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Long-time behaviour for a model of porous-medium equations with variable coefficients

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By analysing the uniform attractor for multi-valued processes, we study the long-time behaviour of the solutions of a model of non-autonomous porous-medium equations. The result is obtained by using the *a priori* estimates and suitable compactness arguments.

Keywords: porous-medium equations; degenerate parabolic equations; non-autonomous; uniform attractors

AMS Subject Classifications: 35B40; 35B41; 35B45; 35D05

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) with a smooth boundary $\partial\Omega$. We consider the following problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(\rho(x)\nabla\varphi(u)) + f(t, u) = g(x, t), \quad x \in \Omega, \quad t > \tau, \quad (1.1)$$

$$u|_{t=\tau} = u_\tau(x), \quad x \in \Omega, \quad (1.2)$$

$$u|_{\partial\Omega} = 0, \quad (1.3)$$

where $\tau \in \mathbb{R}$, and the functions ρ , φ , f and g satisfy some conditions specified later.

The equation of type (1.1) represents the motion of gas through a porous medium where $u(x, t)$ stands for the gas density. The original model was established by the mass conservation law as follows. Let \mathcal{V} be the velocity. Then

$$u_t + \operatorname{div}(u\mathcal{V}) = f(u), \quad (1.4)$$

where $f(u)$ models the effects of reaction or absorption. In the case of a non-homogeneous medium, $\mathcal{V} = -\rho(x)\nabla P(u)$. Here, P is the pressure and ρ is a given function.

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There are literature works with different views on this type of equation. In many works, it is usually assumed that $P(u) = \mu u^m$ with μ and m as constants and then Equation (1.4) assumes the form

$$u_t - \frac{\mu m}{m+1} \operatorname{div}(\rho(x) \nabla u^{1+m}) = f(u).$$

For a systematic investigation in the case of $\rho = 1$, we refer the readers to [23]. In addition, for the problem with $f = 0$, there are several studies in the situation when $m \rightarrow +\infty$, in relation to the Hele–Shaw problem. See [3–5, 10, 12] and related works. Recently, many works have been carried out for the problem with variants of nonlinearity f . Let us introduce some relevant studies in [7, 15, 19], among others. For more generalized equations, when m is a variable exponent, see [2, 20]. Using an approach similar to that in [1], we study, in this article problem (1.1)–(1.3) when the derivative φ' is a generalization of homogeneous function, ρ may have zeros at some points in Ω and the nonlinearity of f has the form of a power without upper bound. Precisely, we assume that

- (H1) The function $\varphi \in C^1(\mathbb{R})$ and satisfies $m|u|^{p-2} \leq \varphi'(u) \leq M|u|^{p-2}$, where $m, M > 0$ and $p > 2$.
- (H2) $\rho \in L^1_{\text{loc}}(\Omega)$ such that $\liminf_{x \rightarrow z} |x - z|^{-\alpha} \rho(x) > 0$ for some $\alpha \in (0, 2)$ and for every $z \in \overline{\Omega}$.
- (H3) $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $|f(t, u)| \leq C_f(|u|^{q-1} + 1)$ for $C_f > 0$, $q \geq p$ and for all $t \in \mathbb{R}$.
- (H4) There exists $M_f > 0$ such that $f(u)u \geq M_f(|u|^q - 1)$.
For the external force g , we impose a restriction that
- (H5) The function $g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ and satisfies

$$\|g\|_{L^2_b}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(\cdot, s)\|_{L^2(\Omega)}^2 ds < +\infty.$$

The class of functions satisfying (H1) includes the functions such that their derivatives are homogeneous of order $p - 2$, $\varphi'(u) = |u|^{p-2}$ and some generalized forms, such as $\varphi'(u) = |u|^{p-2} \phi(u)$ where $\phi(u)$ is bounded, $m \leq \phi(u) \leq M$ for all $u \in \mathbb{R}$.

Hypothesis (H2) is motivated by the work [6] in which a semi-linear elliptic problem was studied. It ensures that ρ has at most finite zeros in Ω . In various processes, $\rho(x) \sim |x|^\alpha$. It is worth noting that the degeneracy made by ρ prevents us from using the regularization method as in [2] to get the existence result. Moreover, since q has no upper bounds, we may have a set of solutions to (1.1)–(1.3) corresponding to each initial datum from the phase space. That makes our problem, in general, generating a multi-valued process (MVP). The aim of our work is to prove the existence result for (1.1)–(1.3) and study the asymptotic behaviour of solutions over a large amount of time. To this end, we employ the notion of uniform attractors for MVPs, with respect to the symbol (f, g) . For the analysis details, see [13, 17, 18] and a similar approach in [8]. This framework can be seen as an extension for the theory of global attractors which was developed in [9, 11, 21].

In order to study the problem (1.1)–(1.3), we introduce some weighted Sobolev spaces. By $\mathcal{D}_0^{1,2}(\Omega, \rho)$ we denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{\mathcal{D}_0^{1,2}(\Omega, \rho)} = \left(\int_{\Omega} \rho(x) |\nabla v|^2 \right)^{\frac{1}{2}}$$

and let V be the space of all functions $w \in L^p(\Omega)$ satisfying

$$w|_{\partial\Omega} = 0 \quad \text{and} \quad \frac{\partial}{\partial x_i} \left(\rho \frac{\partial w}{\partial x_i} \right) \in L^p(\Omega); \quad i = 1, \dots, N, \tag{1.5}$$

equipped with the norm

$$\|w\|_V = \left(\int_{\Omega} |\operatorname{div}(\rho \nabla w)|^p \right)^{\frac{1}{p}}. \tag{1.6}$$

Putting

$$a(u, \xi) = \int_{\Omega} \rho(x) \nabla \varphi(u) \nabla \xi, \quad \text{for } \xi \in V, \tag{1.7}$$

one can write

$$a(u, \xi) = - \int_{\Omega} \varphi(u) \operatorname{div}(\rho \nabla \xi), \tag{1.8}$$

$$a(u, u) = \int_{\Omega} \rho(x) \varphi'(u) |\nabla u|^2. \tag{1.9}$$

One observes from (H1) that $|\varphi(u)| \leq |u|^{p-1} + |\varphi(0)|$, and $a(u, \xi)$ is well-defined if $u \in L^p(\Omega)$ and $\xi \in V$.

