Approximating fixed points of α-nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces

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Abstract. An existence theorem for a fixed point of an α-nonexpansive mapping of a nonempty bounded, closed and convex subset of a uniformly convex Banach space is recently established by Aoyama and Kohsaka with a non-constructive argument. In this paper, we show that appropriate Ishihawa iterate algorithms ensure weak and strong convergence to a fixed point of such a mapping. Our theorems are also extended to CAT(0) spaces.

Keywords. α-nonexpansive mapping; fixed point; Ishihawa iteration algorithm; uniformly convex Banach space; CAT(0) space.

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1. Introduction

The purpose of this paper is to study fixed point theorems of α-nonexpansive mappings of CAT(0) spaces. A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane (see Section 4 for the precise definition). Our approach is to prove firstly weak and strong convergence theorems for Ishikawa iterations of α-nonexpansive mappings in uniformly convex Banach spaces. Then, we extend the results to CAT(0) spaces.

Here are the details. Let E be a (real) Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. Denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. We say that T is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all x, y in C, and that T is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all x in C and y in $F(T)$.

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The concept of nonexpansivity of a map T from a convex set C into C plays an important role in the study of the *Mann-type iteration* given by

$$x_{n+1} = \beta_n T x_n + (1-\beta_n) x_n, \qquad x_1 \in C. \tag{1.1}$$

Here, $\{\beta_n\}$ is a real sequence in $[0,1]$ satisfying some appropriate conditions, which is usually called a *control sequence*. A more general iteration scheme is the *Ishikawa iteration*, given by

$$\begin{cases} y_n = \beta_n T x_n + (1-\beta_n) x_n, \\ x_{n+1} = \gamma_n T y_n + (1-\gamma_n) x_n, \end{cases} \tag{1.2}$$

where the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy some appropriate conditions. In particular, when all $\beta_n = 0$, the Ishikawa iteration (1.2) becomes the standard Mann iteration (1.1). Let T be nonexpansive and let C be a nonempty closed and convex subset of a uniformly convex Banach space E satisfying the Opial property. Takahashi and Kim [1] proved that, for any initial data x_1 in C, the iterates $\{x_n\}$ defined by the Ishikawa iteration (1.2) converges weakly to a fixed point of T, with appropriate choices of control sequences $\{\beta_n\}$ and $\{\gamma_n\}$.

Following Aoyama and Kohsaka [2], a mapping $T : C \to E$ is said to be *α-nonexpansive* for some real number $\alpha < 1$ if

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 + (1-2\alpha)\|x - y\|^2, \quad \forall x, y \in C.$$

Clearly, 0-nonexpansive maps are exactly nonexpansive maps. Moreover, T is Lipschitz continuous whenever $\alpha \leq 0$. An example of a discontinuous α-nonexpansive mapping (with $\alpha > 0$) has been given in [2]. See also Example 3.6(b) below.

An existence theorem for a fixed point of an α-nonexpansive mapping T of a nonempty bounded, closed and convex subset C of a uniformly convex Banach space E is established in [2] with a non-constructive argument. In Section 3, we show that, under mild conditions on the control sequences $\{\beta_n\}$ and $\{\gamma_n\}$, the fixed point set $F(T)$ is nonempty if and only if the sequence $\{x_n\}$ obtained by the Ishikawa iteration (1.2) is bounded and $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$. In this case, $\{x_n\}$ converges weakly or strongly to a fixed point in $F(T)$.

In Section 4, together with other elementary generalizations we establish the existence result, Theorem 4.7, of fixed points of an α-nonexpansive mapping in a CAT(0)-space in parallel to [2]. In Section 5, we extend the convergence theorems for Ishikawa iterations obtained in Section 3 to the case of CAT(0) spaces, as we plan.

2. Preliminaries

Let E be a (real) Banach space with norm $\|\cdot\|$ and dual space E^*. Denote by $x_n \to x$ the strong convergence of a sequence $\{x_n\}$ to x in E, and by $x_n \rightharpoonup x$ the weak convergence. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $0 < \epsilon \le 2$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each x, y in S, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. If the limit (2.1) is attained uniformly in x, y in S, then E is called *uniformly smooth*. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S$ and $x \ne y$. It is well-known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [3].

A Banach space E is said to satisfy the *Opial property* [4] if for every weakly convergent sequence $x_n \rightharpoonup x$ in E we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all y in E with $y \ne x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces and the Banach spaces l^p $(1 \le p < \infty)$ satisfy the Opial property, while the uniformly convex spaces $L_p[0, 2\pi]$ $(p \ne 2)$ do not; see, for example, [4, 5, 6].

Let $\{x_n\}$ be a bounded sequence in a Banach space E. For any x in E, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|.$$

The *asymptotic radius* of $\{x_n\}$ relative to a nonempty closed and convex subset C of E is defined by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center* of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that if E is uniformly convex then $A(C, \{x_n\})$ consists of exactly one point; see [7, 8].

Lemma 2.1. *Let C be a nonempty subset of a Banach space E. Let $T : C \to E$ be an α-nonexpansive mapping for some $\alpha < 1$ such that $F(T) \ne \emptyset$. Then T is quasi-nonexpansive. Moreover, $F(T)$ is norm closed.*

Proof. Let $x \in C$ and $z \in F(T)$. Then we have

$$\begin{aligned} \|Tx - z\|^2 &= \|Tx - Tz\|^2 \\ &\leq \alpha\|Tx - z\|^2 + \alpha\|Tz - x\|^2 + (1 - 2\alpha)\|x - z\|^2 \\ &= \alpha\|Tx - z\|^2 + \alpha\|z - x\|^2 + (1 - 2\alpha)\|x - z\|^2 \\ &= \alpha\|Tx - z\|^2 + (1 - \alpha)\|x - z\|^2. \end{aligned}$$

Therefore,

$$\|Tx - z\| \leq \|x - z\|.$$

This inequality ensures the closedness of $F(T)$. □

Lemma 2.2. *Let C be a nonempty subset of a Banach space E. Let $T : C \to E$ be an α-nonexpansive mapping for some $\alpha < 1$. Then the following assertions hold.*

