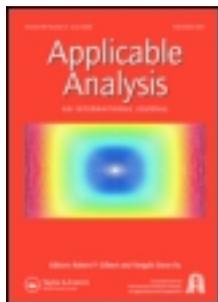


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On a class of fractional order differential inclusions with infinite delays

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On a class of fractional order differential inclusions with infinite delays

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Our aim is to study fractional order differential inclusions with infinite delays in Banach spaces. We impose the regularity condition on multivalued nonlinearity in terms of measures of noncompactness to get the existence result. Some properties of the solution map are proved.

Keywords: fractional order differential inclusion; fractional derivative; functional differential inclusion; infinite delay; initial value problem; continuous dependence; multivalued map; measure of noncompactness; fixed point; condensing map

AMS Subject Classifications: 34K37; 26A33; 34A60; 34G25; 34K09; 34K30; 47H04; 47H08; 47H10; 47H11

1. Introduction

Let E be a Banach space. We are concerned with the following problem:

$${}^C D^\alpha u(t) \in F(t, u_t, \nabla^N u), \quad t \in J := [0, T], \quad (1.1)$$

$$\nabla^N u(0) = U_0, \quad (1.2)$$

$$u(s) = \varphi(s), \quad s \in (-\infty, 0), \quad (1.3)$$

where $N \geq 1$ is an integer, $\alpha \in (N-1, N]$, $u : (-\infty, T] \rightarrow E$ is the unknown function, ${}^C D^\alpha$ denotes the Caputo fractional derivative, $\nabla^N u = (u, u', \dots, u^{(N-1)})$ and $F : [0, T] \times \mathcal{B} \times E^N \rightarrow \mathcal{P}(E)$ is a multivalued map with nonempty compact convex values. Here $\mathcal{P}(E)$ stands for the collection of all subsets of E , \mathcal{B} is a phase space of delays and $u_t \in \mathcal{B}$ is the history of the state function u up to the time t , that is

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$u_i(s) = u(t+s)$ for $s \in (-\infty, 0]$. The initial data $U_0 = (\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{N-1})$ are given in E^N and the initial function $\varphi \in \mathcal{B}$ is such that $\varphi(0) = \tilde{u}_0$.

The subject of fractional differential equations has recently received much attention due to its important applications in modelling phenomena of science and engineering. The employment of differential equations with fractional order allows to deal with many problems in numerous areas including fluid flow, rheology, electrical networks, viscoelasticity, electrochemistry, etc. For complete references, we refer to some significant works, e.g., the monographs of Kilbas et al. [1], Kiryakova [2], Miller and Ross [3] and Podlubny [4]. In the past few years, there has been a great contribution in fractional differential equations. Let us refer to some relevant works in [5–18]. With initial or boundary conditions, some particular cases of (1.1) without delay were studied. The equation

$${}^C D^\alpha u(t) = f(t, u(t))$$

or the inclusion

$${}^C D^\alpha u(t) \in F(t, u(t))$$

in the cases $\alpha \in (0, 1]$ or $\alpha \in (1, 2]$ were considered in [6,8–11,13]. Similar problems with the Riemann–Liouville fractional derivatives were also investigated, for instance, in [12,17,18]. The readers can find more works in the survey of Argawal et al. [19].

In addition, inclusion (1.1) can be seen as a generalized model of high-order ordinary differential equations, an example of which is the equation considered in [20]:

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t), (\mathcal{T}u)(t)),$$

where \mathcal{T} is an integral operator.

There are several approaches that can be used to get the solvability for local, global or extremal solutions of fractional differential equations or inclusions. One can use the method of upper and lower solutions to obtain the results as in [9]. Another approach develops a comparison principle and iteration schemes to receive the existence results as in [14–16]. The method that is widely used consists of transforming problems to corresponding fixed point equations or inclusions, followed by applying some known fixed point theorems.

Our approach is employing the fixed point theory technique for multivalued condensing maps under assumptions expressed in terms of the measure of noncompactness (MNC). The method used in this note allows us to solve the problems of differential inclusions in infinite-dimensional spaces with a general form of nonlinearity. Furthermore, by using this method, we need not impose the Lipschitz condition on the nonlinearity. Instead of Lipschitz assumptions, we suppose that the nonlinearity F satisfies a regularity condition expressed in terms of the Hausdorff MNC. In the sequel, we define a suitable MNC, prove that the solution multioperator is condensing with respect to this new MNC and find solutions by using the fixed point theory of condensing multimaps presented in [21].

Besides [21], an employment of this approach can be found, e.g., in [17,22] and other works. For more applications of multivalued analysis to differential equations

and inclusions, the readers are referred to [23–25]. The reader may also find a complete reference to MNCs in the monographs [26,27].

The rest of this article is organized as follows. In the next section we recall some basic facts related to fractional calculus, measures of noncompactness, multivalued maps and the phase space for delay differential equations. Section 3 is devoted to the formulations and proofs of the local and global existence results. In the last section, we study the continuous dependence of the solution set of the stated problem on initial data.

2. Preliminaries

2.1. Fractional calculus

We start this section with some notation and definitions in fractional calculus. For the motivations of these definitions, see, for example [1–4].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0, T; E)$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the Gamma function.

We use Bochner integral in the foregoing definition and in the rest of this work.

Definition 2.2 For a function $f \in C^N([0, T]; E)$, the Caputo fractional derivative of order $\alpha \in (N-1, N]$ is defined by

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s) ds.$$

It should be noted that there are some notions of fractional derivatives in which the Riemann–Liouville and Caputo definitions have been used widely. Many application problems, expressed by differential equations of fractional order, require initial conditions related to $u(0)$, $u'(0)$, etc., and the Caputo fractional derivative satisfies these demands. For $u \in C^N([0, T]; E)$, we have the following formulae:

$${}^C D_0^\alpha I_0^\alpha u(t) = u(t), \tag{2.1}$$

$$I_0^\alpha {}^C D_0^\alpha u(t) = u(t) - \sum_{k=0}^{N-1} \frac{u^{(k)}(0)}{k!} t^k. \tag{2.2}$$

2.2. Phase space

Let \mathcal{B} be a linear space, with a seminorm $|\cdot|_{\mathcal{B}}$, consisting of functions mapping $(-\infty, 0]$ into E . The definition of the phase space \mathcal{B} , introduced by Hale and Kato [28], can be given by the following axioms. If $v: (-\infty, T] \rightarrow E$ is such that $v|_{[0, T]} \in C([0, T]; E)$ and $v_0 \in \mathcal{B}$, then

(B1) $v_t \in \mathcal{B}$ for all $t \in [0, T]$;

(B2) the function $t \mapsto v_t$ is continuous on $[0, T]$;

(B3) $|v_t|_B \leq K(t) \sup\{\|v(s)\|_E : 0 \leq s \leq t\} + M(t)|v_0|_B$, where $K, M : [0, T] \rightarrow [0, \infty)$, K is continuous, M is bounded and they are independent of v .

We may consider the following examples of phase spaces satisfying all the above properties.

- (1) For $\eta > 0$, let $\mathcal{B} = C_\eta$ be the space of continuous functions $\psi : (-\infty; 0] \rightarrow E$ having a limit $\lim_{\theta \rightarrow -\infty} e^{\eta\theta} \psi(\theta)$ with

$$|\psi|_B = \sup_{-\infty < \theta \leq 0} e^{\eta\theta} \|\psi(\theta)\|.$$

- (2) (*Spaces of 'fading memory'*). Let $\mathcal{B} = C_\rho$ be the space of functions $\psi : (-\infty; 0] \rightarrow E$ such that

- (a) ψ is continuous on $[-r; 0]$, $r > 0$;
- (b) ψ is Lebesgue measurable on $(-\infty; r)$ and there exists a nonnegative Lebesgue integrable function $\rho : (-\infty; -r) \rightarrow \mathbb{R}^+$ such that $\rho\psi$ is Lebesgue integrable on $(-\infty; r)$; moreover, there exists a locally bounded function $P : (-\infty; 0] \rightarrow \mathbb{R}^+$ such that, for all $\xi \leq 0$, $\rho(\xi + \theta) \leq P(\xi)\rho(\theta)$ a.e. $\theta \in (-\infty; -r)$. Then,

$$|\psi|_B = \sup_{-r \leq \theta \leq 0} \|\psi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta) \|\psi(\theta)\| d\theta.$$

A simple example of such a space is given by $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

For more examples of phase spaces, see [28].