Denoting

$$\eta(u) = \int_0^u \sqrt{\varphi'(s)} ds \tag{1.10}$$

for $u \in \mathbb{R}$, it is obvious that η is an increasing function. Furthermore, it follows from (H1) that

$$\sqrt{m}|u|^{\frac{p}{2}} \leq |\eta(u)| \leq \sqrt{M}|u|^{\frac{p}{2}}. \tag{1.11}$$

From (1.9) and (1.10), we have

$$a(u, u) = \int_{\Omega} \rho(x) |\nabla \eta(u)|^2. \tag{1.12}$$

Definition 1.1 We say that a function $u(x, t)$ is a weak solution of (1.1)–(1.3) in $Q_{\tau, T} = \Omega \times [\tau, T]$ if and only if

$$\begin{aligned} u &\in L^q(Q_{\tau, T}) \cap C([\tau, T]; L^2(\Omega)), \\ \eta(u) &\in \mathcal{D}_0^{1,2}(\Omega, \rho), \\ \frac{\partial u}{\partial t} &\in L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau, T}), \\ u|_{t=\tau} &= u_{\tau} \quad \text{a.e. in } \Omega, \end{aligned}$$

and

$$\int_{\tau}^T a(u, \xi) dt = \int_{Q_{\tau, T}} \left(g(x, t) - f(t, u) - \frac{\partial u}{\partial t} \right) \xi \, dx \, dt \tag{1.13}$$

for every test functions $\xi \in L^p(\tau, T; V) \cap L^q(Q_{\tau, T})$.

Equation (1.1) can be seen as an equation in $L^{p'}(\tau, T; V') + L^q(Q_{\tau, T})$. Here V' is the dual space of V ; p' and q' are the conjugate exponents of p and q , respectively. In this definition, we assume that $\eta(u) \in \mathcal{D}_0^{1,2}(\Omega, \rho)$ since we need the definiteness of $a(u, u)$, which is important for our arguments in Section 2.

We first show some compactness results for our purposes in Section 2. In the last section, we state and prove the main results.

2. Preliminaries

We recall the results of Caldiroli and Musina [6] related to the space $\mathcal{D}_0^{1,2}(\Omega, \rho)$.

PROPOSITION 2.1 *Assume that Ω is a bounded domain in \mathbb{R}^N , $N \geq 2$, and ρ satisfies (H2). Then the following embeddings hold:*

- (i) $\mathcal{D}_0^{1,2}(\Omega, \rho) \hookrightarrow L^{2_\alpha^*}(\Omega)$ continuously,
- (ii) $\mathcal{D}_0^{1,2}(\Omega, \rho) \hookrightarrow L^r(\Omega)$ compactly if $r \in [1, 2_\alpha^*)$,

where $2_\alpha^* = \frac{2N}{N-2+\alpha}$.

By Proposition 2.1, $\|\cdot\|_V$ in (1.6) is the well-defined norm. Indeed, it suffices to check that if $\|w\|_V = 0$ then $w = 0$. For $w \in V$, we have

$$-\int_\Omega w \operatorname{div}(\rho \nabla w) = \int_\Omega \rho |\nabla w|^2 = \|w\|_{\mathcal{D}_0^{1,2}(\Omega, \rho)}^2 \geq \tilde{C}_1 \|w\|_{L^2(\Omega)}^2.$$

In addition

$$\begin{aligned} -\int_\Omega w \operatorname{div}(\rho \nabla w) &\leq \|w\|_{L^2(\Omega)} \|\operatorname{div}(\rho \nabla w)\|_{L^2(\Omega)} \\ &\leq \tilde{C}_2 \|w\|_{L^2(\Omega)} \|\operatorname{div}(\rho \nabla w)\|_{L^p(\Omega)} \end{aligned}$$

for some $\tilde{C}_1, \tilde{C}_2 > 0$. Thus

$$\|w\|_V \geq \frac{\tilde{C}_1}{\tilde{C}_2} \|w\|_{L^2(\Omega)}.$$

The following is the important tools for our arguments.

PROPOSITION 2.2 *Let $\{v_n\}$ be a sequence such that $a(v_n, v_n) \leq C$, for some $C > 0$, and for all $n \in \mathbb{N}$. Then $\{v_n\}$ is precompact in $L^p(\Omega)$.*

Proof From (1.12), we have

$$a(v_n, v_n) = \int_\Omega \rho(x) |\nabla \eta(v_n)|^2.$$

Then we observe that the sequence $\{\eta(v_n)\}$ is bounded in $\mathcal{D}_0^{1,2}(\Omega, \rho)$. By Proposition 2.1, $\{\eta(v_n)\}$ is precompact in $L^\beta(\Omega)$ for all β satisfying $1 < \beta < 2_\alpha^*$ and then $\eta(v_n) \rightarrow \chi$ strongly in $L^\beta(\Omega)$, by replacing with a subsequence if necessary. This ensures that $\eta(v_n) \rightarrow \chi$ a.e. in Ω . In addition, by the boundedness of $\{\eta(v_n)\}$ in $L^\beta(\Omega)$ and (1.11), we see that $\{v_n\}$ is bounded in $L^{\frac{p\beta}{2}}(\Omega)$. It follows that $v_n \rightarrow v$ in $L^{\frac{p\beta}{2}}(\Omega)$. By the monotonicity of η , we deduce that $v_n \rightarrow \eta^{-1}(\chi) = v$ a.e. in Ω .

For any $\epsilon > 0$, by Egorov's theorem, there exists a subset $E \subset \Omega$ with measure $|E| < \epsilon$, such that $v_n \rightarrow v$ uniformly in $\Omega \setminus E$. Taking $r < \frac{p\beta}{2}$, we have

$$\begin{aligned} \int_{\Omega} |v_n - v|^r &= \int_{\Omega \setminus E} |v_n - v|^r + \int_E |v_n - v|^r \\ &\leq \int_{\Omega \setminus E} |v_n - v|^r + \left(\int_E |v_n - v|^{\frac{p\beta}{2}} \right)^{\theta} \epsilon^{1-\theta}, \text{ with } \theta = \frac{2r}{p\beta}. \end{aligned}$$

The last inequality shows that $v_n \rightarrow v$ strongly in $L^r(\Omega)$. Choosing β such that $2 < \beta < 2^*$ and then taking $r = p$, we get the conclusion. ■

The next two propositions can be proved by using the arguments as in [16, Section 12, Chapter 1] with some slight modifications. In what follows, we denote by $V^* = V' + L^{q'}(\Omega)$.

PROPOSITION 2.3 *For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|u - v\|_{L^p(\Omega)}^p \leq \epsilon [a(u, u) + a(v, v)] + C_\epsilon \|u - v\|_{V^*}^p$$

for all $u, v \in S = \{w \mid \eta(w) \in \mathcal{D}_0^{1,2}(\Omega, \rho)\}$.

