# TOPOLOGICAL DEGREE METHODS IN BOUNDARY VALUE PROBLEMS FOR DEGENERATE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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ABSTRACT. We consider the general boundary value problem for a degenerate semilinear functional differential inclusion in a Banach space with infinite delay. We construct the multivalued integral operator whose fixed points are mild solutions of the above problem and study its properties. We apply the topological degree method to obtain the general existence principle and consider some particular cases, including Cauchy and periodic problems.

## 1. INTRODUCTION

In the recent time two directions in the theory of differential equations and inclusions in Banach spaces are intensively developing and attract the attention of many researchers. The first one is the theory of degenerate (or Sobolev type) differential equations and inclusions (see, e.g., [1], [2], [5], [14], [16], [17] and the references therein). One of the reasons of growing interest to this branch is the fact that many types of PDEs arising in problems of mathematical physics and applied sciences may be naturally presented in this form.

The second direction is connected with the study of functional differential equations and inclusions with infinite delay. Starting from the work of J.K. Hale and J. Kato [8], who suggested the axiomatic approach to the definition of the phase space of distributed infinite delays, this subject is investigated very actively (see, e.g., [6], [7], [9], [10], [15] and the references therein).

In the present paper, generalizing some results of the works [1], [14], we suggest a version of synthesis of both theories, considering the general type boundary value problem for a degenerate semilinear functional differential inclusion in a Banach space with infinite delay. We construct the multivalued integral operator whose fixed points are mild solutions of the above problem and study its properties. In particular, we give conditions under which this multioperator is condensing w.r.t. the vector measure of noncompactness of

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a special form. This allows us to apply the methods of topological degree theory for condensing multimaps (see, e.g. [3], [12]) and to obtain the general existence result (Theorem 24). We consider some particular cases including Cauchy and periodic problems.

## 2. Preliminaries

### 2.1. **Multivalued linear operators.**

We begin with some necessary definitions and results from the theory of multivalued linear operators. Details can be found in [1], [2], and [5].

Let $E$ be a complex Banach space.

**Definition 1.** *A multivalued map (multimap) $A : E \to 2^E$ is said to be a **multivalued linear operator** (MLO) on $E$ if:*
*1) $D(A) = \{x \in E : Ax \neq \emptyset\}$ is a linear subspace of $E$;*
*2)*

$$\begin{cases} Ax + Ay \subset A(x+y), & \forall x, y \in D(A) \ ; \\ \lambda\, Ax \subseteq A(\lambda\, x) \ , & \forall \lambda \in \mathbb{C}, x \in D(A) \ . \end{cases}$$

It is an easy consequence of the definition to note that $Ax + Ay = A(x+y)$ for all $x, y \in D(A)$ and $\lambda\, Ax = A(\lambda\, x)$ for all $x \in D(A)$, $\lambda \neq 0$. It is also clear that $A$ is a MLO on $E$ if and only if its graph $\Gamma_A$ is a linear subspace of $E \times E$. A MLO $A$ is said to be **closed** if $\Gamma_A$ is the closed subspace of $E \times E$. The collection of all closed MLO's in $E$ will be denoted by $ML(E)$.

**Definition 2.** *The inverse $A^{-1}$ of a MLO is defined as:*
*1) $D\left(A^{-1}\right) = R(A)$ ;*
*2) $A^{-1}y = \{x \in D(A) : y = Ax\}$ .*

It is obvious that $(y, x) \in \Gamma_{A^{-1}}$ if and only if $(x, y) \in \Gamma_A$ and hence $A^{-1} \in ML(E)$ if $A \in ML(E)$.

Denote by $\mathcal{L}(E)$ the space of all single-valued bounded operators on $E$.