(i) If $0 \leq \alpha < 1$, then

$$\|x - Ty\|^2 \leq \frac{1+\alpha}{1-\alpha}\|x - Tx\|^2 + \frac{2}{1-\alpha}(\alpha\|x - y\| + \|Tx - Ty\|)\|x - Tx\| + \|x - y\|^2, \quad \forall x, y \in C.$$

(ii) If $\alpha < 0$, then

$$\|x - Ty\|^2 \leq \|x - Tx\|^2 + \frac{2}{1-\alpha}[(-\alpha)\|Tx - y\| + \|Tx - Ty\|]\|x - Tx\| + \|x - y\|^2, \quad \forall x, y \in C.$$

Proof. (i) Observe

$$\begin{aligned} \|x - Ty\|^2 &= \|x - Tx + Tx - Ty\|^2 \\ &\leq (\|x - Tx\| + \|Tx - Ty\|)^2 \\ &= \|x - Tx\|^2 + \|Tx - Ty\|^2 + 2\|x - Tx\|\|Tx - Ty\| \\ &\leq \|x - Tx\|^2 + \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2 \\ &\quad + 2\|x - Tx\|\|Tx - Ty\| \\ &\leq \|x - Tx\|^2 + \alpha(\|Tx - x\| + \|x - y\|)^2 \\ &\quad + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2 + 2\|x - Tx\|\|Tx - Ty\| \\ &\leq \|x - Tx\|^2 + \alpha\|Tx - x\|^2 + \alpha\|x - y\|^2 \\ &\quad + 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - Ty\|^2 \\ &\quad + (1 - 2\alpha)\|x - y\|^2 + 2\|x - Tx\|\|Tx - Ty\| \\ &= (1 + \alpha)\|x - Tx\|^2 + 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - Ty\|^2 \\ &\quad + (1 - \alpha)\|x - y\|^2 + 2\|x - Tx\|\|Tx - Ty\|. \end{aligned}$$

This implies that

$$\|x - Ty\|^2 \leq \frac{1+\alpha}{1-\alpha}\|x - Tx\|^2 + \frac{2}{1-\alpha}(\alpha\|x - y\| + \|Tx - Ty\|)\|x - Tx\| + \|x - y\|^2.$$

(ii) Observe

$$
\begin{aligned}
\|x-Ty\|^2 &= \|x-Tx+Tx-Ty\|^2 \\
&\le (\|x-Tx\|+\|Tx-Ty\|)^2 \\
&= \|x-Tx\|^2+\|Tx-Ty\|^2+2\|x-Tx\|\|Tx-Ty\| \\
&\le \|x-Tx\|^2+\alpha\|Tx-y\|^2+\alpha\|x-Ty\|^2+(1-2\alpha)\|x-y\|^2 \\
&\quad +2\|x-Tx\|\|Tx-Ty\| \\
&= \|x-Tx\|^2+\alpha\|Tx-y\|^2+\alpha\|x-Ty\|^2 \\
&\quad +(1-\alpha)\|x-y\|^2-\alpha\|x-y\|^2+2\|x-Tx\|\|Tx-Ty\| \\
&\le \|x-Tx\|^2+\alpha\|Tx-y\|^2+\alpha\|x-Ty\|^2 \\
&\quad +(1-\alpha)\|x-y\|^2-\alpha[\|x-Tx\|^2+\|Tx-y\|^2+2\|x-Tx\|\|Tx-y\|] \\
&\quad +2\|x-Tx\|\|Tx-Ty\| \\
&= (1-\alpha)\|x-Tx\|^2+\alpha\|x-Ty\|^2 \\
&\quad +(1-\alpha)\|x-y\|^2-2\alpha\|x-Tx\|\|Tx-y\|+2\|x-Tx\|\|Tx-Ty\| \\
&= (1-\alpha)\|x-Tx\|^2+\alpha\|x-Ty\|^2 \\
&\quad +(1-\alpha)\|x-y\|^2+2[(-\alpha)\|Tx-y\|+\|Tx-Ty\|]\|x-Tx\|.
\end{aligned}
$$

This implies that

$$
\|x-Ty\|^2 \le \|x-Tx\|^2+\frac{2}{1-\alpha}[(-\alpha)\|Tx-y\|+\|Tx-Ty\|]\|x-Tx\|+\|x-y\|^2.
$$

□

Proposition 2.3 (Demiclosedness Principle). *Let C be a subset of a Banach space E with the Opial property. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.*

Proof. Since $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, both $\{x_n\}$ and $\{Tx_n\}$ are bounded. Let $M_1 = \sup\{\|x_n\|, \|Tx_n\|, \|z\|, \|Tz\| : n \in \mathbb{N}\} < \infty$. If $0 \le \alpha < 1$ then, in view of Lemma 2.2(i),

$$
\begin{aligned}
&\|x_n-Tz\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\|x_n-Tx_n\|^2+\frac{2}{1-\alpha}(\alpha\|x_n-z\|+\|Tx_n-Tz\|)\|x_n-Tx_n\|+\|x_n-z\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\|x_n-Tx_n\|^2+\frac{4M_1(1+\alpha)}{1-\alpha}\|x_n-Tx_n\|+\|x_n-z\|^2.
\end{aligned}
$$

If $\alpha < 0$ then, in view of Lemma 2.2(ii),

$$
\begin{aligned}
&\|x_n-Tz\|^2 \\
\le\ & \|x_n-Tx_n\|^2+\frac{2}{1-\alpha}[(-\alpha)\|Tx_n-z\|+\|Tx_n-Tz\|]\|x_n-Tx_n\|+\|x_n-z\|^2 \\
\le\ & \|x_n-Tx_n\|^2+4M_1\|x_n-Tx_n\|+\|x_n-z\|^2.
\end{aligned}
$$

These imply

$$\limsup_{n\to\infty} \|x_n - Tz\| \le \limsup_{n\to\infty} \|x_n - z\|.$$

From the Opial property, we obtain $Tz = z$. □

The following result has been proved in [9].