Proof Assume on contrary that there exist $\epsilon_0 > 0$ and two sequences $u_n, v_n \in S$ such that

$$\|u_n - v_n\|_{L^p(\Omega)}^p > \epsilon_0 [a(u_n, u_n) + a(v_n, v_n)] + n \|u_n - v_n\|_{V^*}^p.$$

Putting

$$\tilde{u}_n = \frac{u_n}{[a(u_n, u_n) + a(v_n, v_n)]^{\frac{1}{p}}}, \quad \tilde{v}_n = \frac{v_n}{[a(u_n, u_n) + a(v_n, v_n)]^{\frac{1}{p}}},$$

we have

$$\|\tilde{u}_n - \tilde{v}_n\|_{L^p(\Omega)}^p > \epsilon_0 + n \|\tilde{u}_n - \tilde{v}_n\|_{V^*}^p. \tag{2.1}$$

Noting that

$$\begin{aligned} a(tu, tu) &= |t|^2 \int_{\Omega} \rho(x) \varphi'(tu) |\nabla u|^2 \leq M |t|^p \int_{\Omega} \rho(x) |u|^{p-2} |\nabla u|^2 \\ &\leq \frac{M}{m} |t|^p a(u, u) \end{aligned}$$

for all $t \in \mathbb{R}$, we get

$$a(\tilde{u}_n, \tilde{u}_n) \leq \frac{M}{m}, \quad a(\tilde{v}_n, \tilde{v}_n) \leq \frac{M}{m}.$$

Therefore, by Proposition 2.2, there exist two subsequences \tilde{u}_{n_m} and \tilde{v}_{n_m} such that

$$\tilde{u}_{n_m} \rightarrow \tilde{u}, \quad \tilde{v}_{n_m} \rightarrow \tilde{v} \text{ strongly in } L^p(\Omega).$$

Since $p > 2$ and $V \subset L^p(\Omega)$, it follows that $L^p(\Omega) \subset L^{p'}(\Omega) \subset V'$. Moreover, since $q \geq p$, one has $L^p(\Omega) \subset L^{p'}(\Omega) \subset L^{q'}(\Omega)$. This implies that $L^p(\Omega) \subset V^*$. By (2.1) we have $\|\tilde{u} - \tilde{v}\|_{V^*} = 0$, and therefore $\tilde{u} = \tilde{v}$. Then $\|\tilde{u} - \tilde{v}\|_{L^p(\Omega)} = 0$, which contradicts (2.1). ■

PROPOSITION 2.4 Assume that $\int_{\tau}^T a(u_n(t), u_n(t))dt \leq C$ and $\{\frac{du_n}{dt}\}$ is bounded in $L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau, T})$. Then $\{u_n\}$ is precompact in $L^p(Q_{\tau, T})$.

Proof From Proposition 2.3, we have

$$\int_{\tau}^T \|u_{n+\ell} - u_n\|_{L^p(\Omega)}^p \leq \epsilon \int_{\tau}^T [a(u_{n+\ell}, u_{n+\ell}) + a(u_n, u_n)] + C_{\epsilon} \int_{\tau}^T \|u_{n+\ell} - u_n\|_{V^*}^p.$$

Hence it suffices to show that the sequence $\{u_n\}$ contains a subsequence, which is a Cauchy sequence in $L^p(\tau, T; V^*)$.

We will prove a stronger claim: There exists a subsequence u_{μ} such that

$$u_{\mu} \rightarrow u \quad \text{in } C([\tau, T]; V^*). \tag{2.2}$$

By the hypotheses, there exists a set $Z \subset [\tau, T]$ with measure $|Z|=0$ such that, for $t \in [\tau, T] \setminus Z$,

$$a(u_n(t), u_n(t)) \leq K_t < \infty.$$

Hence for any $t \notin Z$, there exists a subsequence (dependent of t) such that

$$u_k(t) \rightarrow u(t) \quad \text{strongly in } V^*. \tag{2.3}$$

Now let $\{t_1, t_2, \dots\}$ be a sequence which is dense in $[\tau, T]$, and $t_i \notin Z$. Using (2.3) and a diagonal procedure, we can extract a subsequence u_{μ} such that

$$u_{\mu}(t_i) \rightarrow u(t_i) \quad \text{strongly in } V^* \text{ for all } i. \tag{2.4}$$

Noting that $L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau, T}) \subset L^{q'}(\tau, T; V^*)$, for all $t \in [\tau, T]$, we have

$$\begin{aligned} \|u_{\mu}(t_i) - u_{\mu}(t)\|_{V^*} &= \left\| \int_t^{t_i} u'_{\mu}(s) ds \right\|_{V^*} \\ &\leq \left(\int_t^{t_i} \|u'_{\mu}(s)\|_{V^*}^{q'} ds \right)^{\frac{1}{q'}} |t_i - t|^{\frac{1}{q}} \leq C |t_i - t|^{\frac{1}{q}}. \end{aligned}$$

Then

$$\begin{aligned} \|u_{\mu+r}(t) - u_{\mu}(t)\|_{V^*} &\leq \|u_{\mu+r}(t) - u_{\mu+r}(t_i)\|_{V^*} \\ &\quad + \|u_{\mu+r}(t_i) - u_{\mu}(t_i)\|_{V^*} + \|u_{\mu}(t_i) - u_{\mu}(t)\|_{V^*}, \\ &\leq \|u_{\mu+r}(t_i) - u_{\mu}(t_i)\|_{V^*} + 2C |t_i - t|^{\frac{1}{q}} \end{aligned}$$

for all $t \in [\tau, T]$. By the density of $\{t_i\}$ in $[\tau, T]$, it follows that $\{u_{\mu}\}$ is a Cauchy sequence in V^* uniformly in $t \in [\tau, T]$. The proof is complete. ■

We use the next proposition, which makes the initial condition in problem (1.1)–(1.3) meaningful.

PROPOSITION 2.5 If $u \in L^p(\tau, T; V) \cap L^q(Q_{\tau, T})$ and $\frac{du}{dt} \in L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau, T})$ then $u \in C([\tau, T]; L^2(\Omega))$.

For the proof, we refer the readers to [14].