2.3. Measures of noncompactness and multivalued maps

Let us recall some basic facts from the multivalued analysis, which will be used in this article. Let \mathcal{E} be a Banach space. We denote

- $\mathcal{P}(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \emptyset\}$,
- $Pv(\mathcal{E}) = \{A \in \mathcal{P}(\mathcal{E}) : A \text{ is convex}\}$,
- $K(\mathcal{E}) = \{A \in \mathcal{P}(\mathcal{E}) : A \text{ is compact}\}$,
- $Kv(\mathcal{E}) = Pv(\mathcal{E}) \cap K(\mathcal{E})$.

We will use the following definition of the MNC (see, e.g. [21]).

Definition 2.3 Let (\mathcal{A}, \geq) be a partially ordered set. A function $\beta : \mathcal{P}(\mathcal{E}) \rightarrow \mathcal{A}$ is called an MNC in \mathcal{E} if

$$\beta(\overline{\text{co}} \Omega) = \beta(\Omega) \quad \text{for every } \Omega \in \mathcal{P}(\mathcal{E}),$$

where $\overline{\text{co}} \Omega$ is the closure of the convex hull of Ω . An MNC β is called

- (i) monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
- (ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}$, $\Omega \in \mathcal{P}(\mathcal{E})$;

(iii) invariant with respect to union with compact set, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{P}(\mathcal{E})$.

If \mathcal{A} is a cone in a normed space, we say that β is

(iv) algebraically semi-additive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;

(v) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω .

An important example of MNC is the Hausdorff MNC, which satisfies all the above properties:

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

It should be mentioned that the Hausdorff MNC has also the following properties:

- Semihomogeneity: $\chi(t\Omega) \leq |t|\chi(\Omega)$ for any $\Omega \in \mathcal{P}(\mathcal{E})$ and $t \in \mathbb{R}$;
- in a separable Banach space \mathcal{E} , $\chi(\Omega) = \lim_{m \rightarrow \infty} \sup_{x \in \Omega} d(x, \mathcal{E}_m)$, where $\{\mathcal{E}_m\}$ is a sequence of finite-dimensional subspaces of \mathcal{E} such that $\mathcal{E}_m \subset \mathcal{E}_{m+1}$, $m = 1, 2, \dots$ and $\bigcup_{m=1}^{\infty} \mathcal{E}_m = \mathcal{E}$.

Let X be a metric space.

Definition 2.4 A multivalued map (multimap) $\mathcal{F} : X \rightarrow \mathcal{P}(\mathcal{E})$ is said to be:

- (i) upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for every open set $V \subset \mathcal{E}$;
- (ii) closed if its graph $\Gamma_{\mathcal{F}} = \{(x, y) : y \in \mathcal{F}(x)\}$ is a closed subset of $X \times \mathcal{E}$;
- (iii) compact if its range $\mathcal{F}(X)$ is relatively compact in \mathcal{E} ;
- (iv) quasicompact if its restriction to any compact subset $A \subset X$ is compact.

Definition 2.5 A multimap $\mathcal{F} : X \subset \mathcal{E} \rightarrow K(\mathcal{E})$ is said to be condensing with respect to an MNC β (β -condensing) if for every bounded set $\Omega \subset X$ that is not relatively compact, we have

$$\beta(\mathcal{F}(\Omega)) \not\leq \beta(\Omega).$$

Suppose that $D \subset \mathcal{E}$ is a nonempty closed convex subset of \mathcal{E} and \mathcal{U}_D is a nonempty relatively open subset of D . We denote by $\overline{\mathcal{U}}_D$ and $\partial\mathcal{U}_D$, the closure and the boundary of \mathcal{U}_D in the relative topology of D , respectively.

Let β be a monotone nonsingular MNC in \mathcal{E} . The application of the topological degree theory for condensing multimaps (see, e.g. [21]) yields the following fixed point theorems.

THEOREM 2.1 [21, Corollary 3.3.1] *Let \mathcal{M} be a bounded convex closed subset of \mathcal{E} and $\mathcal{F} : \mathcal{M} \rightarrow K_V(\mathcal{M})$ an u.s.c. β -condensing multimap. Then the fixed point set $\text{Fix } \mathcal{F} := \{x : x \in \mathcal{F}(x)\}$ is a nonempty compact set.*

The following theorem presents a version for multimaps of the classical Leray–Schauder alternative.

THEOREM 2.2 [21, Corollary 3.3.3] *Let \mathcal{U}_D be a bounded open neighbourhood of $a \in D$ and $\mathcal{F} : \overline{\mathcal{U}}_D \rightarrow K_V(D)$ an u.s.c β -condensing multimap satisfying the boundary condition*

$$x - a \notin \lambda(\mathcal{F}(x) - a)$$

for all $x \in \partial\mathcal{U}_D$ and $0 < \lambda \leq 1$. Then $\text{Fix } \mathcal{F}$ is a nonempty compact set.

Definition 2.6 Let $G: [0, T] \rightarrow K(E)$ be a multifunction and $p \geq 1$. Then G is said to be

- L^p -integrable, if it admits a Bochner L^p -integrable selection. That is there exists $g: [0, T] \rightarrow E$, $g(t) \in G(t)$ for a.e. $t \in [0, T]$ such that $\int_0^T \|g(s)\|_E^p ds < \infty$;
- L^p -integrably bounded, if there exists a function $\xi \in L^p([0, T])$ such that

$$\|G(t)\| := \sup\{\|g\|_E : g \in G(t)\} \leq \xi(t) \quad \text{for a.e. } t \in [0, T].$$

The set of all L^p -integrable selections of G will be denoted by S_G^p .

The multifunction G is called *measurable* if $G^{-1}(V)$ measurable (with respect to the Lebesgue measure on $J := [0, T]$) for any open subset V of E . We say that G is *strongly measurable* if there exists a sequence $G_n: [0, T] \rightarrow K(E)$, $n = 1, 2, \dots$ of step multifunctions such that

$$\lim_{n \rightarrow \infty} \mathcal{H}(G_n(t), G(t)) = 0 \quad \text{for a.e. } t \in [0, T],$$

where \mathcal{H} is the Hausdorff metric in $K(E)$.

It is known that, when E is a separable Banach space, the notion of measurable multifunctions has some equivalences. More precisely, for a multifunction $G: [0, T] \rightarrow K(E)$, the following conditions are equivalent to each other (see, e.g. [21]):

- (1) G is measurable;
- (2) for every countable dense set $\{x_n\}$ of E , the functions $\varphi_n: [0, T] \rightarrow \mathbb{R}$, defined by

$$\varphi_n(t) = d(x_n, G(t))$$

are measurable;

- (3) G has a Castaing presentation: there is a countable family $\{g_n\}$ of measurable selections of G such that

$$\overline{\bigcup_{n=1}^{\infty} g_n(t)} = G(t)$$

for a.e. $t \in [0, T]$;

- (4) G is strongly measurable.

Furthermore, if G is measurable and L^p -integrably bounded, then it is L^p -integrable. If G is L^p -integrable on $[0, d]$ for some $p \geq 1$, then G is also L^1 -integrable. In this case, we have a multifunction $t \mapsto \int_0^t G(s) ds$ defined by

$$\int_0^t G(s) ds := \left\{ \int_0^t g(s) ds : g \in S_G^1 \right\} \quad \forall t \in [0, d].$$

The following χ -estimate (χ is the Hausdorff MNC), which is similar to [21, Theorem 4.2.3] will be used in the sequel.

LEMMA 2.3 *Assume that E is a separable Banach space. Let $G: [0, d] \rightarrow \mathcal{P}(E)$ be L^p -integrable, L^p -integrably bounded multifunction such that*

$$\chi(G(t)) \leq q(t)$$

for a.e. $t \in [0, d]$. Here, $q \in L^p_+([0, d])$. Then

$$\chi\left(\int_0^t G(s)ds\right) \leq \int_0^t q(s)ds$$

for all $t \in [0, d]$. In particular, if the multifunction $G : [0, d] \rightarrow K(E)$ is measurable and L^p -integrably bounded then the function $\chi(G(\cdot))$ is integrable and, moreover,

$$\chi\left(\int_0^t G(s)ds\right) \leq \int_0^t \chi(G(s))ds$$

for all $t \in [0, d]$.

Now we consider the multimap $F : [0, T] \times \mathcal{B} \times E^N \rightarrow K_V(E)$ in our problem (1.1)–(1.3).