Lemma 2.4. *Let $r > 0$ be a fixed real number. If E is a uniformly convex Banach space, then there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \le \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),$$

for all x, y in $B_r(0) = \{u \in E : \|u\| \le r\}$ and $\lambda \in [0, 1]$.

Recently, Aoyama and Kohsaka [2] proved the following fixed point theorem for α-nonexpansive mappings of Banach spaces.

Lemma 2.5. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Then the following conditions are equivalent.*

(i) There exists x in C such that $\{T^n x\}_{n=1}^{\infty}$ is bounded.

(ii) $F(T) \neq \emptyset$.

3. Fixed Point and Convergence Theorems in Banach Spaces

Lemma 3.1. *Let C be a nonempty closed and convex subset of a Banach space E. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by the Ishikawa iteration (1.2) such that $\{\beta_n\}$ and $\{\gamma_n\}$ are arbitrary sequences in $[0, 1]$. Suppose that the fixed point set $F(T)$ contains an element z. Then the following assertions hold.*
(1) $\max\{\|x_{n+1} - z\|, \|y_n - z\|\} \le \|x_n - z\|$ for all $n = 1, 2, \ldots$.
(2) $\lim_{n\to\infty} \|x_n - z\|$ exists.
(3) $\lim_{n\to\infty} d(x_n, F(T))$ exists, where $d(x, F(T))$ denotes the distance from x to $F(T)$.

Proof. In view of Lemma 2.1, we conclude that

$$\begin{aligned}\|y_n - z\| &= \|\beta_n T x_n + (1-\beta_n)x_n - z\| \\ &\leq \beta_n \|Tx_n - z\| + (1-\beta_n)\|x_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_{n+1} - z\| &= \|\gamma_n T y_n + (1-\gamma_n)x_n - z\| \\ &\leq \gamma_n \|Ty_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &\leq \gamma_n \|y_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &\leq \gamma_n \|x_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned}$$

This implies that $\{\|x_n - z\|\}$ is a bounded and nonincreasing sequence. Thus, $\lim_{n\to\infty} \|x_n - z\|$ exists.

In the same manner, we see that $\{d(x_n, F(T))\}$ is also a bounded nonincreasing real sequence, and thus converges. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$, and let $\{x_n\}$ be a sequence with x_1 in C defined by the Ishikawa iteration (1.2).*

1. *If $\{x_n\}$ is bounded and $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$, then the fixed point set $F(T) \neq \emptyset$.*
2. *Assume $F(T) \neq \emptyset$. Then $\{x_n\}$ is bounded, and the following hold.*

Case 1: $0 < \alpha < 1$**.**

(a) $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$ *when* $\limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.

(b) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ *when* $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.

Case 2: $\alpha \leq 0$**.**

(a) $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$ *when*

$$\begin{cases} \liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \liminf_{n\to\infty} \beta_n < 1, \end{cases} \quad \text{or} \quad \begin{cases} \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \limsup_{n\to\infty} \beta_n < 1. \end{cases}$$

(b) $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ *when* $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ *and* $\limsup_{n\to\infty}\beta_n < 1$.

Proof. Assume that $\{x_n\}$ is bounded and $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. There is a bounded subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$. Suppose $A(C, \{x_{n_k}\}) = \{z\}$. Let $M_1 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : k \in \mathbb{N}\} < \infty$. If $0 \le \alpha < 1$, then, by Lemma 2.2 (i), we have

$$\begin{aligned}
&\|x_{n_k} - Tz\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\|x_{n_k} - Tx_{n_k}\|^2 + \frac{2}{1-\alpha}(\alpha\|x_{n_k} - z\| + \|Tx_{n_k} - Tz\|)\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\|x_{n_k} - Tx_{n_k}\|^2 + \frac{4M_1(1+\alpha)}{1-\alpha}\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\limsup_{k\to\infty}\|x_{n_k} - Tz\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\limsup_{k\to\infty}\|x_{n_k} - Tx_{n_k}\|^2 + \frac{4M_1(1+\alpha)}{1-\alpha}\limsup_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| + \limsup_{k\to\infty}\|x_{n_k} - z\|^2 \\
=\ & \limsup_{k\to\infty}\|x_{n_k} - z\|^2.
\end{aligned}$$

If $\alpha < 0$, then, by Lemma 2.2 (ii), we have

$$\begin{aligned}
&\|x_{n_k} - Tz\|^2 \\
\le\ & \|x_{n_k} - Tx_{n_k}\|^2 + \frac{2}{1-\alpha}[(-\alpha)\|Tx_{n_k} - z\| + \|Tx_{n_k} - Tz\|)\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\|x_{n_k} - Tx_{n_k}\|^2 + \frac{4M_1(1+\alpha)}{1-\alpha}\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|^2.
\end{aligned}$$

This implies again that

$$\begin{aligned}
&\limsup_{k\to\infty}\|x_{n_k} - Tz\|^2 \\
\le\ & \frac{1+\alpha}{1-\alpha}\limsup_{k\to\infty}\|x_{n_k} - Tx_{n_k}\|^2 + \frac{4M_1(1+\alpha)}{1-\alpha}\limsup_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| + \limsup_{k\to\infty}\|x_{n_k} - z\|^2 \\
=\ & \limsup_{k\to\infty}\|x_{n_k} - z\|^2.
\end{aligned}$$

Thus, we have in all cases

$$\begin{aligned}
r(Tz, \{x_{n_k}\}) &= \limsup_{n\to\infty}\|x_{n_k} - Tz\| \\
&\le \limsup_{n\to\infty}\|x_{n_k} - z\| \\
&= r(z, \{x_{n_k}\}).
\end{aligned}$$

This means that $Tz \in A(C, \{x_{n_k}\})$. By the uniform convexity of E we conclude that $Tz = z$.