3. Main results

3.1. Existence of a global solution

THEOREM 3.1 *Under the assumptions (H1)–(H5), the problem (1.1)–(1.3) has at least one weak solution for each $u_\tau \in L^2(\Omega)$.*

Proof Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^n u_{nk}(t)e_k,$$

where $\{e_j\}_{j=1}^\infty$ is a basis of $\mathcal{D}_0^{1,2}(\Omega, \rho) \cap L^p(\Omega)$, which is orthogonal in $L^2(\Omega)$. We get u_n from solving the problem

$$\begin{aligned} \left\langle \frac{du_n}{dt}, e_k \right\rangle &= -\langle Au_n, e_k \rangle - \langle f(u_n), e_k \rangle + \langle g, e_k \rangle, \\ (u_n(\tau), e_k) &= (u_\tau, e_k), \end{aligned}$$

where $Au = \text{div}(\rho(x)\varphi'(u)\nabla u)$.

Since $f, \varphi' \in C(\mathbb{R})$, the Peano theorem ensures the local existence of u_n . We now establish some *a priori* estimates for u_n . We have

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + a(u_n, u_n) + \int_\Omega f(u_n)u_n = \int_\Omega g u_n.$$

Using hypothesis (H4) and the Cauchy inequality, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + a(u_n, u_n) + M_f \int_\Omega |u_n|^q \\ \leq M_f |\Omega| + \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_n\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.1}$$

It follows that

$$\frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 \leq \|u_n\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 + 2M_f |\Omega|.$$

Then

$$\frac{d}{dt} (e^{-t} \|u_n\|_{L^2(\Omega)}^2) \leq e^{-t} (\|g\|_{L^2(\Omega)}^2 + 2M_f |\Omega|).$$

Integrating the last inequality from τ to t , we get

$$\begin{aligned} \|u_n(t)\|_{L^2(\Omega)}^2 &\leq e^{t-\tau} \|u_n(\tau)\|_{L^2(\Omega)}^2 + \int_\tau^t e^{t-s} \|g(\cdot, s)\|_{L^2(\Omega)}^2 ds + 2M_f (e^{t-\tau} - 1) \\ &\leq e^{t-\tau} \|u_n(\tau)\|_{L^2(\Omega)}^2 + 2M_f |\Omega| (e^{t-\tau} - 1) \\ &\quad + \int_\tau^{\tau+1} e^{t-s} \|g(\cdot, s)\|_{L^2(\Omega)}^2 ds + \int_{\tau+1}^{\tau+2} e^{t-s} \|g(\cdot, s)\|_{L^2(\Omega)}^2 ds + \dots \\ &\leq e^{t-\tau} \|u_n(\tau)\|_{L^2(\Omega)}^2 + 2M_f |\Omega| (e^{t-\tau} - 1) + e^{t-\tau} (1 + e^{-1} + e^{-2} + \dots) \|g\|_{L_b^2}^2 \\ &\leq e^{t-\tau} \|u_n(\tau)\|_{L^2(\Omega)}^2 + 2M_f |\Omega| (e^{t-\tau} - 1) + \frac{e^{t-\tau}}{1 - e^{-1}} \|g\|_{L_b^2}^2. \end{aligned}$$

This allows us to state that $\{u_n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega))$, thanks to the fact that $\|u_n(\tau)\|_{L^2(\Omega)} \leq \|u_\tau\|_{L^2(\Omega)}$. Integrating (3.1) on $[\tau, T]$, we have

$$\begin{aligned} & \|u_n(t)\|_{L^2(\Omega)}^2 + 2 \int_\tau^T a(u_n, u_n) dt + 2M_f \|u_n\|_{L^q(Q_{\tau,T})}^q \\ & \leq \|u_n(\tau)\|_{L^2(\Omega)} + \int_\tau^T \|g(\cdot, s)\|_{L^2(\Omega)}^2 ds + \|u_n\|_{L^2(Q_{\tau,T})}^2 + 2M_f |\Omega|(T - \tau). \end{aligned}$$

The last inequality implies that

$$\{u_n\} \text{ is bounded in } L^q(Q_{\tau,T}), \tag{3.2}$$

$$\left\{ \int_\tau^T a(u_n(t), u_n(t)) dt \right\} \text{ is bounded.} \tag{3.3}$$

Taking (H3) into account, we get the estimate

$$\int_{Q_{\tau,T}} |f(u_n)|^{q'} \leq \int_{Q_{\tau,T}} C(1 + |u_n|^{q-1})^{q'} \leq \int_{Q_{\tau,T}} C(1 + |u_n|^q).$$

Then $\{f(u_n)\}$ is bounded in $L^{q'}(Q_{\tau,T})$ and

$$f(u_n) \rightharpoonup \chi \text{ in } L^{q'}(Q_{\tau,T}). \tag{3.4}$$

On the other hand, we rewrite the equation as

$$\frac{du_n}{dt} = a(u_n, \cdot) - f(t, u_n) + g(x, t)$$

and implement the following estimates:

$$\begin{aligned} \left| \int_\tau^T a(u_n, v) dt \right| &= \left| \int_\tau^T dt \int_\Omega \varphi(u_n) \operatorname{div}(\rho(x)\nabla v) \right| \\ &\leq \int_\tau^T dt \int_\Omega (|u_n|^{p-1} + |\varphi(0)|) |\operatorname{div}(\rho\nabla v)| \\ &\leq C \int_\tau^T \left(\|u_n\|_{L^p(\Omega)}^{\frac{p}{p-1}} + 1 \right) \|v\|_V dt \\ &\leq C \left(\|u_n\|_{L^p(Q_{\tau,T})}^{\frac{p}{p-1}} + 1 \right) \|v\|_{L^p(\tau,T;V)}, \\ |\langle f(u_n), v \rangle| &\leq \|f(u_n)\|_{L^{q'}(Q_{\tau,T})} \|v\|_{L^q(Q_{\tau,T})}, \\ |\langle g, v \rangle| &\leq \|g\|_{L^{q'}(Q_{\tau,T})} \|v\|_{L^q(Q_{\tau,T})} \end{aligned}$$

for all $v \in L^p(\tau, T; V) \cap L^q(Q_{\tau,T})$. Then it follows that $\left\{ \frac{du_n}{dt} \right\}$ is bounded in $L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau,T})$. Combining this with (3.3) and using Proposition 2.4 we conclude that $\{u_n\}$ is precompact in $L^p(Q_{\tau,T})$. Hence we can assume that

- $u_n \rightarrow u$ strongly in $L^p(Q_{\tau,T})$,
- $u_n \rightarrow u$ a.e. in $Q_{\tau,T}$.