**Definition 3.** *The **resolvent set** $\rho(A)$ of a MLO $A$ is defined as the collection of all $\lambda \in \mathbb{C}$ for which:*
*1) $R(\lambda I - A) = D\left((\lambda I - A)^{-1}\right) = E$;*

*2) $(\lambda I - A)^{-1} \in \mathcal{L}(E)$.*

**Definition 4.** *The operator-valued function $R(\cdot, A) : \rho(A) \to \mathcal{L}(E)$*

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

*is called the **resolvent** of a MLO $A$.*

**Remark 5.** *If $E$ is a real Banach space and $A$ is a MLO on $E$, we may consider the **complexification** $\widetilde{E} = E + iE$ and $\widetilde{A}$ defined by*

$$\Gamma_{\widetilde{A}} = \{(x, y_1) + i\,(x, y_2) : x \in D(A), y_1, y_2 \in Ax\}.$$

*Then we set, by definition, $\rho(A) = \rho(\widetilde{A})$.*

Let $U : \mathbb{R}_+ = [0, +\infty) \to \mathcal{L}(E)$ be a $C_0$-semigroup of operators, i.e., we suppose the following conditions:

$(i)$ $U(t+s) = U(t)\, U(s)\,,\ \ \forall t, s \in \mathbb{R}_+;$
$(ii)$ for each $x \in E$, the function $t \to U(t)x$ is continuous on $\mathbb{R}_+$.

Notice that the usual condition $U(0) = I$ is absent here. From assumption $(i)$ it follows that $U(0) = P \in \mathcal{L}(E)$ is the projector. In case $P \neq I$ the semigroup $U$ is called **generalized** (or **degenerate**).

It is easy to verify that there exist constants $C \geq 1$ and $\gamma \geq 0$ such that

$$\|U(t)\|_{\mathcal{L}(E)} \leq Ce^{\gamma t}, \quad t \in \mathbb{R}_+. \tag{1}$$

Therefore, for each $\lambda \in \mathbb{C}_\gamma = \{\mu \in \mathbb{C} : \ Re\, \mu > \gamma\}$ the bounded linear operator $R(\lambda)$ may be defined by the following Laplace transformation:

$$R(\lambda)\, x = \int_0^\infty U(\tau)\, xe^{-\lambda\tau} d\tau.$$

The function $R : \mathbb{C}_\gamma \to \mathcal{L}(E)$ satisfies Hilbert equality and it is the resolvent of a certain (unique) $A \in ML(E)$. This MLO A is called the **generator** of the generalized semigroup $U$.

Let $E^*$ be the dual space of $E$. For $A \in ML(E)$, we denote by $A^*$ a MLO on $E^*$ defined in the following way: for $h, g \in E^*$, the relation $h \in A^*(g)$ means that $g(y) = h(x)$ for all pairs $(x, y) \in \Gamma_A$. It is easy to verify that $A^* 0^* = \{h \in E^* : \overline{D(A)} \subset Kerh\} = \overline{D(A)}^{\perp}$.

Consider the following assumptions on $A \in ML(E)$.

$(A_1)$ functionals from $A^*0^*$ are separated by vectors of $A0$, i.e., for each $h \in A^*0^*$, $h \neq 0^*$ there exists $y \in A0$ such that $h(y) \neq 0$;

$(A_2)$ the Hille–Yosida condition: there exist a constant $C > 0$ and $\gamma \in \mathbb{R}$ such that $\mathbb{C}_\gamma \subset \rho(A)$ and

$$\|R(\lambda, A)^n\|_{\mathcal{L}(E)} \leq \frac{C}{(Re\, \lambda - \gamma)^n}, \quad n = 1, 2, ..., \quad \lambda \in \mathbb{C}_\gamma.$$

**Remark 6.** *In* [1] *it was shown that each of the following conditions implies* $(A_1)$*:* $(i)$ *the space* $E$ *is reflexive;* $(ii)$ $dim\, A0 = dim A^* 0^* < \infty$.

The following result holds true (cfr. [1], [5]).