Conversely, let $F(T) \neq \emptyset$ and let $z \in F(T)$. It follows from Lemma 3.1 that $\lim_{n\to\infty} \|x_n - z\|$ exists and hence $\{x_n\}$ is bounded. In view of Lemmas 2.1 and 2.4, we obtain a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\gamma_n T y_n + (1-\gamma_n)x_n - z\|^2 \\ &\leq \gamma_n \|Ty_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|Ty_n - x_n\|) \\ &\leq \gamma_n \|y_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|Ty_n - x_n\|) \\ &\leq \gamma_n \|x_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|Ty_n - x_n\|) \\ &= \|x_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|Ty_n - x_n\|). \end{aligned} \tag{3.1}$$

In view of (3.1), we conclude with Lemma 3.1 that

$$\begin{aligned} \gamma_n(1-\gamma_n) g(\|Ty_n - x_n\|) &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

It follows that

$$\liminf_{n\to\infty} g(\|Ty_n - x_n\|) = 0 \quad \text{whenever} \quad \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0.$$

From the property of g we deduce that

$$\liminf_{n\to\infty} \|Ty_n - x_n\| = 0 \quad \text{in case} \quad \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0. \tag{3.2}$$

In the same manner, we also obtain that

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0 \quad \text{in case} \quad \liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0. \tag{3.3}$$

On the other hand, from (1.2) we get

$$Tx_n - y_n = (1-\beta_n)(Tx_n - x_n), \quad x_n - y_n = \beta_n(x_n - Tx_n). \tag{3.4}$$

Observing (3.4), we see that the assertions about the case $\alpha \leq 0$ follow from (3.2) and (3.3).

In the following, we discuss the case $0 < \alpha < 1$. Assuming first $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. By Lemma 2.1 and (3.3) we see that $M_2 := \sup\{\|Tx_n\|, \|Ty_n\| : n \in \mathbb{N}\} < \infty$. Since T is α-nonexpansive, in view of (3.4), we obtain

$$\begin{aligned} & \|Tx_n - x_n\|^2 \\ = \ & \|Tx_n - Ty_n + Ty_n - x_n\|^2 \\ \leq \ & (\|Tx_n - Ty_n\| + \|Ty_n - x_n\|)^2 \\ = \ & \|Tx_n - Ty_n\|^2 + \|Ty_n - x_n\|^2 + 2\|Tx_n - Ty_n\|\|Ty_n - x_n\| \\ \leq \ & \alpha\|Tx_n - y_n\|^2 + \alpha\|Ty_n - x_n\|^2 + (1-2\alpha)\|x_n - y_n\|^2 + \|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\| \\ \leq \ & \alpha\|(1-\beta_n)(Tx_n - x_n)\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + (1-2\alpha)\|\beta_n(x_n - Tx_n)\|^2 + 4M_2\|Ty_n - x_n\| \\ \leq \ & [\alpha(1-\beta_n)^2 + (1-2\alpha)\beta_n^2]\|Tx_n - x_n\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\|. \end{aligned} \tag{3.5}$$

Case (i): If $0 < \alpha < \frac{1}{2}$, then (3.5) becomes

$$\begin{aligned}
& \|Tx_n - x_n\|^2 \\
\leq\ & [\alpha(1-\beta_n)^2 + (1-2\alpha)\beta_n^2]\|Tx_n - x_n\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\| \\
=\ & (1-\alpha)\|Tx_n - x_n\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\|,
\end{aligned}$$

since all β_n are in $[0,1]$. We then derive from (3.3) that

$$\|Tx_n - x_n\|^2 \leq \tfrac{1+\alpha}{\alpha}\|Ty_n - x_n\|^2 + \tfrac{4M_2}{\alpha}\|Ty_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{3.6}$$

Case (ii): If $\frac{1}{2} \leq \alpha < 1$, then (3.5) becomes

$$\begin{aligned}
& \|Tx_n - x_n\|^2 \\
\leq\ & [\alpha(1-\beta_n)^2 + (1-2\alpha)\beta_n^2]\|Tx_n - x_n\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\| \\
\leq\ & \alpha\|Tx_n - x_n\|^2 + (\alpha+1)\|Ty_n - x_n\|^2 + 4M_2\|Ty_n - x_n\|.
\end{aligned}$$

We then derive from (3.3) again that

$$\|Tx_n - x_n\|^2 \leq \tfrac{1+\alpha}{1-\alpha}\|Ty_n - x_n\|^2 + \tfrac{4M_2}{1-\alpha}\|Ty_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{3.7}$$

Finally, we assume $\limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ instead. By (3.2) we have subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$\lim_{k\to\infty} \|Ty_{n_k} - x_{n_k}\| = 0.$$

Replacing M_2 by the number $\sup\{\|Tx_{n_k}\|, \|Ty_{n_k}\| : k \in \mathbb{N}\} < \infty$ and dealing with the subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ in (3.6) and (3.7), we will arrive at the desired conclusion that $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. This gives $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E with the Opial property. Let $T : C \to C$ be an α-nonexpansive mapping with a nonempty fixed point set $F(T)$ for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$, and let $\{x_n\}$ be a sequence with x_1 in C defined by the Ishikawa iteration (1.2).*

Assume that $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$, and assume, in addition, $\limsup_{n\to\infty} \beta_n < 1$ if $\alpha \leq 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. It follows from Theorem 3.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The uniform convexity of E implies that E is reflexive; see, for example, [3]. Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. In view of Proposition 2.3, we conclude that $p \in F(T)$. We claim that $x_n \rightharpoonup p$ as $n \to \infty$. Suppose on contrary that there existed a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some q in C with $p \neq q$. By

Proposition 2.3, we see that $q \in F(T)$. Lemma 3.1 says that $\lim_{n\to\infty} \|x_n - z\|$ exists for all z in $F(T)$. The Opial property then implies

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{i\to\infty} \|x_{n_i} - p\| &< \lim_{i\to\infty} \|x_{n_i} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| &= \lim_{j\to\infty} \|x_{n_j} - q\| \\ &< \lim_{j\to\infty} \|x_{n_j} - p\| &= \lim_{n\to\infty} \|x_n - p\|. \end{aligned}$$

This is a contradiction. Thus $p = q$, and the desired assertion follows. ☐

Theorem 3.4. *Let C be a nonempty compact and convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$.*

When $0 < \alpha < 1$, we assume $\limsup_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$. When $\alpha \le 0$, we assume either

$$\begin{cases} \liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0, \\ \liminf_{n\to\infty} \beta_n < 1, \end{cases} \quad \textit{or} \quad \begin{cases} \limsup_{n\to\infty} \gamma_n(1 - \gamma_n) > 0, \\ \limsup_{n\to\infty} \beta_n < 1. \end{cases}$$

Let $\{x_n\}$ be a sequence with x_1 in C defined by the Ishikawa iteration (1.2). Then $\{x_n\}$ converges strongly to a fixed point z of T.