Since $f \in C(\mathbb{R})$, it follows that $f(u_n) \rightarrow f(u)$ a.e. in $Q_{\tau,T}$. Thanks to (3.4), one has

$$f(u_n) \rightharpoonup f(u) \quad \text{weakly in } L^q(Q_{\tau,T}). \tag{3.5}$$

Analogously, since $\{u_n\}$ is bounded in $L^p(Q_{\tau,T})$, one can see that $\{\varphi(u_n)\}$ is bounded in $L^{p'}(Q_{\tau,T}) \subset L^{p'}(0, T; V')$. Therefore $\varphi(u_n) \rightharpoonup \Phi$ weakly in $L^{p'}(0, T; V')$. Putting this together with the fact that $\varphi(u_n) \rightarrow \varphi(u)$ a.e. in $Q_{\tau,T}$, we have $\varphi(u_n) \rightharpoonup \varphi(u) = \Phi$ weakly in $L^{p'}(0, T; V')$.

Finally, passing to the limit as $n \rightarrow \infty$ in the relation

$$\int_{\tau}^T \langle u'_n, v \rangle + \int_{\tau}^T dt \int_{\Omega} \varphi(u_n) \operatorname{div}(\rho \nabla v) + \int_{\tau}^T dt \int_{\Omega} f(u_n) v = \int_{\tau}^T dt \int_{\Omega} g v$$

for $v \in L^p(\tau, T; V) \cap L^q(Q_{\tau,T})$, and using Proposition 2.5 we conclude that u is a weak solution of (1.1)–(1.3). ■

3.2. Existence of the uniform attractor

Let us recall some definitions and related results. The pair of functions $(f(s, \cdot), g(\cdot, s)) = \sigma(s)$ is called a symbol of (1.1). We consider (1.1) with a family of symbols including the shifted forms $\sigma(s+h) = (f(s+h, \cdot), g(\cdot, s+h))$ and the limits of some sequence $\{\sigma(s+h_n)\}_{n \in \mathbb{N}}$ in an appropriate topological space Ξ . The family of such symbols is said to be the hull of σ in Ξ and is denoted by $\mathcal{H}(\sigma)$, i.e.

$$\mathcal{H}(\sigma) = cl_{\Xi} \{ \sigma(s+h) \mid h \in \mathbb{R} \}.$$

If the hull $\mathcal{H}(\sigma)$ is a compact set in Ξ , we say that σ is *translation compact* in Ξ .

Let E be a Banach space (which in our case will be $L^2(\Omega)$), $\mathbb{R}_d = \{(t, \tau) \in \mathbb{R}^2 \mid t \geq \tau\}$ and $\mathcal{P}(E) = \{B \subset E \mid B \neq \emptyset\}$. Assume that Σ is a subspace of Ξ .

Definition 3.1 A family of mappings $\{U_{\sigma} : \mathbb{R}_d \times E \rightarrow \mathcal{P}(E)\}_{\sigma \in \Sigma}$ is called an MVP if there exists a continuous group $\{T(h) : \Sigma \rightarrow \Sigma\}_{h \in \mathbb{R}}$ such that for all $\sigma \in \Sigma$, $x \in E$ we have

- (1) $U_{\sigma}(\tau, \tau, x) = x$, for all $\tau \in \mathbb{R}$;
- (2) $U_{\sigma}(t, \tau, x) \subset U_{\sigma}(t, s, U_{\sigma}(s, \tau, x))$ for all $t \geq s \geq \tau$;
- (3) $U_{\sigma}(t+h, \tau+h, x) \subset U_{T(h)\sigma}(t, \tau, x)$ for all $(t, \tau) \in \mathbb{R}_d$, $h \in \mathbb{R}$.

Denote by

$$C_q = \{ \psi \in C(\mathbb{R}; \mathbb{R}) : |\psi(u)| \leq C_{\psi}(1 + |u|^{q-1}) \text{ for } C_{\psi} > 0 \},$$

$$\|\psi\|_{C_q} = \sup_{u \in \mathbb{R}} \frac{|\psi(u)|}{1 + |u|^{q-1}}.$$

Then C_q is a Banach space. We say that $f_n \rightarrow f$ in the space $C(\mathbb{R}; C_q)$ if

$$\lim_{n \rightarrow +\infty} \sup_{s \in [t, t+r]} \|f_n(s, \cdot) - f(s, \cdot)\|_{C_q} = 0$$

for all $t \in \mathbb{R}$, $r > 0$.

Let $f_0 \in C(\mathbb{R}; C_q)$, $g_0 \in L_{loc}^{2,w}(\mathbb{R}; L^2(\Omega))$ and

$$\begin{aligned} \mathcal{H}(f_0) &= cl_{C(\mathbb{R}; C_q)}\{f_0(s+h) \mid h \in \mathbb{R}\}, \\ \mathcal{H}(g_0) &= cl_{L_{loc}^{2,w}(\mathbb{R}; L^2(\Omega))}\{g_0(s+h) \mid h \in \mathbb{R}\}, \end{aligned}$$

where the topology in $L_{loc}^{2,w}(\mathbb{R}; L^2(\Omega))$ is equipped by local weak convergence, i.e. $g_n \rightarrow g$ in $L_{loc}^{2,w}(\mathbb{R}; L^2(\Omega))$ if

$$\lim_{n \rightarrow +\infty} \int_t^{t+r} \int_{\Omega} (g_n(x, s) - g(x, s))\phi(x, s) dx ds = 0$$

for all $t \in \mathbb{R}$, $r > 0$ and $\phi \in L^2(Q_{t, t+r})$.

Let us take $\Sigma = \mathcal{H}(f_0) \times \mathcal{H}(g_0)$. For each $\sigma = (f, g) \in \Sigma$, we define

$$U_{\sigma}(t, \tau, v) = \{u = u(t) \mid u \text{ is the solution of (1.1)–(1.3), } u(\tau) = v\}.$$

Then $\{U_{\sigma}\}_{\sigma \in \Sigma}$ is an MVP with respect to the translation group $T(h)\sigma = \sigma(\cdot + h)$ by the arguments in [22]. In addition, U_{σ} is a strict MVP, that is $U_{\sigma}(t, \tau, x) = U_{\sigma}(t, s, U(s, \tau, x))$ for all $t \geq s \geq \tau$.

Denote by

$$U_{\Sigma}(t, \tau, v) = \bigcup_{\sigma \in \Sigma} U_{\sigma}(t, \tau, v)$$

and $B_R = B(0, R)$, the ball in E centred at 0 with radius R .