Definition 2.7 We say that F satisfies the upper Carathéodory conditions if

- (1) the multifunction $F(\cdot, \zeta, \mathcal{U}) : [0, T] \rightarrow K_V(E)$ admits a strongly measurable selection for each $(\zeta, \mathcal{U}) \in \mathcal{B} \times E^N$, and
- (2) the multimap $F(t, \cdot, \cdot) : \mathcal{B} \times E^N \rightarrow K_V(E)$ is u.s.c for a.e. $t \in [0, T]$.

The multimap F is said to be L^p -locally integrably bounded if for each $r > 0$, there exists a function $\omega_r \in L^p([0, T])$ such that

$$\|F(t, \zeta, \mathcal{U})\| = \sup\{\|z\|_E : z \in F(t, \zeta, \mathcal{U})\} \leq \omega_r(t)$$

for all $(\zeta, \mathcal{U}) \in \mathcal{B} \times E^N$ satisfying $|\zeta|_{\mathcal{B}} + \|\mathcal{U}\|_{E^N} \leq r$.

Let $\mathcal{C}_E(-\infty, T)$ denote the linear topological space of functions $u : (-\infty, T] \rightarrow E$ satisfying that

$$u_0 \in \mathcal{B} \quad \text{and} \quad u|_{[0, T]} \in C^{N-1}([0, T]; E),$$

endowed with the seminorm

$$\|u\|_{\mathcal{C}_E(-\infty, T)} = |u_0|_{\mathcal{B}} + \|u\|_{C^{N-1}([0, T]; E)}.$$

For $u \in \mathcal{C}_E(-\infty, T)$, consider the superposition multifunction

$$\Phi_F : [0, T] \rightarrow K_V(E), \quad \Phi_F(t) = F(t, u_t, \nabla^N u(t)).$$

By the axioms of the phase space, we see that $t \mapsto u_t \in \mathcal{B}$ is a continuous function. Further, the function $\nabla^N u : [0, T] \rightarrow E^N$ is continuous, too. Then Φ_F is L^p -integrable provided F satisfies upper Carathéodory conditions and is L^p -locally bounded. The proof can be made in the same way as in [21, Theorem 1.3.5].

As the consequence, we can define on $\mathcal{C}_E(-\infty, T)$ the superposition multi-operator \mathcal{P}_F by

$$\mathcal{P}_F(u) = \{\phi \in L^p(0, T; E) : \phi(t) \in F(t, u_t, \nabla^N u(t)) \text{ for a.e. } t \in [0, T]\}.$$

We have the following property of weak closedness of \mathcal{P}_F , whose proof can be proceeded as in [21, Lemma 5.1.1].

LEMMA 2.4 Let $\{u_n\}$ be a sequence in $\mathcal{C}_E(-\infty, T)$ converging to $u^* \in \mathcal{C}_E(-\infty, T)$ and suppose that the sequence $\{\phi_n\} \subset L^p(0, T; E)$, $\phi_n \in \mathcal{P}_F(u_n)$ weakly converges to a function ϕ^* . Then $\phi^* \in \mathcal{P}_F(u^*)$.

3. Existence results

In the sequel, let E be a separable Banach space. We give the definition of a mild solution for (1.1)–(1.3), in accordance with formula (2.2), as follows.

Definition 3.1 For a given $\tau \in (0, T]$, a function $u \in \mathcal{C}_E(-\infty, \tau)$ is called a *mild solution* to problem (1.1)–(1.3) on the interval $(-\infty, \tau]$ if it satisfies the integral equation

$$u(t) = \begin{cases} \varphi(t), & \text{for } t \leq 0, \\ \sum_{k=0}^{N-1} \frac{t^k}{k!} \tilde{u}_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) ds & \text{for } t \in [0, \tau], \end{cases}$$

where $\phi \in \mathcal{P}_F(u)$.

We assume that the multimap F in problem (1.1)–(1.3) satisfies the following hypotheses:

(F1) $F: [0, T] \times \mathcal{B} \times E^N \rightarrow K_V(E)$ satisfies the upper Carathéodory conditions;

(F2) F is L^p -locally integrably bounded for $p > \frac{1}{\alpha-N+1}$;

(F3) there exists a function $k \in L^p(0, T; E)$ such that for any bounded subsets $\mathcal{Q} \subset \mathcal{B}$ and $\Omega_j \subset E$, $j=0, \dots, N-1$, we have

$$\chi \left(F \left(t, \mathcal{Q}, \prod_{j=0}^{N-1} \Omega_j \right) \right) \leq k(t) \left(\psi(\mathcal{Q}) + \sum_{j=0}^{N-1} \chi(\Omega_j) \right),$$

where

$$\psi(\mathcal{Q}) = \sup_{\theta \leq 0} \chi(\mathcal{Q}(\theta)), \quad (3.1)$$

$$\mathcal{Q}(\theta) = \{q(\theta) : q \in \mathcal{Q}\}.$$

Remark 3.1 In the case $E = \mathbb{R}^m$, condition (F3) follows from (F2). In fact, the L^p -locally integrably boundedness of F implies that the set $F(t, \mathcal{Q}, \prod_{j=0}^{N-1} \Omega_j)$ is bounded in \mathbb{R}^m for a.e. $t \in [0, T]$, and hence it is a precompact set.

If $\dim(E) = +\infty$, then a particular case of fulfilling (F3) is the following condition:

$$F(t, \cdot, \cdot) : \mathcal{B} \times E^N \rightarrow K_V(E)$$

is completely continuous for a.e. $t \in [0, T]$, i.e. $F(t, \cdot, \cdot)$ maps each bounded set in $\mathcal{B} \times E^N$ into a precompact set in E .

For a given $\tau \in (0, T]$, we set

$$\begin{aligned}
 S &: L^p(0, \tau; E) \rightarrow C([0, \tau]; E), \\
 S(\phi)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) ds, \\
 u^*(t) &= \begin{cases} \varphi(t), & \text{if } t \leq 0, \\ \sum_{k=0}^{N-1} \frac{t^k}{k!} \tilde{u}_k, & \text{if } 0 \leq t \leq \tau, \end{cases}
 \end{aligned} \tag{3.2}$$

and

$$\mathcal{G}(u) = u^* + S \circ \mathcal{P}_F(u). \tag{3.3}$$

One sees that $u \in \mathcal{C}_E(-\infty, \tau)$ is a mild solution of problem (1.1)–(1.3) on interval $(-\infty, \tau]$ if and only if it is a fixed point of \mathcal{G} . From now on, we can restrict \mathcal{G} on the subset $\mathcal{D}_\tau \subset C^{N-1}([0, \tau]; E)$ defined as

$$\mathcal{D}_\tau = \{v \in C^{N-1}([0, \tau]; E), v(0) = \tilde{u}_0 = \varphi(0)\}, \tag{3.4}$$

by setting $\mathcal{G}(v) = \mathcal{G}(v[\varphi])$, where

$$v[\varphi](t) = \begin{cases} \varphi(t), & \text{if } t \leq 0, \\ v(t), & \text{if } 0 \leq t \leq \tau. \end{cases}$$

We first verify some features of S which will be used to obtain some important properties of the multioperator \mathcal{G} . Define the following collection of operators $S_j: L^p(0, \tau; E) \rightarrow C([0, \tau]; E), j=0, 1, \dots, N-1$:

$$S_0 = S. \tag{3.5}$$

If $N > 1$ then

$$S_j(\phi)(t) = \frac{d^j}{dt^j} S(\phi)(t) = \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-s)^{\alpha-j-1} \phi(s) ds, \quad j = 1, \dots, N-1. \tag{3.6}$$

PROPOSITION 3.1 *The operators $S_j, j=0, 1, \dots, N-1$ have the following properties:*

(S1) *There exist constants $C_j > 0, j=0, 1, \dots, N-1$, such that*

$$\|S_j(\xi)(t) - S_j(\eta)(t)\|_E^p \leq C_j^p \int_0^t \|\xi(s) - \eta(s)\|_E^p ds, \quad \xi, \eta \in L^p(0, \tau; E).$$

(S2) *For each compact set $K \subset E$ and sequence $\{\xi_n\} \subset L^p(0, \tau; E)$ such that $\{\xi_n(t)\} \subset K$ for a.e. $t \in [0, \tau]$, the weak convergence $\xi_n \rightharpoonup \xi_0$ implies $S_j(\xi_n) \rightarrow S_j(\xi_0)$ in $C([0, \tau]; E)$.*

Proof

(i) By using the Hölder inequality we have

$$\begin{aligned}
 \|S_j(\xi)(t) - S_j(\eta)(t)\|_E &\leq \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-s)^{\alpha-j-1} \|\xi(s) - \eta(s)\|_E ds \\
 &\leq \frac{1}{\Gamma(\alpha-j)} \left[\int_0^t (t-s)^{(\alpha-j-1)p/(p-1)} ds \right]^{\frac{p-1}{p}} \left[\int_0^t \|\xi(s) - \eta(s)\|_E^p ds \right]^{\frac{1}{p}}.
 \end{aligned}$$

Then

$$\|S_j(\xi)(t) - S_j(\eta)(t)\|_E^p \leq C_j^p \int_0^t \|\xi(s) - \eta(s)\|_E^p ds,$$

where

$$C_j = \left[\frac{p-1}{(\alpha-j)p-1} \right]^{\frac{p-1}{p}} \frac{T^{\alpha-j-\frac{1}{p}}}{\Gamma(\alpha-j)}.$$

Notice that $(\alpha-j)p-1 > 0$, $j=0, \dots, N-1$ since $p > \frac{1}{\alpha-N+1}$.