**Theorem 7.** *Conditions* $(A_1)$ *and* $(A_2)$ *are necessary and sufficient for* $A \in ML(E)$ *to be the generator of a* $C_0$*-semigroup* $U$*. Moreover, the semigroup* $U$ *is generalized iff* $A$ *is not single-valued. In this case the space* $E$ *may be represented as* $E = E_1 \oplus E_1$*, where* $E_0 = \overline{D(A)}$, $E_1 = A0$ *and the restriction of* $U(t)$ *on* $E_0$ *defines the usual* $C_0$*-semigroup on* $E_0$ *whereas the restriction on* $E_1$ *vanishes.*

2.2. **Multivalued maps and measures of noncompacness.** Let us recall some notions (see, e.g., [3], [12]). Let $X$ be a metric space; $\mathcal{E}$ a normed space; $P(\mathcal{E})$ denote the collection of all nonempty subsets of $\mathcal{E}$. By symbols $K(\mathcal{E})$ and $Kv(\mathcal{E})$ we denote the collections of all nonempty compact and, respectively, compact convex subsets of $\mathcal{E}$.

**Definition 8.** *A multivalued map (multimap) $\mathcal{F} : X \to K(\mathcal{E})$ is said to be* ***upper semicontinuous (u.s.c.)*** *if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of $X$ for every open $V \subset \mathcal{E}$.*

**Definition 9.** *A multivalued map (multimap) $\mathcal{F} : X \to K(\mathcal{E})$ is said to be* ***compact*** *if its range $\mathcal{F}(X)$ is a relatively compact subset of $\mathcal{E}$. If a u.s.c. multimap $\mathcal{F}$ is compact on bounded subsets of $X$ it is called* ***completely continuous.***

**Definition 10.** *Let $\mathcal{E}$ be a normed space; $(\mathcal{A}, \geq 0)$ a (partially) ordered set. A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a* ***measure of noncompactness (MNC)*** *in $\mathcal{E}$ if*

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

*for every $\Omega \in P(\mathcal{E})$.*

A MNC $\beta$ is called:

(a) *monotone* if $\Omega_1 \subseteq \Omega_2$ implies $\beta(\Omega_1) \leq \beta(\Omega_2)$;
(b) *nonsingular* if $\beta(\Omega \cup \{a\}) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;
(c) *invariant with respect to union with compact sets* if $\beta(\Omega \cup K) = \beta(\Omega)$ for every $\Omega \in P(\mathcal{E})$, $K$ is relatively compact in $\mathcal{E}$ ;
(d) *invariant with respect to reflection through the origin* if $\beta(-\Omega) = \beta(\Omega)$ for every $\Omega \in P(\mathcal{E})$;
(e) *real* if $\mathcal{A} = [0, +\infty]$ with natural ordering.

If $\mathcal{A}$ is a cone in a Banach space, we say that the MNC $\beta$ is:

(f) *algebraically semiadditive* if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0$, $\Omega_1 \in P(\mathcal{E})$;
(g) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega$.

As the example of the MNC possessing all these properties, we may consider the **Hausdorff MNC**:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon \text{ - net } \}.$$

Let $I \subseteq \mathbb{R}$ be any closed interval. The examples of real measures of noncompactness defined on the space of continuous functions $C(I; E)$ with the values in a Banach space $E$ are presented by the following characteristics:

1) *modulus of equicontinuity*:

$$\operatorname{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| < \delta} \|x(t_1) - x(t_2)\|_E.$$

2) *modulus of fiber noncompactness*:

$$\varphi(\Omega) = \sup_{t \in I} \chi_E(\Omega(t)),$$

where $\Omega(t) = \{x(t) : x \in \Omega\}$.

Let $\mathcal{E}$ and $\mathcal{E}'$ be normed spaces with MNC $\beta$ and $\beta'$ respectively; $\mathcal{N} : \mathcal{E} \to \mathcal{E}'$ a continuous linear operator.

**Definition 11.** *The operator $\mathcal{N}$ is said to be $(\beta, \beta')$-**bounded** provided there exists $C \geq 0$ such that*

$$\beta'(\mathcal{L}\Omega) \leq C\beta(\Omega) \quad \textit{for all bounded sets} \quad \Omega \subset \mathcal{E}.$$

*The value* $\|\mathcal{N}\|^{(\beta,\beta')}$ *which is equal to the infimum of all such coefficients is called the* $(\beta,\beta')$-***norm*** *of operator* $\mathcal{L}$.