Definition 3.2 The set $\mathcal{A}_{\Sigma} \subset E$ is called a uniform attractor of MVP $\{U_{\sigma}\}_{\sigma \in \Sigma}$ if $\mathcal{A}_{\Sigma} \neq E$ and

- (1) \mathcal{A}_{Σ} is a uniformly attracting set, that is, for any $R > 0$, for all $\tau \in \mathbb{R}$,

$$\text{dist}(U_{\Sigma}(t, \tau, B_R), \mathcal{A}_{\Sigma}) \rightarrow 0 \text{ as } t \rightarrow +\infty;$$

- (2) \mathcal{A}_{Σ} is a minimal uniformly attracting set, that is, if Λ_{Σ} is a uniformly attracting set then $\mathcal{A}_{\Sigma} \subset cl_E \Lambda_{\Sigma}$.

Here the notation $\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_E$, that is the Hausdorff semi-distance between two sets in E .

We use the following theorem as a sufficient condition for the existence of a uniform attractor described above. For a proof, see [18].

THEOREM 3.2 *If the family of MVP $\{U_{\sigma}\}_{\sigma \in \Sigma}$ satisfies the following conditions*

- (1) *there exists $R_0 > 0$ such that, for all $R > 0$, $\text{dist}(U_{\Sigma}(t, 0, B_R), B_{R_0}) \rightarrow 0$ as $t \rightarrow +\infty$,*
- (2) *for all $R > 0$ and $\{t_n \mid t_n \nearrow +\infty\}$, $\{\xi_n \mid \xi_n \in U_{\Sigma}(t_n, 0, B_R)\}$ is precompact in E ,*

then $\{U_{\sigma}\}_{\sigma \in \Sigma}$ has a uniform attractor

$$\mathcal{A}_{\Sigma} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} U_{\Sigma}(t, 0, B_{R_0+1})},$$

which is compact in E .

In order to deal with a uniform attractor, with respect to the family of symbols, one usually requires the translation compact property. Let us recall some discussions on this requirement. It is known that the hypothesis (H5) ensures that g is translation compact in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$, (see [8] for details). In addition, the following statement gives a sufficient condition for the translation compact property in $C(\mathbb{R}; \mathcal{C}_q)$.

PROPOSITION 3.3 [8] *The function $f \in C(\mathbb{R}; \mathcal{C}_q)$ is translation compact if, and only if, for all $R > 0$ one has*

- (1) $|f(t, v)| \leq C(R)$ for all $t \in \mathbb{R}, v \in [-R, R]$,
- (2) $|f(t_1, v_1) - f(t_2, v_2)| \leq \alpha(|t_1 - t_2| + |v_1 - v_2|, R)$ for all $t_1, t_2 \in \mathbb{R}; v_1, v_2 \in [-R, R]$.

Here $C(R) > 0$ and $\alpha(s, R) \rightarrow 0$ as $s \rightarrow 0^+$.

It is easy to see that, if $f \in C^1(\mathbb{R}^2; \mathbb{R})$ such that $|f(t, v)| \leq C(R), \left| \frac{\partial f}{\partial v}(t, v) \right| \leq C(R)$ and $\left| \frac{\partial f}{\partial t}(t, v) \right| \leq C(R)$ for all $t \in \mathbb{R}$ and for all $v \in [-R, R]$ then the hypotheses in Proposition 3.3 are satisfied. For instance, the function $f(t, u) = |u|^{q-2}u \arctan t$ is well-checked.

From now on, we suppose that f is translation compact. Together with the fact that g is translation compact in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$, one has that Σ is a compact set in $C(\mathbb{R}; \mathcal{C}_q) \times L^2_{loc}(\mathbb{R}; L^2(\Omega))$. Then it follows from [8] that $T(h) : \Sigma \rightarrow \Sigma$ is continuous and $T(h)\Sigma \subset \Sigma$ for all $h \in \mathbb{R}$.

We need the following lemma to prove the dissipative property of MVP.

LEMMA 3.4 *If $u(t)$ is a weak solution of (1.1)–(1.3) then*

$$\|u(t)\|_{L^2(\Omega)}^2 \leq e^{-(t-\tau)} \|u_\tau\|_{L^2(\Omega)}^2 + M_0(1 - e^{-(t-\tau)}) + \frac{\|g\|_{L^2_b}^2}{1 - e^{-1}},$$

where $M_0 = M_0(q, |\Omega|, M_f) > 0$.

Proof From (1.1), using (H4) and Cauchy inequality, we have

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + 2a(u, u) + 2M_f \|u\|_{L^q(\Omega)}^q \\ \leq 2M_f |\Omega| + \|u(t)\|_{L^2(\Omega)}^2 + \|g(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.6}$$

Since $q > 2$, using Young's inequality, one has

$$2\|u(t)\|_{L^2(\Omega)}^2 \leq C\|u(t)\|_{L^q(\Omega)}^2 \leq 2M_f \|u(t)\|_{L^q(\Omega)}^q + M_q,$$

where $M_q > 0$. Putting this into (3.6), we obtain

$$\frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^2(\Omega)}^2 \leq 2M_f |\Omega| + M_q + \|g(t)\|_{L^2(\Omega)}^2.$$

Then

$$\frac{d}{dt} (e^t \|u(t)\|_{L^2(\Omega)}^2) \leq e^t (2M_f |\Omega| + M_q) + e^t \|g(t)\|_{L^2(\Omega)}^2.$$

Taking integration over $[\tau, t]$, we arrive at

$$\begin{aligned}
 \|u(t)\|_{L^2(\Omega)}^2 &\leq e^{-(t-\tau)}\|u_n\|_{L^2(\Omega)}^2 + (1 - e^{-(t-\tau)})(2M_f|\Omega| + M_q) \\
 &\quad + \int_{\tau}^t e^{-(t-s)}\|g(s)\|_{L^2(\Omega)}^2 ds \\
 &\leq e^{-(t-\tau)}\|u_n\|_{L^2(\Omega)}^2 + (1 - e^{-(t-\tau)})(2M_f|\Omega| + M_q) \\
 &\quad + \int_{t-1}^t e^{-(t-s)}\|g(s)\|_{L^2(\Omega)}^2 ds + \int_{t-2}^{t-1} e^{-(t-s)}\|g(s)\|_{L^2(\Omega)}^2 ds + \cdots \\
 &\leq e^{-(t-\tau)}\|u_n\|_{L^2(\Omega)}^2 + (1 - e^{-(t-\tau)})(2M_f|\Omega| + M_q) \\
 &\quad + (1 + e^{-1} + e^{-2} + \cdots) \sup_{t \in \mathbb{R}} \|g(s)\|_{L^2(\Omega)}^2 \\
 &\leq e^{-(t-\tau)}\|u_n\|_{L^2(\Omega)}^2 + (1 - e^{-(t-\tau)})(2M_f|\Omega| + M_q) + \frac{\|g\|_{L_b^2}^2}{1 - e^{-1}}.
 \end{aligned}$$

So we complete the proof. \blacksquare

LEMMA 3.5 *Let the hypotheses (H1)–(H5) hold. Assume that $\{u_n\}_{n \in \mathbb{N}}$ is a sequence of weak solutions of (1.1)–(1.3) with respect to the sequence of symbols $\{\sigma_n\}_{\sigma_n \in \Sigma, n \in \mathbb{N}}$ such that*

- (1) $u_n(\tau) \rightharpoonup u_\tau$ in $L^2(\Omega)$,
- (2) $\sigma_n \rightarrow \sigma$ in Σ ,

then there exists a solution u of (1.1)–(1.3) with respect to the symbol σ such that $u(\tau) = u_\tau$ and $u_n(t^*) \rightarrow u(t^*)$ in $L^2(\Omega)$ for any $t^* > \tau$.