(ii) To prove (S2), notice that, without loss of generality, $\{\xi_n(t)\} \subset E'$ for all $t \in [0, \tau]$, where $E' = \overline{\text{sp} K}$ is the separable Banach space spanned by the compact set K . Moreover, it is clear also that $\{S_j(\xi_n)(t)\} \subset E'$ for all $t \in [0, \tau]$ and $j=0, 1, \dots, N-1$. Then, applying Lemma 2.3 we obtain

$$\chi(\{S_j(\xi_n)(t)\}) \leq \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-s)^{\alpha-j-1} \chi(\{\xi_n(s)\}) ds = 0.$$

Hence, the sequence $\{S_j(\xi_n)(t)\}_{n=1}^\infty \subset E$ is relatively compact for every $t \in [0, \tau]$.

On the other hand, we have

$$\begin{aligned} & \|S_j(\xi_n)(t_2) - S_j(\xi_n)(t_1)\|_E \\ &= \frac{1}{\Gamma(\alpha-j)} \left\| \int_0^{t_2} (t_2-s)^{\alpha-j-1} \xi_n(s) ds - \int_0^{t_1} (t_1-s)^{\alpha-j-1} \xi_n(s) ds \right\|_E \\ &\leq \frac{1}{\Gamma(\alpha-j)} \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-j-1} \xi_n(s) ds \right\|_E \\ &\quad + \frac{1}{\Gamma(\alpha-j)} \left\| \int_0^{t_1} [(t_2-s)^{\alpha-j-1} - (t_1-s)^{\alpha-j-1}] \xi_n(s) ds \right\|_E. \end{aligned}$$

Since $\{\xi_n(s)\} \subset K$ for a.e. $s \in [0, \tau]$, the right side of this inequality tends to zero as $t_2 \rightarrow t_1$ uniformly with respect to n . So $\{S_j(\xi_n)\}$ is an equicontinuous set. Thus from the Arzela–Ascoli theorem, we obtain that the sequence $\{S_j(\xi_n)\} \subset C([0, \tau]; E)$ is relatively compact.

Property (S1) ensures that each $S_j: L^p(0, \tau; E) \rightarrow C([0, \tau]; E)$, $j=0, \dots, N-1$, is a bounded linear operator. Then it is continuous with respect to the topology of weak sequential convergence, that is the weak convergence $\xi_n \rightharpoonup \xi_0$ ensuring $S_j(\xi_n) \rightharpoonup S_j(\xi_0)$. Taking into account that $\{S_j(\xi_n)\}$ is relatively compact, we arrive at the conclusion that $S_j(\xi_n) \rightarrow S_j(\xi_0)$ strongly in $C([0, \tau]; E)$. ■

We have the following technical result, whose proof constitutes of the arguments from [21, Theorem 4.2.1, Corollary 4.2.1, Remarks 4.2.1 and 4.2.2].

PROPOSITION 3.2 *Let the sequence of functions $\{\xi_n\} \subset L^p(0, \tau; E)$ be L^p -integrably bounded:*

$$\|\xi_n(t)\|_E \leq v(t), \quad \text{for all } n = 1, 2, \dots, \text{ and a.e. } t \in [0, \tau],$$

where $v \in L^p(0, \tau)$. Assume that

$$\chi(\{\xi_n(t)\}) \leq q(t)$$

for a.e. $t \in [0, \tau]$, where $q \in L^p(0, \tau)$. Then for every $\delta > 0$ there exists a compact set $K_\delta \subset E$, a set $m_\delta \subset [0, \tau]$, $\text{meas}(m_\delta) < \delta$ and a sequence of functions $G_\delta \subset L^p(0, \tau; E)$ with values in K_δ , such that for every $n \geq 1$ there exists $b_n \in G_\delta$ for which

$$\|\xi_n(t) - b_n(t)\|_E \leq 2q(t) + \delta, \quad t \in [0, \tau] \setminus m_\delta.$$

In addition, one can choose the sequence $\{b_n\}$ so that $b_n = 0$ on m_δ and this sequence is weakly compact.

Now using this result, we will prove the following proposition.

PROPOSITION 3.3 *Let the sequence $\{\xi_n\} \subset L^p(0, \tau; E)$ satisfy the conditions of Proposition 3.2. Then we have*

$$\chi(\{S_j(\xi_n)(t)\}) \leq 2C_j \left(\int_0^t |q(s)|^p ds \right)^{\frac{1}{p}}, \quad j = 1, \dots, N-1,$$

for any $t \in [0, \tau]$.

Proof We follow the arguments in [21, Theorem 4.2.2] with some slight modifications. For any $\epsilon > 0$, choose $\delta \in (0, \epsilon)$ such that for all $m \subset [0, \tau]$, with $\text{meas}(m) < \delta$, we have

$$\int_m |v(s)|^p ds < \epsilon.$$

Taking m_δ and $\{b_n\}$ corresponding to $\{\xi_n\}$ from Proposition 3.2, we have by Proposition 3.1 that the sequences $\{S_j(b_n)\}$, $j = 0, \dots, N-1$ are relatively compact in $C([0, \tau]; X)$. Furthermore

$$\begin{aligned} & \|S_j(\xi_n)(t) - S_j(b_n)(t)\|_E^p \\ & \leq C_j^p \int_0^t \|\xi_n(s) - b_n(s)\|_E^p ds \\ & \leq C_j^p \int_{[0,t] \setminus m_\delta} \|\xi_n(s) - b_n(s)\|_E^p ds + C_j^p \int_{[0,t] \cap m_\delta} \|\xi_n(s)\|_E^p ds \\ & \leq C_j^p \int_{[0,t] \setminus m_\delta} [2q(s) + \delta]^p ds + C_j^p \int_{m_\delta} |v(s)|^p ds \\ & \leq C_j^p \left(\int_0^t |2q(s) + \epsilon|^p ds + \epsilon \right). \end{aligned}$$

Therefore, the relatively compact set $S_j G_\delta(t)$ forms a $C_j(\int_0^t |2q(s) + \epsilon|^p ds + \epsilon)^{\frac{1}{p}}$ -net for the set $\{S_j(\xi_n)(t)\}$. This proves the proposition since $\epsilon > 0$ is arbitrary. ■

Definition 3.2 A sequence $\{\xi_n\}$ in $L^p(0, \tau; E)$ is called semicompact if it is L^p -integrably bounded and the set $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0, \tau]$.

PROPOSITION 3.4 *Let $\{\xi_n\}$ be a semicompact sequence in $L^p(0, \tau; E)$. Then $\{\xi_n\}$ is weakly compact in $L^p(0, \tau; E)$. For each $0 \leq j \leq N-1$, $\{S_j(\xi_n)\}$ is relatively compact in $C([0, \tau]; E)$. Moreover, if $\xi_n \rightharpoonup \xi_0$ then $S_j(\xi_n) \rightarrow S_j(\xi_0)$, $j = 0, \dots, N-1$.*

Proof The weak compactness of $\{\xi_n\}$ in $L^p(0, \tau; E)$ is the consequence of the results in [29, Corollary 3.4]. Since $\{\xi_n(t)\}$ is relatively compact in E for a.e. $t \in [0, \tau]$,

from Proposition 3.3, we have that for each $j=0, \dots, N-1$, the sequence $\{S_j(\xi_n)(t)\}$ is relatively compact in E for a.e. $t \in [0, \tau]$.