In particular, if $\mathcal{E}=\mathcal{E}'$ and $\beta=\beta'$ then $\|\mathcal{N}\|^{(\beta,\beta)}$ is denoted by $\|\mathcal{N}\|^{(\beta)}$ and called the $\beta$-*norm of the operator* $\mathcal{N}$. For the evaluation of the $\chi$-norm of the operator $\mathcal{N}$ we can apply the formula

$$\|\mathcal{N}\|^{(\chi)}=\chi(\mathcal{N}S)=\chi(\mathcal{N}B),$$

where $S$ and $B$ are the unit sphere and the unit ball in $\mathcal{E}$, respectively. It is easy to see that

$$\|\mathcal{N}\|^{(\chi)}\leq\|\mathcal{N}\|.$$

**Definition 12.** *A multimap* $\mathcal{F}:X\subseteq\mathcal{E}\to K(\mathcal{E})$ *is called* ***condensing w.r.t. a MNC*** $\beta$ *in* $\mathcal{E}$ *(or* $\beta$-***condensing****), if for every* $\Omega\subseteq X$*, that is not relatively compact, we have*

$$\beta(\mathcal{F}(\Omega))\not\geq\beta(\Omega).$$

Let $U$ be an open set in $\mathcal{E}$, $\mathcal{K}\subseteq\mathcal{E}$ a convex closed set, such that $U_{\mathcal{K}}=U\cap\mathcal{K}$ is nonempty, bounded and $\beta$ a monotone nonsingular MNC in $\mathcal{E}$.

Let $\mathcal{F}:\overline{U}_{\mathcal{K}}\to Kv(\mathcal{K})$ be compact or $\beta$-condensing, u.s.c. multimap, moreover, let $x\notin\mathcal{F}(x)$ for all $x\in\partial U_{\mathcal{K}}$, where $\overline{U}_{\mathcal{K}}$ and $\partial U_{\mathcal{K}}$ denote, respectively, the closure and the boundary of the set $U_{\mathcal{K}}$ in the relative topology of the space $\mathcal{K}$. In this situation for the corresponding multifield $i-\mathcal{F}$ the characteristic

$$deg_{\mathcal{K}}(i-\mathcal{F},\overline{U}_{\mathcal{K}}),$$

called the **relative topological degree**, having all standart properties, is defined (see, e.g., [3], [12]). In particular, the difference of this characteristic from zero implies the existence of at least one fixed point $x\in U_{\mathcal{K}},\ x\in\mathcal{F}(x)$.

We will use the following notion. Let $E$ be a Banach space; for $T>0$ by the symbol $L^1([0,T];E)$ we will denote the space of all Bochner summable functions.

**Definition 13.** *The sequence* $\{f_n\}_{n=1}^{\infty}\subset L^1([0,T];E)$ *is said to be* ***semicompact*** *if it is integrably bounded and the set* $\{f_n(t)\}_{n=1}^{\infty}\subset E$ *is relatively compact for a.e.* $t\in[0,T]$.

**Theorem 14.** *(see, e.g.* [12]*). Every semicompact sequence is weakly compact in the space* $L^1([0,d];E)$.

2.3. **Phase space.** We will employ the axiomatical definition of the **phase space** $\mathcal{B}$, introduced by J.K.Hale and J.Kato (see [8], [10]). The space $\mathcal{B}$ will be considered as a linear topological space of functions mapping $(-\infty,0]$ into a Banach space $E$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ .

For any function $x:(-\infty;T]\to E$ and for every $t\in(-\infty;T]$, $x_t$ represents the function from $(-\infty,0]$ into $E$ defined by

$$x_t(\theta)=x(t+\theta),\ \theta\in(-\infty;0].$$

We will assume that $\mathcal{B}$ satisfies the following axioms.