Proof. Since C is bounded, it follows from Lemma 2.5 that the fixed point set $F(T)$ of T is nonempty. In view of Theorem 3.2, the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$. By the compactness of C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strongly to some z in C, and $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. In particular, $\{Tx_{n_k}\}$ is bounded. Let $M_3 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : k \in \mathbb{N}\} < \infty$. If $0 \le \alpha < 1$ then, in view of Lemma 2.2(i), we obtain

$$\begin{aligned} &\|x_{n_k} - Tz\|^2 \\ &\le \tfrac{1+\alpha}{1-\alpha}\|x_{n_k} - Tx_{n_k}\|^2 + \tfrac{2}{1-\alpha}(\alpha\|x_{n_k} - z\| + \|Tx_{n_k} - Tz\|)\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|^2 \\ &\le \tfrac{1+\alpha}{1-\alpha}\|x_{n_k} - Tx_{n_k}\|^2 + \tfrac{4M_3(1+\alpha)}{1-\alpha}\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{k\to\infty} \|x_{n_k} - Tz\|^2 \\ \le\ &\tfrac{1+\alpha}{1-\alpha} \limsup_{k\to\infty} \|x_{n_k} - Tx_{n_k}\|^2 + \frac{4M_3(1+\alpha)}{1-\alpha} \limsup_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| + \limsup_{k\to\infty} \|x_{n_k} - z\|^2. \end{aligned}$$

If $\alpha < 0$ then, in view of Lemma 2.2(ii), we obtain

$$\begin{aligned} &\|x_{n_k} - Tz\|^2 \\ &\le \|x_{n_k} - Tx_{n_k}\|^2 + \tfrac{2}{1-\alpha}[(-\alpha)\|Tx_{n_k} - z\| + \|Tx_{n_k} - Tz\|]\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|^2 \\ &\le \|x_{n_k} - Tx_{n_k}\|^2 + \tfrac{4M_3(1-\alpha)}{1-\alpha}\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{k\to\infty}\|x_{n_k}-Tz\|^2\\ \le\ &\limsup_{k\to\infty}\|x_{n_k}-Tx_{n_k}\|^2+4M_3\limsup_{k\to\infty}\|Tx_{n_k}-x_{n_k}\|+\limsup_{k\to\infty}\|x_{n_k}-z\|^2.\end{aligned}$$

It follows $\lim_{k\to\infty}\|x_{n_k}-Tz\|=0$. Thus we have $Tz=z$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n-z\|$ exists. Therefore, z is the strong limit of the sequence $\{x_n\}$. □

Let C be a nonempty closed and convex subset of a Banach space E. A mapping $T : C \to C$ is said to satisfy *condition* (I) [10] if

> there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that
>
> $$d(x,Tx) \ge f(d(x,F(T))), \quad \forall x \in C.$$

Using Theorem 3.2, we can prove the following result.

Theorem 3.5. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be an α-nonexpansive mapping with a nonempty fixed point set $F(T)$ for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. When $0 < \alpha < 1$, we assume $\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0$. When $\alpha \le 0$, we assume either*

$$\begin{cases}\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0,\\ \liminf_{n\to\infty}\beta_n < 1,\end{cases}\quad \text{or} \quad \begin{cases}\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0,\\ \limsup_{n\to\infty}\beta_n < 1.\end{cases}$$

Let $\{x_n\}$ be a sequence with x_1 in C defined by the Ishikawa iteration (1.2). If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point z of T.

Proof. It follows from Theorem 3.2 that

$$\liminf_{n\to\infty}\|Tx_n-x_n\|=0.$$

Therefore, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\|Tx_{n_k}-x_{n_k}\|=0.$$

Since T satisfies condition (I), with respect to the sequence $\{x_{n_k}\}$, we obtain

$$\lim_{k\to\infty}d(x_{n_k},F(T))=0.$$

This implies that, there exist a subsequence of $\{x_{n_k}\}$, denoted also by $\{x_{n_k}\}$, and a sequence $\{z_k\}$ in $F(T)$ such that

$$d(x_{n_k}, z_k) < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}. \tag{3.8}$$

In view of Lemma 3.1, we have

$$\|x_{n_{k+1}} - z_k\| \leq \|x_{n_k} - z_k\| < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}.$$

This implies

$$\begin{aligned} \|z_{k+1} - z_k\| &\leq \|z_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - z_k\| \\ &\leq \tfrac{1}{2^{(k+1)}} + \tfrac{1}{2^k} \\ &< \tfrac{1}{2^{(k-1)}}, \quad \forall k = 1, 2, \ldots. \end{aligned}$$

Consequently, $\{z_k\}$ is a Cauchy sequence in $F(T)$. Due to the closedness of $F(T)$ in E (see Lemma 2.1), we deduce that $\lim_{k\to\infty} z_k = z$ for some z in $F(T)$. It follows from (3.8) that $\lim_{k\to\infty} x_{n_k} = z$. By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - z\|$ exists. This forces $\lim_{n\to\infty} \|x_n - z\| = 0$. □

The following examples explain why we need to impose some conditions on the control sequences in previous theorems.

Examples 3.6 (a) Let $T : [-1, 1] \to [-1, 1]$ be defined by $Tx = -x$. Then T is a 0-nonexpansive (i.e. nonexpansive) mapping. Setting all $\beta_n = 1$, the Ishikawa iteration (1.2) provides a sequence

$$x_{n+1} = \gamma_n T^2 x_n + (1 - \gamma_n)x_n = x_n, \quad \forall n = 1, 2, \ldots,$$

no matter how we choose $\{\gamma_n\}$. Unless $x_1 = 0$, we can never reach the unique fixed point 0 of T via $\{x_n\}$.