On the other hand, from the assumptions, there exists a function $v \in L^p(0, \tau)$ such that

$$\|\xi_n(t)\| \leq v(t), \quad \text{for all } n = 1, 2, \dots, \text{ and } t \in [0, \tau].$$

Thanks to the Hölder inequality, one thus has

$$\begin{aligned} & \|S_j(\xi_n)(t_2) - S_j(\xi_n)(t_1)\|_E \\ &= \frac{1}{\Gamma(\alpha - j)} \left\| \int_0^{t_2} (t_2 - s)^{\alpha-j-1} \xi_n(s) ds - \int_0^{t_1} (t_1 - s)^{\alpha-j-1} \xi_n(s) ds \right\|_E \\ &\leq \frac{1}{\Gamma(\alpha - j)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-j-1} \xi_n(s) ds \right\|_E \\ &\quad + \frac{1}{\Gamma(\alpha - j)} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-j-1} - (t_1 - s)^{\alpha-j-1}] \xi_n(s) ds \right\|_E \\ &\leq \frac{1}{\Gamma(\alpha - j)} \left(\int_{t_1}^{t_2} (t_2 - s)^{(\alpha-j-1)p'} ds \right)^{\frac{1}{p'}} \left(\int_{t_1}^{t_2} |v(s)|^p ds \right)^{\frac{1}{p}} \\ &\quad + \frac{1}{\Gamma(\alpha - j)} \left(\int_0^{t_1} |(t_2 - s)^{\alpha-j-1} - (t_1 - s)^{\alpha-j-1}|^{p'} ds \right)^{\frac{1}{p'}} \left(\int_0^{t_1} |v(s)|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

where p' is the conjugate of p . The last inequality implies that $\{S_j(\xi_n)\}$ is equicontinuous in $C([0, \tau]; E)$, and then it is relatively compact in $C([0, \tau]; E)$. The last conclusion follows the same lines as in the proof of Proposition 3.1. ■

LEMMA 3.5 *Suppose that F satisfies (F1)–(F3). Then the composition*

$$\mathcal{G} = u^* + S \circ \mathcal{P}_F$$

is a closed multioperator with compact values.

Proof It suffices to prove the assertions for $S \circ \mathcal{P}_F$. Let $\{v_n\}$ be such that $v_n \rightarrow v^*$ in \mathcal{D}_τ . Assume that $\xi_n \in \mathcal{P}_F(v_n[\varphi])$ and $z_n = S(\xi_n) \in S(\mathcal{P}_F(v_n[\varphi]))$, $z_n \rightarrow z^*$ in $C^{N-1}([0, \tau]; E)$. We prove that $z^* \in S(\mathcal{P}_F(v^*[\varphi]))$. Since

$$\{\xi_n(t)\} \subset F(t, (v_n[\varphi])_t, \nabla^N v_n(t)),$$

we see that $\{\xi_n\}$ is integrably bounded by (F2) and the following inequality holds by (F3)

$$\chi(\{\xi_n(t)\}) \leq k(t) \left(\psi(\{(v_n[\varphi])_t\}) + \sum_{j=0}^{N-1} \chi(\{v_n^{(j)}(t)\}) \right).$$

For each $j=0, \dots, N-1$, the sequence $\{v_n^{(j)}\}$ converges in $C([0, \tau]; E)$. Then $\chi(\{v_n^{(j)}(t)\}) = 0$ for a.e. $t \in [0, \tau]$. On the other hand,

$$\begin{aligned} \psi(\{(v_n[\varphi])_t\}) &= \sup_{\theta \leq 0} \chi(\{v_n[\varphi](t + \theta)\}) \\ &\leq \sup_{s \in [0, t]} \chi(\{v_n(s)\}) = 0. \end{aligned} \tag{3.7}$$

Thus

$$\chi(\{\xi_n(t)\}) = 0, \quad \text{for a.e. } t \in [0, \tau],$$

and then $\{\xi_n\}$ is a semicompact sequence. By Proposition 3.4, we may assume, without loss of generality, that there exists $\xi^* \in L^p(0, \tau; E)$ such that

$$\xi_n \rightharpoonup \xi^* \quad \text{and} \quad z_n = S(\xi_n) \rightarrow S(\xi^*) = z^*.$$

Using Lemma 2.4, we obtain $\xi^* \in \mathcal{P}_F(v^*[\varphi])$ and we deduce that $z^* = S(\xi^*) \in S(\mathcal{P}_F(v^*[\varphi]))$.

It remains to show that, for $v \in \mathcal{D}_\tau$ and $\{\xi_n\}$ chosen in $\mathcal{P}_F(v[\varphi])$, the sequence $\{S(\xi_n)\}$ is relatively compact in $C^{N-1}([0, \tau]; E)$. Hypotheses (F2)–(F3) imply that $\{\xi_n\}$ is semicompact. Using Proposition 3.4, we obtain that $S_j(\xi_n)$, $j=0, \dots, N-1$, is relatively compact in $C([0, \tau]; E)$. Therefore, $\{S(\xi_n)\}$ is relatively compact in $C^{N-1}([0, \tau]; E)$. The proof is completed. ■

In order to prove the u.s.c. property of $S \circ \mathcal{P}_F$, we need the following result.

THEOREM 3.6 [21] *Let X and Y be metric spaces and $F: X \rightarrow K(Y)$ a closed quasicompact multimap. Then F is u.s.c.*

LEMMA 3.7 *Let the conditions of Lemma 3.5 hold. Then the multioperator \mathcal{G} is u.s.c.*

Proof In view of Theorem 3.6 and the result in Lemma 3.5, it suffices to check that \mathcal{G} is a quasicompact multimap. Let $A \subset \mathcal{D}_\tau$ be a compact set. We prove that $\mathcal{G}(A)$ is a relatively compact subset of $C^{N-1}([0, \tau]; E)$. Assume that $\{z_n\} \subset \mathcal{G}(A)$. Then $z_n = u^* + S(\xi_n)$, where $\xi_n \in \mathcal{P}_F(v_n[\varphi])$, for a certain sequence $\{v_n\} \subset A$. Hypotheses (F2)–(F3) yield the fact that $\{\xi_n\}$ is semicompact and then it is a weakly compact sequence in $L^p(0, \tau; E)$. Similar arguments as in the proof of Lemma 3.5 imply that $\{S(\xi_n)\}$ is relatively compact in $C^{N-1}([0, \tau]; E)$. Thus we have the desired conclusion. ■

We are in a position to prove that \mathcal{G} is a condensing multioperator. We first need an MNC constructed suitably for our problem. Denote

$$\begin{aligned} \gamma: \mathcal{P}(C^{N-1}([0, \tau]; E)) &\rightarrow \mathbb{R}_+, \\ \gamma(\Omega) &= \sup_{t \in [0, \tau]} e^{-Lt} \sum_{j=0}^{N-1} \chi\left(\frac{d^j}{dt^j} \Omega(t)\right). \end{aligned} \tag{3.8}$$

Here,

$$\frac{d^j}{dt^j} \Omega(t) := \left\{ \frac{d^j}{dt^j} v(t) : v \in \Omega \right\},$$

and the constant L is chosen so that

$$\ell := 4 \sum_{j=0}^{N-1} C_j \sup_{t \in [0, T]} \left(\int_0^t e^{-Lp(t-s)} k^p(s) ds \right)^{\frac{1}{p}} < 1, \tag{3.9}$$

where $k(\cdot)$ is the function from condition (F3).

Further, consider

$$\begin{aligned} \text{mod}_C &: \mathcal{P}(C^{N-1}([0, \tau]; E)) \rightarrow \mathbb{R}_+, \\ \text{mod}_C(\Omega) &= \limsup_{\delta \rightarrow 0} \sup_{v \in \Omega} \max_{0 \leq j \leq N-1} \max_{|t_1 - t_2| < \delta} \|v^{(j)}(t_1) - v^{(j)}(t_2)\|, \end{aligned} \quad (3.10)$$

which is called the *modulus of equicontinuity* of Ω in $C^{N-1}([0, \tau]; E)$. Consider the function

$$\begin{aligned} v &: \mathcal{P}(C^{N-1}([0, \tau]; E)) \rightarrow \mathbb{R}_+^2, \\ v(\Omega) &= \max_{D \in \Delta(\Omega)} (\gamma(D), \text{mod}_C(D)), \end{aligned} \quad (3.11)$$

where $\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the ordering in the cone \mathbb{R}_+^2 . By the same arguments as in [21], one can see that v is well-defined. That is, the maximum archives in $\Delta(\Omega)$, and v is an MNC in the space $C^{N-1}([0, \tau]; E)$, which fulfils all properties in Definition 2.3 (see [21, Example 2.1.3] for details).