($\mathcal{B}1$) If $x:(-\infty;T]\to E$ is continuous on $[0;T]$ and $x_0\in\mathcal{B}$, then for every $t\in[0;T]$ we have

$(i)$ $x_t \in \mathcal{B}$;
$(ii)$ function $t \mapsto x_t$ is continuous;
$(iii)$ $\|x_t\|_{\mathcal{B}} \le K(t) \sup_{0\le\tau\le t} \|x(\tau)\| + N(t)\|x_0\|_{\mathcal{B}}$, where $K, N : [0;\infty) \to [0;\infty)$ are independent on $x$, $K$ is strictly positive and continuous, and $N$ is locally bounded.

($\mathcal{B}2$) There exists $l > 0$ such that

$$\|\psi(0)\|_E \le l\|\psi\|_{\mathcal{B}}$$

for all $\psi \in \mathcal{B}$.

Let us mention that under above hypotheses the space $C_{00}$ of all continuous functions from $(-\infty, 0]$ into $E$ with compact support is a subset of each phase space $\mathcal{B}$ ([10], Proposition 1.2.1). We will assume, additionally, that the following hypothesis holds true.

($\mathcal{BC}1$) If a uniformly bounded sequence $\{\psi_n\}_{n=1}^{+\infty} \subset C_{00}$ converges to a function $\psi$ compactly (i.e. uniformly on each compact subset of $(-\infty, 0]$), then $\psi \in \mathcal{B}$ and

$$\lim_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0.$$

The hypothesis ($\mathcal{BC}1$) yields that the Banach space $BC = BC((-\infty, 0]; E)$ of bounded continuous functions is continuously imbedded into $\mathcal{B}$. More exactly, the following proposition holds true.

**Theorem 15.** [[10], Proposition 7.1.1].
*(i)* $BC \subset \overline{C_{00}}$, *where* $\overline{C_{00}}$ *denotes the closure of* $C_{00}$ *in* $\mathcal{B}$;
*(ii) if a uniformly bounded sequence* $\{\psi_n\}$ *in* $BC$ *converges to a function* $\psi$ *compactly on* $(-\infty, 0]$ *then* $\psi \in \mathcal{B}$ *and* $\lim_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0$;
*(iii)* $\|\psi\|_{\mathcal{B}} \le L\|\psi\|_{BC}, \quad \psi \in BC$ *for some constant* $L > 0$.

At last, we will assume the following.

($\mathcal{BC}2$) If $\psi \in BC$ and $\|\psi\|_{BC} \ne 0$, then $\|\psi\|_{\mathcal{B}} \ne 0$.

This hypothesis implies that that the space $BC$ endowed with $\|\cdot\|_{\mathcal{B}}$ is a normed space. We will denote it by $\mathcal{BC}$.

We may consider the following examples of phase spaces satisfying all above properties.

(1) For $\nu > 0$ let $\mathcal{B} = C_\nu$ be a space of continuous functions $\varphi : (-\infty; 0] \to E$ having a limit $\lim_{\theta\to-\infty} e^{\nu\theta}\varphi(\theta)$ with

$$\|\varphi\|_{\mathcal{B}} = \sup_{-\infty<\theta\le 0} e^{\nu\theta}\|\varphi(\theta)\|.$$

(2) (*Spaces of "fading memory"*). Let $\mathcal{B} = C_\rho$ be a space of functions $\varphi : (-\infty; 0] \to E$ such that

(a) $\varphi$ is continuous on $[-r; 0], r > 0$;

(b) $\varphi$ is Lebesgue measurable on $(-\infty; r)$ and there exists a positive Lebesgue integrable function $\rho : (-\infty; -r) \to \mathbb{R}^+$ such that $\rho\varphi$ is Lebesgue integrable on $(-\infty; r)$; moreover, there exists a locally bounded function $P : (-\infty; 0] \to \mathbb{R}^+$ such that, for all $\xi \leq 0$, $\rho(\xi + \theta) \leq P(\xi)\rho(\theta)$ a.e. $\theta \in (-\infty; -r)$. Then,

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r\leq\theta\leq 0} \|\varphi(\theta)\| + \int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\| d\theta.$$

A simple example of such a space is given by $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

## 3. Boundary value problem for a degenerate functional differential inclusion with infinite delay

Let $M : D(M) \subseteq E \to E$ be a bounded linear operator and $L : D(L) \subseteq E \to E$ a closed linear operator in a real separable Banach space $E$ satisfying the condition

$(ML)$ $D(L) \subseteq D(M)$ and $\overline{M(D(L))} \subseteq R(M)$.