(b) Let $T : [0, 4] \to [0, 4]$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \neq 4, \\ 2 & \text{if } x = 4. \end{cases}$$

Then T is a $\frac{1}{2}$-nonexpansive mapping. Indeed, for any x in $[0, 4)$ and $y = 4$, we have

$$|Tx - Ty|^2 = 4 \leq 8 + \frac{1}{2}|x - 2|^2 = \frac{1}{2}|Tx - y|^2 + \frac{1}{2}|x - Ty|^2.$$

The other cases can be verified similarly. It is worth mentioning that T is neither nonexpansive nor continuous. Setting all $\beta_n = 1$, the Ishikawa iteration (1.2) provides a sequence

$$x_{n+1} = \gamma_n T^2 x_n + (1 - \gamma_n)x_n, \quad \forall n = 1, 2, \ldots.$$

For any arbitrary starting point x_1 in $[0,4]$, we have $T^2x_n = 0$ and

$$\begin{aligned} x_{n+1} &= (1-\gamma_n)x_n \\ &= (1-\gamma_1)(1-\gamma_2)...(1-\gamma_n)x_1 \\ &= \prod_{k=1}^{n}(1-\gamma_k)x_1, \qquad \forall n = 1,2,.... \end{aligned}$$

Consider two possible choices of the values of γ_n:

Case 1. If we set $\gamma_n = \frac{1}{2}$, $\forall n = 1,2,...$, then $\lim_{n\to\infty}\gamma_n(1-\gamma_n) = 1/4 > 0$, and $x_n \to 0$, the unique fixed point of T.

Case 2. If we set $\gamma_n = \frac{1}{(n+1)^2}$, $\forall n = 1,2,...$, then $\lim_{n\to\infty}\gamma_n(1-\gamma_n) = 0$, and $x_n = \frac{n+2}{2n+2}x_1 \to x_1/2$. Unless $x_1 = 0$, we can never reach the unique fixed point 0 of T via x_n.

4. Preliminaries on CAT(0) Spaces

Let (X,d) be a metric space. A *geodesic path* joining x to y in X (or briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ into X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t-t'|$ for all t,t' in $[0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be a *uniquely geodesic* if there exists exactly one geodesic joining x and y for each x,y in X. A subset Y of X is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in X (the vertices of Δ), together with a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ together with a one-to-one correspondence $x \mapsto \bar{x}$ from Δ onto $\bar{\Delta}$ such that it is an isometry on each of the three segments. A geodesic space X is said to be a CAT(0) *space* if all geodesic triangles Δ satisfy the CAT(0) *inequality*:

$$d(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y}), \quad \forall x,y \in \Delta.$$

It is easy to see that a CAT(0) space is uniquely geodesic.

It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include inner product spaces, $\mathbb{R}$-trees (see, for example [11]), Euclidean building (see, for example [12]), and the complex Hilbert ball with a hyperbolic metric (see, for example [8]). For a thorough discussion of other spaces and of the fundamental role they play in geometry, see, for example, [12, 13, 14].

We collect some properties in CAT(0) spaces. For more details, we refer the readers to [15, 16, 17].

Lemma 4.1 *([16]). Let (X, d) be a* CAT(0) *space. Then the following assertions hold.*
(i) For x, y in X and t in $[0, 1]$, there exists a unique point z in $[x, y]$ such that

$$d(x, z) = td(x, y) \qquad \text{and} \qquad d(y, z) = (1-t)d(x, y). \tag{4.1}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (4.1).
(ii) For x, y in X and t in $[0, 1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

The notion of asymptotic centers in a Banach space can be extended to a CAT(0) space as well, by simply replacing the distance defined by $\| \cdot - \cdot \|$ with the one by the metric $d(\cdot, \cdot)$. In particular, in a CAT(0) space, $A(C, \{x_n\})$ consists of exactly one point whenever C is a closed and convex set and $\{x_n\}$ is a bounded sequence; see [18, Proposition 7].

Definition 4.2 *([19, 20]). A sequence $\{x_n\}$ in a* CAT(0) *space X is said to Δ-converge to x in X if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$, and we call x the Δ-limit of $\{x_n\}$.*

Lemma 4.3 *([19]). Every bounded sequence in a complete* CAT(0) *space X always has a Δ-convergent subsequence.*

Lemma 4.4 *([21]). Let C be a closed and convex subset of a complete* CAT(0) *space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 4.5 *([22]). Let X be a complete* CAT(0) *space and let $x \in X$. Suppose that $0 < b \leq t_n \leq c < 1$, and $x_n, y_n \in X$ for $n = 1, 2, \ldots$. If for some $r \geq 0$ we have*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \ \limsup_{n\to\infty} d(y_n, x) \leq r, \ \text{and} \ \lim_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) = r,$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Recall that the *Ishikawa iteration* in CAT(0) spaces is described as follows: for any initial point x_1 in C, we define the iterates $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n T x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} = \gamma_n T y_n \oplus (1-\gamma_n)x_n, \end{cases} \tag{4.2}$$

where the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy some appropriate conditions.

We introduce the notion of α-nonexpansive mappings of CAT(0) spaces.

Definition 4.6. *Let C be a nonempty subset of a* CAT(0) *space X and let $\alpha < 1$. A mapping $T : C \to X$ is said to be α-nonexpansive if*

$$d(Tx,Ty)^2 \le \alpha d(Tx,y)^2 + \alpha d(x,Ty)^2 + (1-2\alpha)d(x,y)^2, \quad \forall x,y \in C.$$

The following is the CAT(0) counterpart to Lemma 2.5. However, we do not know if the compactness assumption can be removed from the negative α case.