LEMMA 3.8 *Under conditions of Lemma 3.5, the multioperator $\mathcal{G}: \mathcal{D}_\tau \rightarrow K(\mathcal{D}_\tau)$ is v -condensing.*

Proof Let $\Omega \subset \mathcal{D}_\tau$ be such that

$$v(\mathcal{G}(\Omega)) \geq v(\Omega). \quad (3.12)$$

We show that Ω is relatively compact. Let $v(\mathcal{G}(\Omega))$ be achieved on a sequence $\{z_n\} \subset \mathcal{G}(\Omega)$, i.e.

$$v(\mathcal{G}(\Omega)) = (\gamma(\{z_n\}), \text{mod}_C(\{z_n\})).$$

Then

$$z_n = u^* + S(\xi_n), \quad \xi_n \in \mathcal{P}_F(v_n[\varphi]) \quad \text{where } \{v_n\} \subset \Omega.$$

Now inequality (3.12) implies

$$\gamma(\{z_n\}) \geq \gamma(\{v_n\}). \quad (3.13)$$

It follows from (F3) that

$$\chi(\{\xi_n(s)\}) \leq k(s) \left(\psi(\{(v_n[\varphi])_s\}) + \sum_{j=0}^{N-1} \chi(\{v_n^{(j)}(s)\}) \right)$$

for every $s \in [0, \tau]$. In view of (3.1), we have

$$\psi(\{(v_n[\varphi])_s\}) = \sup_{\theta \leq 0} \chi(\{v_n[\varphi](s + \theta)\}) = \sup_{\xi \in [0, s]} \chi(\{v_n(\xi)\}).$$

Then

$$\begin{aligned} \chi(\{\xi_n(s)\}) &\leq k(s) e^{Ls} \left(\sup_{\xi \in [0, s]} e^{-L\xi} \chi(\{v_n(\xi)\}) + e^{-Ls} \sum_{j=0}^{N-1} \chi(\{v_n^{(j)}(s)\}) \right) \\ &\leq 2e^{Ls} k(s) \gamma(\{v_n\}). \end{aligned}$$

Now the application of Proposition 3.3 for $S_j, j=0, \dots, N-1$, yields

$$\chi(\{S_j(\xi_n)(t)\}) \leq 4C_j \left(\int_0^t e^{pLs} k^p(s) ds \right)^{\frac{1}{p}} \gamma(\{v_n\}) \quad \text{for any } t \in [0, \tau].$$

Recalling that

$$S_j(\xi_n)(t) = \frac{d^j}{dt^j} S(\xi_n)(t) = \frac{d^j}{dt^j} z_n(t) - \frac{d^j}{dt^j} u^*(t) = z_n^{(j)}(t) - \frac{d^j}{dt^j} u^*(t), \quad t \geq 0,$$

we arrive at

$$\begin{aligned} e^{-Lt} \sum_{j=0}^{N-1} \chi(\{z_n^{(j)}(t)\}) &= e^{-Lt} \sum_{j=0}^{N-1} \chi\left(\left\{z_n^{(j)}(t) - \frac{d^j}{dt^j} u^*(t)\right\}\right) \\ &= e^{-Lt} \sum_{j=0}^{N-1} \chi(\{S_j(\xi_n)(t)\}) \leq 4 \sum_{j=0}^{N-1} C_j \left(\int_0^t e^{-Lp(t-s)} k^p(s) ds \right)^{\frac{1}{p}} \gamma(\{v_n\}). \end{aligned}$$

Putting this relation together with (3.13), we obtain

$$\gamma(\{v_n\}) \leq \gamma(\{z_n\}) = \sup_{t \in [0, \tau]} e^{-Lt} \sum_{j=0}^{N-1} \chi(\{z_n^{(j)}(t)\}) \leq \ell \gamma(\{v_n\}).$$

Therefore $\gamma(\{v_n\}) = 0$. This implies

$$\chi(\{v_n^{(j)}(t)\}) = 0, \quad j = 0, \dots, N-1 \quad \text{for all } t \in [0, \tau].$$

Using (F2)–(F3) again, one gets that $\{\xi_n\}$ is a semicompact sequence. Then, Proposition 3.4 ensures that $\{S_j(\xi_n)\}$ is relatively compact in $C([0, \tau]; E)$ for $j=0, \dots, N-1$. This yields that $\{z_n\}$ is relatively compact in $C^{N-1}([0, \tau]; E)$. Hence $\text{mod}_C(\{z_n\}) = 0$. Finally,

$$v(\Omega) = (0, 0). \quad \blacksquare$$

Now we go to the main results of this section. The following assertion is the local existence result for (1.1)–(1.3).

THEOREM 3.9 *Suppose that conditions (F1)–(F3) are satisfied. Then there exists $\tau \in (0, T]$ such that problem (1.1)–(1.3) has at least one mild solution in $C_E(-\infty, \tau)$.*

Proof We take an arbitrary number $\rho > 0$ and denote

$$\begin{aligned} \rho_0 &= (K+1)\rho + M|\varphi|_B, \quad K = \max_{t \in [0, T]} K(t), \quad M = \sup_{t \in [0, T]} M(t), \\ p' &= \frac{p}{p-1}, \quad \text{and} \quad Q = \left(\int_0^T |\omega_{\rho_0}(s)|^p ds \right)^{\frac{1}{p}}, \end{aligned}$$

where ω_{ρ_0} is given in Definition 2.7. Taking into account that $p > \frac{1}{\alpha - N + 1}$, it is obvious that $\alpha - j - \frac{1}{p} > 0$ for all $j=0, \dots, N-1$. Thus we can take $\tau \in (0, T]$, which is small

so that

$$\frac{1}{\Gamma(\alpha-j)} \mathcal{Q} \left[\frac{1}{(\alpha-j-1)p' + 1} \right]^{\frac{1}{p'}} t^{\alpha-j-\frac{1}{p}} \leq \frac{\rho}{N}, \quad (3.14)$$

for all $j=0, \dots, N-1$.

Denote by \bar{B}_ρ the closed convex bounded subset of \mathcal{D}_τ defined by

$$\bar{B}_\rho = \{u \in \mathcal{D}_\tau : \|u - u^*\|_{C^{N-1}} \leq \rho\}.$$

Then for $u \in \bar{B}_\rho$ and v chosen from

$$\mathcal{G}(u) = u^* + S \circ \mathcal{P}_F(u[\varphi]),$$

we have

$$\begin{aligned} \|v(t) - u^*(t)\|_E &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|F(s, u_s, \nabla^N u(s))\| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega_{\rho_0}(s) ds, \end{aligned} \quad (3.15)$$

for all $t \in (0, \tau]$. Here we use assumption (F2) with a note that

$$\begin{aligned} \|u_s|_B + \|\nabla^N u(s)\|_{E^N} &\leq K(s) \|u\|_{C([0,s];E)} + M(s) |\varphi|_B + \|u\|_{C^{N-1}([0,\tau];E)} \\ &\leq (K+1) \|u\|_{C^{N-1}([0,\tau];E)} + M |\varphi|_B \\ &\leq \rho_0. \end{aligned}$$

An application of the Hölder inequality to (3.15) gives

$$\begin{aligned} \|v(t) - u^*(t)\|_E &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)p'} ds \right)^{\frac{1}{p'}} \left(\int_0^t |\omega_{\rho_0}(s)|^p ds \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\Gamma(\alpha)} \mathcal{Q} \left[\frac{1}{(\alpha-1)p' + 1} \right]^{\frac{1}{p'}} t^{\alpha-\frac{1}{p}} \leq \frac{\rho}{N}, \end{aligned}$$

thanks to (3.14). Similarly, for $j=1, \dots, N-1$, we have

$$\begin{aligned} \left\| \frac{d^j}{dt^j} v(t) - \frac{d^j}{dt^j} u^*(t) \right\|_E &\leq \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-s)^{\alpha-j-1} \omega_{\rho_0}(s) ds \\ &\leq \frac{1}{\Gamma(\alpha-j)} \left(\int_0^t (t-s)^{(\alpha-j-1)p'} ds \right)^{\frac{1}{p'}} \left(\int_0^t |\omega_{\rho_0}(s)|^p ds \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\Gamma(\alpha-j)} \mathcal{Q} \left[\frac{1}{(\alpha-j-1)p' + 1} \right]^{\frac{1}{p'}} t^{\alpha-j-\frac{1}{p}} \\ &\leq \frac{\rho}{N}. \end{aligned}$$

So, the following estimate holds

$$\sum_{j=0}^{N-1} \sup_{t \in [0, \tau]} \left\| \frac{d^j}{dt^j} (v(t) - u^*(t)) \right\| \leq \rho,$$

or equivalently,

$$\|v - u^*\|_{C^{N-1}([0, \tau]; E)} \leq \rho.$$

Therefore, \mathcal{G} maps \overline{B}_ρ into \overline{B}_ρ . Hence, the proof is completed by invoking the conclusion of Theorem 2.1. ■

In order to get the global existence result, we need to replace assumption (F2) with a stronger one. Precisely, we impose the assumption that

(F2') There exists a function $\omega \in L^p([0, T])$ such that

$$\|F(t, \zeta, \mathcal{U})\| := \sup\{\|f\|_E : f \in F(t, \zeta, \mathcal{U})\} \leq \omega(t)(1 + |\zeta|_B + \|\mathcal{U}\|_{E^N}),$$

for all $\zeta \in B$ and $\mathcal{U} \in E^N$.

