B_r is the closed ball in R^m with radius r and center at 0. We put $\Omega_r = X \cap B_r$ and define the multifunction $\Gamma_r : \Omega_r \to 2^{\Omega_r}$ by setting $\Gamma_r(x) = \Gamma(x) \cap B_r$. According to Lemma 3.1 in [18], Γ_r is l.s.c on Ω_r. Put $\Phi_r = \Phi \mid_{\Omega_r}, \psi_r = \psi \mid_{\Omega_r \times C \times \Omega_r}$. It is easy to check that $IQVI(\Omega_r, \Gamma_r, \Phi_r, \psi_r)$ satisfies all conditions of Theorem 3.1. Hence there exists $\hat{x} \in \Omega_r$ such that

$$\hat{x} \in \Gamma_r(\hat{x}) \text{ and } \inf_{z \in \Phi_r(\hat{x})} \psi_r(\hat{x}, z, y) \le 0 \ \forall y \in \Gamma_r(\hat{x}).$$

Since $\Phi_r(\hat{x}) = \Phi(\hat{x})$ and $\psi_r(\hat{x}, z, y) = \psi(\hat{x}, z, y)$ we get

$$\hat{x} \in \Gamma(\hat{x}) \text{ and } \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \le 0 \ \forall y \in \Gamma_r(\hat{x}). \tag{4}$$

By (viii), we obtain $\hat{x} \in K_0 \subset \mathrm{int}B_r$. Taking any $y \in \Gamma(\hat{x})$ we have $(1-\lambda)\hat{x} + \lambda y \in \Gamma(\hat{x}) \cap B_r$ for a sufficiently small $\lambda \in (0,1)$. From (4) and (vi) one has

$$\begin{aligned} &\lambda \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) + (1-\lambda) \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, \hat{x}) \le \\ &\le \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, \lambda y + (1-\lambda)\hat{x}) \\ &\le 0. \end{aligned}$$

This implies that $\inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \le 0$. Therefore we obtain

$$\sup_{y \in \Gamma(\hat{x})} \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \le 0.$$

By a similar argument as in the proof of Theorem 3.1, we can show that there exists $(\hat{x}, \hat{z}) \in K_0 \times \Phi(\hat{x})$ such that

$$\sup_{y \in \Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

This implies that $(\hat{x}, \hat{z})$ is a solution of $IQVI(K, \Gamma, \Phi, \psi)$. The proof is complete. $\square$

The following example shows that our result can be still applied even in the case $\mathrm{int}_{\mathrm{aff(K)}}(\Gamma(x)) = \emptyset$.

Example 3.2 Let $X = Y = R^2$, $K = [0, 2] \times [0, 2]$ and $\psi(x, z, y) = \langle z, x - y \rangle$. Let Γ and Φ be defined by:

$$\Gamma(x) = \begin{cases} \{(0,0)\} & \text{if } x = (0,0) \\ \{(0,t) : 0 \leq t \leq 1\} & \text{if } x \neq (0,0) \end{cases}$$

$$\Phi(x) = \begin{cases} [2,3] \times [2,3] & \text{if } x = (0,0) \\ \{(1,1)\} & \text{if } x \neq (0,0). \end{cases}$$

Then the set $\{(0,0)\} \times ([2,3] \times [2,3])$ is a solution set of IQVI(K, Γ, Φ, ψ). Moreover Φ is not upper semicontinuous on K.
In fact, it is easy to check that conditions (i) and (ii) of Theorem 3.2 are fulfilled, where $M := \{x \in K : x \in \Gamma(x)\} = \{(0,0)\}$. Conditions (iii)-(vii) are also straightforward to verify. Since K is a compact set, Condition (viii) is automatically fulfilled.

Taking $\hat{x} = (0,0)$ and $\hat{z} \in \Phi(\hat{x})$, we have $\hat{x} \in \Gamma(\hat{x})$ and

$$\langle \hat{z}, \hat{x} - y \rangle \leq 0 \ \forall y \in \Gamma(\hat{x}).$$

Hence the set $\{(0,0)\} \times ([2,3] \times [2,3])$ is a solution set of the problem. We notice that Φ is not u.s.c. at $\hat{x} = (0,0)$. Moreover, $\mathrm{int}_{\mathrm{aff(K)}}(\Gamma(x)) = \emptyset$ because one has $\mathrm{aff}(K) = R^2$.

The following theorem presents a solution existence result of implicit quasivariational inequalities in normed spaces.