Proof We will adapt the technique as in [13,22] to prove this statement. Let $\sigma_n = (f_n, g_n)$. Since f satisfies (H3)–(H4) for all $t \in \mathbb{R}$ and $f_n \in \mathcal{H}(f)$, one sees that $f_n, n \in \mathbb{N}$ also satisfies (H3)–(H4) with the same constants C_f and M_f . From (1.1), we have

$$\begin{aligned}
 \frac{d}{dt} \|u_n(t)\|_{L^2(\Omega)}^2 + a(u_n, u_n) + 2M_f \|u_n\|_{L^q(\Omega)}^q \\
 \leq 2M_f |\Omega| + \|g_n(t)\|_{L^2(\Omega)}^2 + \|u_n(t)\|_{L^2(\Omega)}^2
 \end{aligned}$$

for all $t > \tau$. Using the same arguments as in the proof of Lemma 3.4, we have

$$\|u_n(t)\|_{L^2(\Omega)}^2 \leq e^{-(t-\tau)}\|u_n(\tau)\|_{L^2(\Omega)}^2 + M_0(1 - e^{-(t-\tau)}) + \frac{1}{1 - e^{-1}}\|g_n\|_{L_b^2}^2.$$

Since $\{u_n(\tau)\}$ is bounded and $\|g_n\|_{L_b^2}^2 \leq \|g\|_{L_b^2}^2$, we have $\{u_n\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$ for given $T > \tau$. In particular,

$$u_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega) \text{ for each } t \in [\tau, T] \tag{3.7}$$

up to a subsequence. Now using the arguments as in the proof of Theorem 3.1, we have

- $\int_{\tau}^T a(u_n, u_n) dt$ is bounded,
- $\{u_n\}$ is bounded in $L^q(Q_{\tau, T})$,

- $\{f_n(u_n)\}$ is bounded in $L^{q'}(Q_{\tau,T})$,
- $\{u'_n\}$ is bounded in $L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau,T})$.

Using Proposition 2.4, one gets that $\{u_n\}$ is precompact in $L^p(Q_{\tau,T})$ and then, by replacing $\{u_n\}$ with a subsequence if necessary, we have

- $u_n \rightarrow u$ a.e in $Q_{\tau,T}$,
- $\varphi(u_n) \rightharpoonup \varphi(u)$ in $L^{p'}(\tau, T; V')$,
- $u'_n \rightharpoonup u'$ in $L^{p'}(\tau, T; V') + L^{q'}(Q_{\tau,T})$.

Let $\sigma_n \rightarrow \sigma = (\bar{f}, \bar{g})$ in Σ . In order to prove that u is a weak solution of (1.1)–(1.3) with respect to the symbol σ , we need to pass to the limit in following relation:

$$\int_{\tau}^T \langle u'_n, v \rangle + \int_{\tau}^T dt \int_{\Omega} \varphi(u_n) \operatorname{div}(\rho \nabla v) + \int_{\tau}^T dt \int_{\Omega} f_n(u_n)v = \int_{\tau}^T dt \int_{\Omega} g_n v$$

for $v \in L^p(\tau, T; V) \cap L^q(Q_{\tau,T})$. Since $g_n \rightharpoonup \bar{g}$ in $L^2(\tau, T; L^2(\Omega))$, it remains to prove that $f_n(u_n) \rightharpoonup \bar{f}(u)$ in $L^{q'}(Q_{\tau,T})$. We have the stronger claim, that is $f_n(u_n) \rightarrow \bar{f}(u)$ strongly in $L^{q'}(Q_{\tau,T})$. Indeed,

$$\begin{aligned} \int_{\tau}^T \int_{\Omega} |f_n(t, u_n) - \bar{f}(t, u)|^{q'} dx dt &\leq 2^{q'} \int_{\tau}^T \int_{\Omega} |\bar{f}(t, u_n) - \bar{f}(t, u)|^{q'} dx dt \\ &\quad + 2^{q'} \int_{\tau}^T \int_{\Omega} \frac{|f_n(t, u_n) - \bar{f}(t, u_n)|^{q'}}{(1 + |u_n|^{q-1})^{q'}} (1 + |u_n|^{q-1})^{q'} dx dt \\ &\leq 2^{q'} \int_{\tau}^T \int_{\Omega} |\bar{f}(t, u_n) - \bar{f}(t, u)|^{q'} dx dt \\ &\quad + \left(4 \sup_{[\tau, T]} |f_n - \bar{f}|_{C_q}\right)^{q'} \int_{\tau}^T \int_{\Omega} (1 + |u_n|^q) dx dt. \end{aligned}$$

Then the boundedness of $\{u_n\}$ in $L^q(Q_{\tau,T})$ and the continuity of \bar{f} guarantee our claim. In addition, by Proposition 2.5, we get $u \in C([\tau, T]; L^2(\Omega))$. Then the initial condition makes sense.

We are in a position to show that $u_n(t^*) \rightarrow u(t^*)$ in $L^2(\Omega)$ for any $t^* > \tau$. Taking into account of (3.7), we have to check that $\|u_n(t^*)\|_{L^2(\Omega)} \rightarrow \|u(t^*)\|_{L^2(\Omega)}$ in \mathbb{R} .