In addition, we need the following version of generalized Bellman–Gronwall inequality (see, e.g. [30]).

LEMMA 3.10 *Assume that $f(t)$, $g(t)$ and $y(t)$ are non-negative, integrable functions on $[0, T]$ satisfying the integral inequality*

$$y(t) \leq g(t) + \int_0^t f(s)y(s)ds, \quad t \in [0, T].$$

Then we have

$$y(t) \leq g(t) + \int_0^t \exp\left\{\int_s^t f(\theta)d\theta\right\}f(s)g(s)ds, \quad t \in [0, T].$$

THEOREM 3.11 *Under assumptions (F1), (F2') and (F3), the set of solutions for problem (1.1)–(1.3) on the interval $[0, T]$ is nonempty and compact.*

Proof We apply Theorem 2.2 to prove that $\text{Fix } \mathcal{G}$ is a nonempty compact set. Combining the results of Lemmas 3.5, 3.7 and 3.8, it remains to check that, if $u \in \mathcal{D}_T$ is such that

$$u - u^* \in \lambda(\mathcal{G}(u) - u^*)$$

for $\lambda \in (0, 1]$, then u must belong to a bounded set in $C^{N-1}([0, T]; E)$. Indeed, we have the inequality by using (F2') for $j = 0, \dots, N - 1$

$$\begin{aligned} \left\| \frac{d^j}{dt^j} u(t) - \frac{d^j}{dt^j} u^*(t) \right\|_E &\leq \frac{\lambda}{\Gamma(\alpha - j)} \int_0^t (t - s)^{\alpha - j - 1} \|F(s, u_s, \nabla^N u(s))\| ds \\ &\leq \frac{1}{\Gamma(\alpha - j)} \int_0^t (t - s)^{\alpha - j - 1} \omega(s) (1 + |u_s|_B + \|\nabla^N u(s)\|_{E^N}) ds. \end{aligned}$$

Using the fact that

$$|u_s|_B + \|\nabla^N u(s)\|_{E^N} \leq M|\varphi|_B + (K + 1)\|u\|_{C^{N-1}([0, s]; E)},$$

we have

$$\begin{aligned} \left\| \frac{d^j}{dt^j} u(t) \right\|_E &\leq \|U_0\|_{E^N} \sum_{k=0}^{N-1} \frac{T^k}{k!} + \frac{M|\varphi|_B + 1}{\Gamma(\alpha - j)} \int_0^t (t - s)^{\alpha - j - 1} \omega(s) ds \\ &\quad + \frac{K + 1}{\Gamma(\alpha - j)} \int_0^t (t - s)^{\alpha - j - 1} \omega(s) \|u\|_{C^{N-1}([0, s]; E)} ds \end{aligned}$$

for $j=0, \dots, N-1$. By the Hölder inequality, we get

$$\begin{aligned} \left\| \frac{d^j}{dt^j} u(t) \right\|_E &\leq \|U_0\|_{E^N} \sum_{k=0}^{N-1} \frac{T^k}{k!} \\ &+ \frac{M|\varphi|_B + 1}{\Gamma(\alpha - j)} \left[\frac{1}{(\alpha - j - 1)p' + 1} \right]^{\frac{1}{p'}} T^{\alpha - j - \frac{1}{p}} \left(\int_0^T |\omega(s)|^p ds \right)^{\frac{1}{p}} \\ &+ \frac{K + 1}{\Gamma(\alpha - j)} \left[\frac{1}{(\alpha - j - 1)p' + 1} \right]^{\frac{1}{p'}} T^{\alpha - j - \frac{1}{p}} \left(\int_0^t |\omega(s)|^p \|u\|_{C^{N-1}([0, s]; E)}^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Let

$$\begin{aligned} C_j &= \frac{1}{\Gamma(\alpha - j)} \left[\frac{1}{(\alpha - j - 1)p' + 1} \right]^{\frac{1}{p'}} T^{\alpha - j - \frac{1}{p}}, \\ C &= \max\{C_j : j = 0, \dots, N - 1\}, \\ g_0 &= N \|U_0\|_{E^N} \sum_{k=0}^{N-1} \frac{T^k}{k!} + NC(M|\varphi|_B + 1) \left(\int_0^T |\omega(s)|^p ds \right)^{\frac{1}{p}}, \\ f(s) &= [NC(K + 1)]^{\frac{1}{p}} \omega(s), \quad s \in [0, T]. \end{aligned}$$

Then

$$\|u\|_{C^{N-1}([0, t]; E)} \leq g_0 + \left(\int_0^t |f(s)|^p \|u\|_{C^{N-1}([0, s]; E)}^p ds \right)^{\frac{1}{p}}.$$

This implies

$$v(t) \leq 2^p g_0^p + 2^p \int_0^t |f(s)|^p v(s) ds,$$

where $v(t) = \|u\|_{C^{N-1}([0, t]; E)}^p$. In accordance with Lemma 3.10, we obtain the estimate

$$v(t) = \|u\|_{C^{N-1}([0, t]; E)}^p \leq 2^p g_0^p \left(1 + \int_0^T \exp \left\{ 2^p \int_s^T |f(\theta)|^p d\theta \right\} |f(s)|^p ds \right)$$

for all $t \in [0, T]$. The last inequality leads to the fact that

$$\|u\|_{C^{N-1}([0, T]; E)} \leq R_0,$$

where

$$R_0 = 2g_0 \sqrt[p]{\left(1 + \int_0^T \exp \left\{ 2^p \int_s^T |f(\theta)|^p d\theta \right\} |f(s)|^p ds \right)}.$$

Finally, taking $a = u^* \in \mathcal{D}_T$ and applying Theorem 2.2 for

$$\overline{\mathcal{U}}_{\mathcal{D}_T} = \{u \in \mathcal{D}_T : \|u\|_{C^{N-1}([0, T]; E)} \leq R\}$$

with $R > R_0$, we conclude that $\text{Fix } \mathcal{G}$ is nonempty compact set. ■

4. Properties of solution map

In this section we will study the continuous dependence of the solution set Σ_F of problem (1.1)–(1.3) on the initial data $(\varphi, U) \in \mathcal{B} \times E^N$. To attain these ends we will need two additional hypotheses on the phase space \mathcal{B} .

It will be supposed that

(B4) there exists $l > 0$ such that $\|\psi(0)\| \leq l\|\psi\|_{\mathcal{B}}$ for all $\psi \in \mathcal{B}$;

(B5) there exists $m, 0 \leq m \leq +\infty$, such that for every sequence $\{\psi_n\} \subset \mathcal{B}$ with $\|\psi_n - \psi_0\|_{\mathcal{B}} \rightarrow 0$ the sequence $\{\psi_n(\theta)\}$ is relatively compact in E for every $\theta \in [-m, 0]$.

It is easy to see that both spaces presented as examples in Section 2.2 satisfy these properties. In fact, $l = 1$ in both cases, and $m = +\infty$ in case (1) and $m = r$ in case (2).