THEOREM 3.3 *Let X be a real normed space, Y be a Hausdorff topological space, K be a closed convex subset in X, C be a nonempty subset in Y. Let K_1, K_2 be two nonempty compact subsets of K such that $K_1 \subset K_2$ and K_1 is finite-dimensional and $\gamma > 0$. Assume that:*
(i) the multifunction Γ is l.s.c with nonempty closed convex values and Hausdorff upper semicontinuous;
(ii) $\Gamma(x) \cap K_1 \neq \emptyset$ for all $x \in K$;
(iii) the set $\Phi(x)$ is nonempty, compact and convex for each $x \in K$;
(iv) the set $\{(x, y) \in K \times K : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;
(v) for each $x \in K$ and each $z \in \Phi(x)$ one has $\psi(x, z, x) = 0$;
(vi) for each $x \in K$ and $z \in \Phi(x)$, the function $\psi(x, z, .)$ is concave on $\Gamma(x)$;

(vii) for each $x \in K$ and each $y \in \Gamma(x)$, the function $\psi(x, ., y)$ is lower semicontinuous (in the sense of single-valued maps) and convex on $\Phi(x)$;
(viii) for each $x \in K \setminus K_2$, with $d(x, \Gamma(x)) \leq \gamma$, one has

$$\sup_{y \in \Gamma(x) \cap K_1} \inf_{z \in \Phi(x)} \psi(x, z, y) > 0.$$

Then $IQVI(K, \Gamma, \Phi, \psi)$ has a solution.

Proof We will complete the proof by proving some lemmas.

For each $j > 0$, $j \in N$ we define the multifunction $H_j : K \to 2^K$ by putting

$$H_j(x) = \{w \in K : d(w, \Gamma(x)) < 1/j\}$$

and $\overline{H}_j : K \to 2^K$ by the formula $\overline{H}_j(x) = \overline{H_j(x)}$ for all $x \in K$. We first claim that Γ has a closed graph. Indeed, let $\{x_n\}$ and $\{z_n\}$ be two sequences with $z_n \in \Gamma(x_n)$ and satisfying $z_n \to z$, $x_n \to x$. If $d(z, \Gamma(x)) > 0$ then we can choose $j > 0$ such that $0 < 1/j < d(z, \Gamma(x))$. Since Γ is Hausdorff u.s.c. and $d(z_n, \Gamma(x_n)) \leq 1/j$, Proposition 2.2 implies $d(z, \Gamma(x)) \leq 1/j$. This is absurd. Thus we must have $d(z, \Gamma(x)) = 0$. Since $\Gamma(x)$ is closed, $z \in \overline{\Gamma(x)} = \Gamma(x)$ as we claimed.
We denote by $\mathcal{F}$ the set of all finite-dimensional subspaces of X which contains K_1. Fix any $S \in \mathcal{F}$ and $j > 0$ with $1/j < \gamma$, we define the multifunctions $P_j : K \cap S \to 2^{K \cap S}$ by setting $P_j(x) = H_j(x) \cap S$ and the mapping $\overline{P}_j : X \cap Y \to 2^{X \cap Y}$ by putting $\overline{P}_j(x) = \overline{P_j(x)}$. Here $\overline{P_j(x)}$ is closure of $P_j(x)$ in S. By Proposition 2.1, P_j^S is l.s.c. on $K \cap S$ in the relative topology of S. Hence $\overline{P}_j$ is also l.s.c. on $K \cap S$. We now have

$$\begin{aligned}\overline{P_j}(x) =& \overline{H_j(x) \cap S}_S = \overline{\{w \in K \cap S : d(w, \Gamma(x)) < 1/j\}}_S \\ =& \{w \in K \cap S : d(w, \Gamma(x)) \leq 1/j\} = \overline{H}_j(x) \cap S.\end{aligned}$$

Here $\overline{A}_S$ denote the closure of a set A in S. Put

$$\begin{aligned}&\Gamma_j^S(x) = \overline{P_j}(x),\ \Omega = K \cap S,\ K_0 = K_2 \cap S;\\ &M^S = \{x \in \Omega : x \in \Gamma_j^S(x)\}, \Phi^S = \Phi \mid \Omega,\ \psi^S = \psi \mid \Omega \times C \times \Omega.\end{aligned}$$

The task is now to check that Theorem 3.2 can be applied to $IQVI(\Omega, \Gamma_j^S, \Phi^S, \psi^S)$.
(a_1) It is easily seen that Γ_j^S has closed convex valued. Note that, Γ_j^S is lower semicontinuous on Ω as stated above. By Proposition 2.1, $\overline{H}_j$ has closed graph. Since

$$M^S = \{x \in K : x \in \overline{H}_j(x)\} \cap \Omega,$$

we see that M^S is closed. Hence assumption (i) of Theorem 3.2 is valid.
(a_2) Assumption (ii) of Theorem 3.2 is obvious.