Theorem 4.7. *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. In case $0 \le \alpha < 1$, we have $F(T) \ne \emptyset$ if and only if $\{T^n x\}_{n=1}^\infty$ is bounded for some x in C. If C is compact, we always have $F(T) \ne \emptyset$.*

Proof. Assume first that $0 \le \alpha < 1$. The necessity is obvious. We verify the sufficiency. Suppose that $\{T^n x\}_{n=1}^\infty$ is bounded for some x in C. Set $x_n := T^n x$ for $n = 1, 2, \ldots$. By the boundedness of $\{x_n\}_{n=1}^\infty$, there exists z in X such that $A(C, \{x_n\}) = \{z\}$. It follows from Lemma 4.4 that $z \in C$. Furthermore, we have

$$d(x_n, Tz)^2 \le \alpha d(x_n, z)^2 + \alpha d(x_{n-1}, Tz)^2 + (1-2\alpha)d(x_{n-1}, z)^2, \quad \forall n = 1, 2, \ldots.$$

This implies

$$\limsup_{n\to\infty} d(x_n, Tz)^2 \le \alpha \limsup_{n\to\infty} d(x_n, z)^2 + \alpha \limsup_{n\to\infty} d(x_{n-1}, Tz)^2 + (1-2\alpha) \limsup_{n\to\infty} d(x_{n-1}, z)^2.$$

Thus,

$$\limsup_{n\to\infty} d(x_n, Tz) \le \limsup_{n\to\infty} d(x_n, z).$$

Consequently, $Tz \in A(\{x_n\}) = \{z\}$, ensuring that $F(T) \ne \emptyset$.

Next, we assume $\alpha < 0$ and C is compact. In particular, T is continuous and the sequence of $x_n := T^n x$ for any x in C is bounded. In the following, we adapt the arguments in [1] with slight modifications.

Let μ be a Banach limit, i.e., μ is a bounded unital positive linear functional of ℓ_∞ such that $\mu \circ s = \mu$. Here, s is the left shift operator on ℓ_∞. We write $\mu_n\, a_n$ for the value of $\mu(a)$ with $a = (a_n)$ in ℓ_∞ as usual. In particular, $\mu_n\, a_{n+1} = \mu(s(a)) = \mu(a) = \mu_n\, a_n$. As showed in [1, Lemmas 3.1 and 3.2], we have

$$\mu_n\, d(x_n, Ty)^2 \le \mu_n\, d(x_n, y)^2, \quad \forall y \in C, \tag{4.3}$$

and

$$g(y) := \mu_n \, d(x_n, y)^2$$

defines a continuous function from C into $\mathbb{R}$.

By compactness, there exists y in C such that $g(y) = \inf g(C)$. Suppose that there were another z in C such that $g(z) = g(y)$. Let m be the midpoint in the geodesic segment joining y to z. In view of Lemma 4.1, we see that g is convex. Thus, $g(m) = g(y)$ too. Observing the comparison triangles in $\mathbb{E}^2$, we have

$$d(x_n, y)^2 + d(x_n, z)^2 \geq 2d(x_n, m)^2 + \frac{1}{2}d(y, z)^2, \quad \forall n = 1, 2, \ldots.$$

Consequently,

$$\mu_n \, d(x_n, y)^2 + \mu_n \, d(x_n, z)^2 \geq 2\mu_n \, d(x_n, m)^2 + \frac{1}{2}\mu_n \, d(y, z)^2.$$

This amounts to say

$$g(y) + g(z) \geq 2g(m) + \frac{1}{2}d(y, z)^2.$$

Since $g(y) = g(z) = g(m)$, we have $y = z$. Finally, it follows from (4.3) that $g(Ty) \leq g(y) = \inf g(C)$. By uniqueness, we have $Ty = y \in F(T)$. $\square$

The proofs of the following results are similar to those in Sections 2 and 3.

Lemma 4.8. *Let C be a nonempty subset of a* CAT(0) *space X. Let $T : C \to X$ be an α-nonexpansive mapping for some $\alpha < 1$ such that $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Lemma 4.9. *Let C be a nonempty closed and convex subset of a* CAT(0) *space X. Let $T : C \to X$ be an α-nonexpansive mapping for some $\alpha < 1$. Then the following assertions hold.*
(i) If $0 \leq \alpha < 1$, then

$$d(x, Ty)^2 \leq \tfrac{1+\alpha}{1-\alpha}d(x, Tx)^2 + \tfrac{2}{1-\alpha}(\alpha d(x, y) + d(Tx, Ty))d(x, Tx) + d(x, y)^2, \quad \forall x, y \in C.$$

(ii) If $\alpha < 0$, then

$$d(x, Ty)^2 \leq d(x, Tx)^2 + \frac{2}{1-\alpha}[(-\alpha)d(Tx, y) + d(Tx, Ty)]d(x, Tx) + d(x, y)^2, \quad \forall x, y \in C$$

Lemma 4.10. *Let C be a nonempty closed and convex subset of a* CAT(0) *space X. Let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by (4.2) such that $\{\beta_n\}$ and $\{\gamma_n\}$ are arbitrary sequences in $[0, 1]$. Let $z \in F(T)$.*

Then the following assertions hold.
(1) $\max\{d(x_{n+1}, z), d(y_n, z)\} \leq d(x_n, z)$ *for* $n = 1, 2, \ldots$.
(2) $\lim_{n\to\infty} d(x_n, z)$ *exists.*
(3) $\lim_{n\to\infty} d(x_n, F(T))$ *exists.*

Lemma 4.11 *([15]).* *Let C be a nonempty convex subset of a* CAT(0) *space X, and let $T : C \to C$ be a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(T)$ is closed, convex and hence contractible.*

The following result is deduced from Lemmas 4.8 and 4.11.

Lemma 4.12. *Let C be a nonempty convex subset of a* CAT(0) *space X, and let $T : C \to C$ be an α-nonexpansive mapping with a nonempty fixed point set $F(T)$ for some $\alpha < 1$. Then $F(T)$ is closed, convex, and hence contractible.*

Lemma 4.13. *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. If $\{x_n\}$ is a sequence in C such that $d(Tx_n, x_n) \to 0$ and $\Delta - \lim_{n\to\infty} x_n = z$ for some z in X, then $z \in C$ and $Tz = z$.*

Proof. It follows from Lemma 4.4 that $z \in C$.