Let us denote

$$\begin{aligned} J_n(t) &= \|u_n(t)\|_{L^2(\Omega)}^2 - 2 \int_{\tau}^t (g_n(s), u_n(s))_{L^2(\Omega)} dt - M|\Omega|(t - \tau), \\ J(t) &= \|u(t)\|_{L^2(\Omega)}^2 - 2 \int_{\tau}^t (g(s), u(s))_{L^2(\Omega)} dt - M|\Omega|(t - \tau) \end{aligned}$$

for some $M > 0$. Then $J_n, J \in C([\tau, T]; \mathbb{R})$. Arguing as in the proof of Lemma 3.4, u_n satisfies the estimate

$$\frac{d}{dt} \|u_n(t)\|_{L^2(\Omega)}^2 \leq M|\Omega| + 2(g_n(t), u_n(t))_{L^2(\Omega)},$$

and the same estimate is valid for u . Hence, J_n and J are decreasing on $[\tau, T]$. We first show that

$$J_n(t) \rightarrow J(t) \text{ for a.e. } t \in [\tau, T]. \tag{3.8}$$

Obviously

$$\begin{aligned}
 & |J_n(t) - J(t)| \\
 & \leq \left| \|u_n(t)\|_{L^2(\Omega)}^2 - \|u(t)\|_{L^2(\Omega)}^2 \right| + 2 \left| \int_{\tau}^t (g_n, u_n)_{L^2(\Omega)} - (g, u)_{L^2(\Omega)} dt \right| \\
 & \leq \|u_n(t) - u(t)\|_{L^2(\Omega)} (\|u_n(t)\|_{L^2(\Omega)} + \|u(t)\|_{L^2(\Omega)}) \\
 & \quad + 2 \left| \int_{\tau}^t (g_n, u_n - u)_{L^2(\Omega)} dt \right| + 2 \left| \int_{\tau}^t (g_n - g, u)_{L^2(\Omega)} dt \right|.
 \end{aligned}$$

Moreover, one has

$$\left| \int_{\tau}^t (g_n, u_n - u)_{L^2(\Omega)} dt \right| \leq \|g_n\|_{L^2(Q_{\tau,t})} \|u_n - u\|_{L^2(Q_{\tau,t})} \rightarrow 0$$

as $n \rightarrow \infty$ since $u_n \rightarrow u$ strongly in $L^2(Q_{\tau,t})$ and $\{g_n\}$ is bounded in $L^2(Q_{\tau,t})$. In addition

$$\int_{\tau}^t (g_n - g, u)_{L^2(\Omega)} dt \rightarrow 0$$

as $n \rightarrow \infty$ since $g_n \rightarrow g$ in $L^2(Q_{\tau,t})$. Then (3.8) is proved due to the fact that $u_n(t) \rightarrow u(t)$ in $L^2(\Omega)$ for a.e. $t \in (\tau, T)$.

Suppose that $\{t_m\}$ is an increasing sequence in $[\tau, T]$ such that $t_m \rightarrow t^*$ as $m \rightarrow \infty$. Then

- $J_n(t_m) \rightarrow J_n(t^*)$ as $m \rightarrow \infty$,
- $J_n(t_m) \rightarrow J(t_m)$ as $n \rightarrow \infty$.

So for $\epsilon > 0$, we have eventually

$$J_n(t^*) - J(t^*) \leq J_n(t_m) - J(t^*) = J_n(t_m) - J(t_m) + J(t_m) - J(t^*) < \epsilon.$$

Similarly, $J(t^*) - J_n(t^*) < \epsilon$. Therefore $J_n(t^*) \rightarrow J(t^*)$ and then

$$\|u_n(t^*)\|_{L^2(\Omega)} \rightarrow \|u(t^*)\|_{L^2(\Omega)} \text{ as } n \rightarrow \infty. \quad \blacksquare$$

The following theorem is the main result in this section.

THEOREM 3.6 *Under the hypotheses (H1)–(H5), the MVP $\{U_{\sigma}\}_{\sigma \in \Sigma}$ generated by the problem (1.1)–(1.3) possesses a uniform attractor which is a compact set in $L^2(\Omega)$.*

Proof Note that each symbol $\sigma_n = (f_n, g_n) \in \Sigma$ satisfies the same conditions as in (H3)–(H5). Furthermore, since $g_n \in \mathcal{H}(g)$, we have $\|g_n\|_{L^2_b} \leq \|g\|_{L^2_b}$. Hence it follows from Lemma 3.4 that, if u_n is the weak solution of (1.1)–(1.3) with respect to the symbol σ_n , one has

$$\|u_n(t)\|_{L^2(\Omega)}^2 \leq e^{-(t-\tau)} \|u_n(\tau)\|_{L^2(\Omega)}^2 + M_0(1 - e^{-(t-\tau)}) + \frac{\|g\|_{L^2_b}^2}{1 - e^{-1}}.$$

The last inequality ensures that, if $u_n(\tau) \in B_R$ then there exists $T_0 = T_0(\tau, R)$ such that

$$u_n(t) \in B_{R_0}, \text{ for all } t \geq T_0,$$

where

$$R_0 = 2M_0 + \frac{\|g\|_{L^2_h}^2}{1 - e^{-1}}.$$

That is $U_\Sigma(t, \tau, B_R) \subset B_{R_0}$ for all $t \geq T_0(\tau, R)$. This fulfils the first condition in Theorem 3.2. We now verify the second condition.

Let $t_n \nearrow +\infty$ and $\xi_n \in U_\Sigma(t_n, 0, B_R)$. Then there exists a sequence of solutions $\{u_n\}$ of (1.1)–(1.3) with respect to the sequence of symbols $\{\sigma_n\}$ such that $\xi_n = u_n(t_n)$. For given $t^* > 0$, we have

$$\begin{aligned} U_{\sigma_n}(t_n, 0, B_R) &= U_{\sigma_n}(t^* + t_n - t^*, 0, B_R) \\ &\subset U_{\sigma_n}(t^* + t_n - t^*, t_n - t^*, U_{\sigma_n}(t_n - t^*, 0, B_R)) \\ &\subset U_{\sigma_n}(t^* + t_n - t^*, t_n - t^*, B_{R_0}) \end{aligned}$$

for $t_n \geq T_0 + t^*$.

Hence

$$U_{\sigma_n}(t_n, 0, B_R) \subset U_{T(t_n - t^*)\sigma_n}(t^*, 0, B_{R_0}).$$

Now $\xi_n \in U_{\bar{\sigma}_n}(t^*, 0, B_{R_0})$ where $\bar{\sigma}_n = T(t_n - t^*)\sigma_n \in \Sigma$. That is, $\xi_n = v_n(t^*)$ where v_n is the weak solution of (1.1)–(1.3) with respect to the symbol $\bar{\sigma}_n$. Suppose that $\bar{\sigma}_n \rightarrow \bar{\sigma}$ in Σ . By Lemma 3.5, there exists a solution v of (1.1)–(1.3) with respect to the symbol $\bar{\sigma}$, such that $\xi_n = v_n(t^*) \rightarrow v(t^*)$ in $L^2(\Omega)$ and thus the proof completes. ■

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