(a_3) For each $y \in \Omega$ we have

$$\{x \in M^S : \inf_{z \in \Phi^S(x)} \psi(x, z, y) \leq 0\} = \{x \in K : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \cap M^S$$

which is closed by assumption (iv). Consequently, assumption (iii) of Theorem 3.2 is satisfied.
(a_4) For each $x \in M^S$, we have

$$\{y \in \Omega : \inf_{z \in \Phi^S(x)} \psi(x, z, y) \leq 0\} = \{y \in K : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\} \cap \Omega$$

which is a closed set by assumption (iv). Hence assumption (iv) of Theorem 3.2 is also valid.
(a_5) Conditions (v), (vi) and (vii) of Theorem 3.2 are straightforward to verify.
(a_6) Finally, for each $x \in \Omega \setminus K_0$ and $x \in \Gamma_j^S(x)$, we get $x \in K \setminus K_2$ and $d(x, \Gamma(x)) < 1/j < \gamma$. By (viii), there exists $y \in \Gamma(x) \cap K_1 \subset \Gamma_j^S(x) \cap K_0$ such that

$$\inf_{z \in \Phi(x)} \psi(x, z, y) > 0.$$

Hence condition (viii) of Theorem 3.2 is valid.

Thus all conditions of Theorem 3.2 are fulfilled for the problem $IQVI(\Omega, \Gamma_j^S, \Phi^S, \psi^S)$. By Theorem 3.2, there exists $x_S \in K_0$ such that

$$x_S \in \Gamma_j^S(x_S), \ \inf_{z \in \Phi(x_S)} \psi(x_S, z, y) \leq 0 \ \forall \ y \in \Gamma_j^S(x_S). \tag{5}$$

LEMMA 3.3 *There exists $\hat{x} \in K_2$ such that*

$$\hat{x} \in \overline{H}_j(\hat{x}), \ \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0 \ \forall \ y \in \overline{H}_j(\hat{x}). \tag{6}$$

Proof Put $Q_S = \{x_Y : x_Y \text{ satisfies (5) with } Y \supset S\}$. Then $Q_S \neq \emptyset$ because $x_S \in Q_S$. Moreover the family $\{\overline{Q}_S\}$ has a finite intersection property. Indeed, taking any $Y, Z \in \mathcal{F}$ and putting $M = \text{span}\{Y, Z\}$, we have $Q_M \subset Q_Y \cap Q_Z$. This implies that $\overline{Q}_M \subset \overline{Q_Y \cap Q_Z} \subset \overline{Q}_Y \cap \overline{Q}_Z$. Since $\overline{Q}_S \subset K_2$, $\overline{Q}_S$ is compact for all $S \in \mathcal{F}$. Hence

$$\bigcap_{S \in \mathcal{F}} \overline{Q}_S \neq \emptyset.$$

Consequently, there exists a point $\hat{x} \in K_2$ such that $\hat{x} \in \overline{Q}_S$ for all $S \in \mathcal{F}$.

For any $S \in \mathcal{F}$, there exists a sequence $\{x_n\}$ such that $x_n \in Q_S$ and $x_n \to \hat{x}$. By the definition of Q_S one has

$$x_n \in \overline{H}_j(x_n), \ \inf_{z \in \Phi(x_n)} \psi(x_n, z, y) \leq 0 \ \forall \ y \in \Gamma_j^S(x_n). \tag{7}$$

Note that $\Gamma_j^S(x) = \overline{H}_j(x) \cap S$. Since $\overline{H}_j$ has a closed graph, $\hat{x} \in \overline{H}_j(\hat{x})$. Take any $y \in \Gamma_j^S(\hat{x})$. By the lower semicontinuity of Γ_j^S, there exists a sequence $\{y_n\}$, $y_n \in \Gamma_j^S(x_n)$ such that $y_n \to y$. From (7) one has

$$\inf_{z\in\Phi(x_n)} \psi(x_n, z, y_n\rangle \leq 0.$$

Letting $n \to \infty$ and using (iv) we have $\inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0$. Thus

$$\hat{x} \in \overline{H}_j(\hat{x}), \inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0 \; \forall \; y \in \overline{H}_j(\hat{x}) \cap S. \tag{8}$$

Note that in (8) $\hat{x}$ is independent of S. Take any $v \in \overline{H}_j(\hat{x})$ and put $S' = \text{span}\{v, S\}$. Since $\hat{x}$ satisfies (8) also for S' we have

$$\hat{x} \in \overline{H}_j(\hat{x}), \inf_{z\in\Phi(\hat{x})} \psi(\hat{x}, z, v) \leq 0. \tag{9}$$

Therefore (6) is valid and the lemma is proved.