Let $0 \leq \alpha < 1$. By Lemma 4.9(i), we deduce that

$$d(x_n, Tz)^2 \leq \tfrac{1+\alpha}{1-\alpha} d(x_n, Tx_n)^2 + \tfrac{2}{1-\alpha}(\alpha d(x_n, z) + d(Tx_n, Tz))d(x_n, Tx_n) + d(x_n, z)^2$$

for all n in $\mathbb{N}$. Thus we have

$$\limsup_{n\to\infty} d(x_n, Tz) \leq \limsup_{n\to\infty} d(x_n, z).$$

Let $\alpha < 0$. Then, by Lemma 4.9(ii), we have

$$d(x_n, Tz)^2 \leq d(x_n, Tx_n)^2 + \frac{2}{1-\alpha}[(-\alpha)d(Tx_n, z) + d(Tx_n, Tz)]d(x_n, Tx_n) + d(x_n, z)^2$$

for all n in $\mathbb{N}$. This implies again that

$$\limsup_{n\to\infty} d(x_n, Tz) \leq \limsup_{n\to\infty} d(x_n, z).$$

By the uniqueness of asymptotic centers, $Tz = z$. □

5. Fixed Point and Convergence Theorems in CAT(0) Spaces

In this section, we extend our results in Section 3 to CAT(0) spaces.

Theorem 5.1. *Let C be a nonempty closed and convex subset of a complete* $\mathrm{CAT}(0)$ *space X and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \le \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \le 0$, we assume also that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in C defined by (4.2). Then the fixed point set $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0$.*

Proof. Suppose that $F(T) \neq \emptyset$ and z in $F(T)$ is arbitrarily chosen. By Lemma 4.10, $\lim_{n\to\infty} d(x_n, z)$ exists and $\{x_n\}$ is bounded. Let

$$\lim_{n\to\infty} d(x_n, z) = l. \tag{5.1}$$

It follows from Lemmas 4.8 and 4.1(ii) that

$$\begin{aligned} d(Ty_n, z) &\le d(y_n, z) \\ &= d(\beta_n Tx_n \oplus (1-\beta_n)x_n, z) \\ &\le \beta_n d(Tx_n, z) + (1-\beta_n) d(x_n, z) \\ &\le \beta_n d(x_n, z) + (1-\beta_n) d(x_n, z) \\ &= d(x_n, z). \end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} d(Ty_n, z) \le \limsup_{n\to\infty} d(y_n, z) \le \limsup_{n\to\infty} d(x_n, z) = l. \tag{5.2}$$

On the other hand, it follows from (4.2) and (5.1) that

$$\lim_{n\to\infty} d(\gamma_n Ty_n \oplus (1-\gamma_n)x_n, z) = \lim_{n\to\infty} d(x_{n+1}, z) = l. \tag{5.3}$$

In view of (5.1)-(5.3) and Lemma 4.5, we conclude that

$$\lim_{k\to\infty} d(Ty_{n_k}, x_{n_k}) = 0.$$

By simply replacing $\|\cdot - \cdot\|$ with $d(\cdot,\cdot)$ in the proof of Theorem 3.2, we have the desired result $\lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0$. The proof of the other direction follows similarly. $\square$

Theorem 5.2. *Let C be a nonempty closed and convex subset of a complete* $\mathrm{CAT}(0)$ *space X, and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \le \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \le 0$, we assume also that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in C defined by (4.2). If $F(T) \neq \emptyset$, then $\{x_{n_k}\}$ Δ-converges to a fixed point of T.*

Proof. It follows from Theorem 5.1 that $\{x_n\}$ is bounded and $\lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0$. Denote by $\omega_w(x_{n_k}) := \cup A(C, \{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_{n_k}\}$. We prove that $\omega_w(x_{n_k}) \subset F(T)$. Let $u \in \omega_w(x_{n_k})$. Then there exists a subsequence $\{u_n\}$ of $\{x_{n_k}\}$ such that $A(C, \{u_n\}) = \{u\}$. In view of Lemmas 4.3 and 4.4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some v in C. Since $\lim_{n\to\infty} d(Tv_n, v_n) = 0$, Lemma 4.13 implies that $v \in F(T)$. By Lemma 4.10, $\lim_{n\to\infty} d(x_n, v)$ exists. We claim that $u = v$. For else, the uniqueness of asymptotic centers implies that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \le \limsup_{n\to\infty} d(u_n, u)\\ &< \limsup_{n\to\infty} d(u_n, v) = \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

which is a contradiction. Thus, we have $u = v \in F(T)$ and hence $\omega_w(x_{n_k}) \subset F(T)$.

Now, we prove that $\{x_{n_k}\}$ Δ-converges to a fixed point of T. It suffices to show that $\omega_w(x_{n_k})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_{n_k}\}$. In view of Lemmas 4.3 and 4.4, there exists a subsequences $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some v in C. Let $A(C, \{u_n\}) = \{u\}$ and $A(C, \{x_{n_k}\}) = \{x\}$. By the argument mentioned above we have $u = v$ and $v \in F(T)$. We show that $x = v$. If it is not the case, then the uniqueness of asymptotic centers implies that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

which is a contradiction. Thus we have the desired result. □

Theorem 5.3. *Let C be a nonempty compact convex subset of a complete* CAT(0) *space X, and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \le \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \le 0$, we assume also that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in C defined by (4.2). Then $\{x_n\}$ converges in metric to a fixed point of T.*

Proof. Using Theorem 4.7 and Lemma 4.9, and replacing $\| \cdot - \cdot \|$ with $d(\cdot, \cdot)$ in the proof of Theorem 3.4, we conclude the desired result. □

As in the proof of Theorem 3.5, we can verify the following result.

Theorem 5.4. *Let C be a nonempty compact convex subset of a complete* CAT(0) *space X, and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \le \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \le 0$, we assume also that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in*

C defined by (4.2). If T satisfies condition (I), then $\{x_n\}$ converges in metric to a fixed point of T.

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