Further, we assume that the multimap $F: [0, T] \times \mathcal{B} \times E^N \rightarrow K_V(E)$ satisfies conditions (F1), (F2') and the following slightly modified condition of χ -regularity:

(F3') there exists a function $k \in L^p(0, T; E)$ such that for any bounded subsets $Q \subset \mathcal{B}$ and $\Omega_j \subset E, j = 0, \dots, N - 1$, we have

$$\chi\left(F(t, Q, \prod_{j=0}^{N-1} \Omega_j)\right) \leq k(t) \left(\sup_{-m \leq \theta \leq 0} \chi(Q(\theta)) + \sum_{j=0}^{N-1} \chi(\Omega_j) \right).$$

Now, we consider in the space $\mathcal{B} \times E^N \times C^{N-1}([0, T]; E)$ the subset

$$\Delta = \{(\psi; \tilde{u}_0, \dots, \tilde{u}_{N-1}; v) : \psi(0) = \tilde{u}_0 = v(0), \tilde{u}_1 = v'(0), \dots, \tilde{u}_{N-1} = v^{(N-1)}(0)\},$$

which is closed by (B4). We define a family of multioperators

$$\Psi : \Delta \rightarrow \mathcal{P}(C^{N-1}([0, T]; E))$$

by

$$\Psi(\psi; \tilde{u}_0, \dots, \tilde{u}_{N-1}; v) = \tilde{u}^* + S \circ \mathcal{P}_F(v[\psi]),$$

where

$$\tilde{u}^*(t) = \sum_{k=0}^{N-1} \frac{t^k}{k!} \tilde{u}_k, \quad 0 \leq t \leq T.$$

It is clear that $v \in \Psi(\psi; \tilde{u}_0, \dots, \tilde{u}_{N-1}; v)$ implies that the function $v[\psi] \in \mathcal{C}_E(-\infty, T)$ belongs to the solution set $\Sigma_F(\psi, U)$, where $U = (\tilde{u}_0, \dots, \tilde{u}_{N-1})$.

LEMMA 4.1 *The multioperator Ψ is closed, i.e., for any sequences $\{(\psi_n, U_n, v_n)\} \subset \Delta$ and $w_n \in \Psi(\psi_n, U_n, v_n)$, the conditions $\|\psi_n - \psi_0\|_{\mathcal{B}} \rightarrow 0, \|U_n - U_0\|_{E^N} \rightarrow 0, \|v_n - v_0\|_{C^{N-1}} \rightarrow 0$, and $\|w_n - w_0\|_{C^{N-1}} \rightarrow 0$ imply $w_0 \in \Psi(\psi_0, U_0, v_0)$.*

Proof Consider a sequence $\{f_n\} \subset L^p(0, T; E)$ such that $f_n \in \mathcal{P}_F(v_n[\psi_n])$ and

$$w_n = \tilde{u}_n^* + S(f_n). \tag{4.1}$$

Here,

$$\tilde{u}_n^*(t) = \sum_{k=0}^{N-1} \frac{t^k}{k!} (\tilde{u}_k)_n, \quad 0 \leq t \leq T,$$

and $(\tilde{u}_k)_n, k = 0, \dots, N - 1$ are the components of U_n .

From hypothesis (B3) it follows that the sequence $\{v_n[\psi_n]\}$ is uniformly bounded with respect to $t \in [0, T]$. Then (F2') implies that the sequence $\{f_n\}$ is L^p -integrably bounded.

Further, from condition (F3') we get for a.e. $t \in [0, T]$:

$$\begin{aligned} \chi(\{f_n(t)\}) &\leq \chi(F(t, \{v_n[\psi_n]_t\}, \{U_n\})) \\ &\leq k(t) \left(\sup_{-m \leq \theta \leq 0} \chi(\{v_n[\psi_n]_t(\theta)\}) + \sum_{j=0}^{N-1} \chi(\{\tilde{u}_j\}_n) \right) \\ &= k(t) \sup_{-m \leq \theta \leq 0} \chi(\{v_n[\psi_n]_t(\theta)\}) \\ &= \begin{cases} k(t) \max \left\{ \sup_{0 \leq \tau \leq t} \chi(\{v_n(\tau)\}), \sup_{t-m \leq \tau \leq 0} \chi(\{\psi_n(\tau)\}) \right\}, & \text{if } 0 \leq t < m, \\ k(t) \sup_{t-m \leq \tau \leq t} \chi(\{v_n(\tau)\}), & \text{if } m \leq t. \end{cases} \end{aligned}$$

Applying hypothesis (B5), we conclude that both values are vanishing and so the sequence $\{f_n\}$ is semicompact. By Proposition 3.4 and Lemma 2.4, we may assume, with loss of generality, that $f_n \rightharpoonup f_0$, where $f_0 \in \mathcal{P}_F(v_0[\psi_0])$. Applying again Proposition 3.4, and passing to the limit in (4.1), we obtain

$$w_0 = \tilde{u}_0^* + S(f_0),$$

concluding the proof. ■

Now we are in position to prove the main result of this section. Denote by Ξ the closed subset of the space $\mathcal{B} \times E^N$ defined by

$$\Xi := \{(\psi, U) = (\psi; \tilde{u}_0, \dots, \tilde{u}_{N-1}) : \psi(0) = \tilde{u}_0\}.$$

For $(\varphi, U) \in \Xi$, let $\Sigma_F(\varphi, U) \subset \mathcal{C}_E(-\infty, T)$ be the set of solutions of problem (1.1)–(1.3).

THEOREM 4.2 *Under conditions (B1)–(B5), (F1), (F2') and (F3'), the multimap*

$$\Sigma_F : \Xi \rightarrow \mathcal{P}(\mathcal{C}_E(-\infty, T))$$

is upper semicontinuous.

Proof At first, we prove the upper semicontinuity of the multimap $\Upsilon : \Xi \rightarrow K(C^{N-1}([0, T]; E))$ defined as

$$\Upsilon(\varphi, U) = \{v \in \Psi(\varphi, U, v)\}.$$

Suppose to the contrary that there exist ε_0 , and sequences $\{(\varphi_n, U_n)\} \subset \Xi$, with $\|\varphi_n - \varphi_0\|_{\mathcal{B}} \rightarrow 0$, $\|U_n - U_0\|_{E^N} \rightarrow 0$, $\{v_n\} \subset C^{N-1}([0, T]; E)$, $v_n \in \Upsilon(\varphi_n, U_n)$ such that

$$v_n \notin W_{\varepsilon_0}(\Upsilon(\varphi_0, U_0)), \quad n \geq 1, \quad (4.2)$$

where W_{ε_0} denotes the ε_0 -neighbourhood of a set. We have

$$v_n \in \Psi(\varphi_n, U_n, v_n), \quad n \geq 1, \quad (4.3)$$

i.e.,

$$v_n = \tilde{u}_n^* + S(f_n),$$

where

$$\tilde{u}_n^*(t) = \sum_{k=0}^{N-1} \frac{t^k}{k!} (\tilde{u}_k)_n, \quad 0 \leq t \leq T,$$

and $f_n \in \mathcal{P}_F(v_n[\varphi_n])$.

Applying the same reasoning as in the proof of Theorem 3.11, we may conclude that the sequence $\{v_n\}$ is bounded. Now, using condition (F3') and hypothesis (B5) we obtain the following estimate for a.e. $s \in [0, t]$:

$$\begin{aligned} \chi(\{f_n(s)\}) &\leq \chi(F(s, \{v_n[\varphi_n]_s\}, \{U_n\})) \\ &\leq k(s) \sup_{-m \leq \theta \leq 0} \chi(\{v_n[\varphi_n]_s(\theta)\}) = k(s) \sup_{0 \leq \tau \leq s} \chi(\{v_n(\tau)\}). \end{aligned}$$

From this we obtain, as in the proof of Lemma 3.8, that $\gamma(\{v_n\})=0$, and the set $\{f_n(t)\}$ is relatively compact for a.e. $t \in [0, T]$.

From the boundedness of the sequences $\{\varphi_n\}$, $\{U_n\}$ and $\{v_n\}$, by condition (F2'), it follows easily that the sequence $\{f_n\}$ is L^p -integrably bounded and hence, semicompact. Then from Proposition 3.4 it follows that the sequence $\{v_n\}$ is relatively compact and so we may assume, without loss of generality, that $v_n \rightarrow v_0 \in C^{N-1}([0, T]; E)$. But then from Lemma 4.1 and (4.3) it follows that

$$v_0 \in \Psi(\varphi_0, U_0, v_0),$$

contrary to (4.2).

Now note that the multimap Σ_F may be presented as the composition of the multimap $\Pi: \Xi \rightarrow \mathcal{P}(\Delta)$,

$$\Pi(\varphi, U) = \{\varphi\} \times \{U\} \times \Upsilon(\varphi, U),$$

and the continuous map $\kappa: \Delta \rightarrow \mathcal{C}_E(-\infty, T)$, $\kappa(\varphi, U, v) = v[\varphi]$. From Theorem 3.11 we know that each value $\Upsilon(\varphi, U)$ is compact. Applying the continuity properties of multimaps (see, e.g. [21]), we finally deduce that the multimap Σ_F is u.s.c. ■

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