The following lemma will finish the proof of the theorem.

LEMMA 3.4 *There exists* $(\overline{x}, \overline{z}) \in K_2 \times C$ *such that*

$$\overline{x} \in \Gamma(\overline{x}), \; \overline{z} \in \Phi(\overline{x}), \; \psi(\hat{x}, z, y) \leq 0 \; \forall \; y \in \Gamma(\overline{x}). \tag{10}$$

Proof According to Lemma 3.3, for each j there exists $\hat{x}_j \in K_2$ such that

$$\hat{x}_j \in \overline{H}_j(\hat{x}_j), \inf_{z\in\Phi(\hat{x}_j)} \psi(\hat{x}_j, z, y) \leq 0 \; \forall \; y \in \overline{H}_j(\hat{x}_j). \tag{11}$$

By the compactness of K_2, we can assume that $\hat{x}_j \to \overline{x} \in K_2$ as $j \to \infty$. By the definition of $\overline{H}_j$ we have

$$d(\hat{x}_j, \Gamma(\hat{x}_j)) \leq 1/j < 2/j.$$

Therefore there exists $z_j \in \Gamma(\hat{x}_j)$ such that $\|\hat{x}_j - z_j\| < 2/j$. This implies that $z_j \to \overline{x} in \Gamma(\overline{x})$. Take any $y \in \Gamma(\overline{x})$. By the lower semicontinuity of Γ, there exists a sequence $\{y_j\}$ such that $y_j \in \Gamma(\hat{x}_j)$ and $y_j \to y$. Since $y_j \in \overline{H}_j(\hat{x}_j)$, it follows from (11) that

$$\inf_{z\in\Phi(\hat{x}_j)} \psi(\hat{x}_j, z, y_j) \leq 0.$$

Letting $j \to \infty$ and using (iv) we get $\inf_{z\in\Phi(\overline{x})} \psi(\overline{x}, z, y) \leq 0$. Hence we obtain

$$\sup_{y\in\Gamma(\overline{x})} \inf_{z\in\Phi(\overline{x})} \psi(\overline{x}, z, y) \leq 0.$$

According to Theorem 5 at p. 216 of [1], taking into account assumptions (i), (iv), (vii) and (viii) we then get

$$\inf_{z\in\Phi(\overline{x})} \sup_{y\in\Gamma(\overline{x})} \psi(\overline{x}, z, y) \leq 0. \tag{12}$$

By assumption (vii) the function $z \to \sup_{y\in\Gamma(\overline{x})} \psi(\overline{x}, z, y)$ is l.s.c. on the compact set $\Phi(\overline{x})$ and by (12), there exists $\overline{z} \in \Phi(\overline{x})$ such that

$$\sup_{y\in\Gamma(\overline{x})} \psi(\overline{x}, \overline{z}, y) = \inf_{z\in\Phi(\overline{x})} \sup_{y\in\Gamma(\overline{x})} \psi(\overline{x}, z, y) \leq 0.$$

This means that $(\overline{x}, \overline{z})$ is a solution of $IQVI(X, \Gamma, \Phi, \psi)$. The proof is complete. □

In summary, we have shown that under certain conditions, the problem $IQVI(X, \Gamma, \Phi, \psi)$ has a solution. Our Theorem 3.3 was inspired by Theorem 3.3 in [7], where the assumption on Hausdorff lower semicontinuity of Γ and condition $\mathrm{int}_{\mathrm{aff}(K)}(\Gamma(x)) \neq \emptyset$ were required. These assumptions are necessary for building a section type multifunction $\Gamma(x) \cap S$ which is lower semicontinuous on finite-dimensional subspace S of X. In the infinite-dimensional setting, in general, a lower semicontinuous multifunction does not have such property, even if X is an Hilbert space (see Remark 3.1 of [5]). It is noted that our results are valid not only in a Banach space setting but also in a normed space setting. The reason is that in the proof we do not need the compactness of the convex hull of a compact set. However, in Theorem 3.3, Condition (viii) and Condition (iv) are rather strict. That is the price we have to pay for omitting the Hausdorff lower semicontinuity of Γ and the condition $\mathrm{int}_{\mathrm{aff}(K)}(\Gamma(x)) \neq \emptyset$.

Recently, in [10] the authors have studied a general variational inclusion problem with constraints which covers generalized qusivariational inequalities. An existence theorem is given for the scalar problem which allows them to derive results on solution existence of variational inequalities and Minty variational inequalities. However, these results were obtained under assumptions that Φ is quasimonotone in a certain sense. In our problem neither monotonicity nor continuity of Φ are required.

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