We will consider the following general boundary value problem for a degenerate (Sobolev type) differential inclusion in $E$

$$\frac{dMy(t)}{dt} \in Ly(t) + F(t, My_t), \qquad t \in [0, T] \tag{2}$$

$$\mathcal{Q}(My) \in \mathcal{S}(My). \tag{3}$$

With the change $x(t) = My(t)$ we can rewrite problem (2), (3) into the following form

$$\frac{dx(t)}{dt} \in Ax(t) + F(t, x_t), \qquad t \in [0, T] \tag{4}$$

$$\mathcal{Q}(x) \in \mathcal{S}(x), \tag{5}$$

where $A = LM^{-1}$. It is clear that $A \in ML(E)$ if $M$ is not invertible and that $D(A) = M(D(L))$.

It will be supposed that:

(A) $A = LM^{-1}$ satisfies conditions $(A_1)$, $(A_2)$ of Section 2.1.

It should be mentioned that to guarantee condition $(A_2)$, it is sufficient to assume that:
*(i)* $[Ly, My] \leq \gamma\|My\|^2, \forall y \in D(L)$ for some $\gamma \in \mathbb{R}$, where $[,]$ is a semi-scalar product in $E$
and
*(ii)* $R(\lambda_0 M - L) = E$ for some $\lambda_0 > \gamma$.
(See [1]).

In accordance with [1], [14] we give the following notion.

**Definition 16.** *A function $y:(-\infty,T]\to E$ is a* ***mild solution*** *of differential inclusion (2) if the function $x(t)=My(t)$, $t\in[0,T]$ has the form*

$$x(t)=U(t)x(0)+\int_0^t U(t-s)f(s)\,ds, \tag{6}$$

*where $U$ is the generalized semigroup generated by $A$ and $f\in L^1([0,T];E)$ is a selection of the multifunction $t\multimap F(t,x_t)$.*

The definition is motivated by the following facts. At first, following [5], Theorem 2.6, it is easy to verify that given a function $f\in L^1([0,T];E)$ every Caratheodory solution to the problem with the MLO $A$

$$\begin{aligned}\frac{dx(t)}{dt} &\in Ax(t)+f(t)\\ x(0) &= x_0\in\overline{D(A)}\end{aligned}$$

is necessarily of the form

$$x(t)=U(t)x_0+\int_0^t U(t-s)f(s)\;ds\,.$$

Further, the function $t\to U(t)x_0+\int_0^t U(t-s)f(s)\,ds$ takes its values in the subspace $\overline{D(A)}=\overline{M(D(L))}\subseteq R(M)$ (see condition $(ML)$). At last, in the non-degenerate case $M=I$, the given definition agrees with the notion of mild solution for a semilinear differential inclusion (see, e.g., [12]).

We will also say that a function $x$ satisfying integral equation (6) is the mild solution of inclusion (4).

In the sequel, we consider the phase space $\mathcal{B}$ of functions $\psi:(-\infty,0]\to E_0$, with $E_0=\overline{D(A)}=\overline{M(D(L))}$, satisfying all axioms of Section 2.3.

We will assume that the multimap $F:[0,T]\times\mathcal{B}\to Kv(E)$ obeys the following conditions:

- (F1) for each $\psi\in\mathcal{BC}$, the multifunction $F(\cdot,\psi):[0;T]\to Kv(E)$ admits a measurable selection;
- (F2) for a.e. $t\in[0;T]$, the multimap $F(t,\cdot):\mathcal{BC}\to Kv(E)$ is u.s.c.;
- (F3) for each nonempty, bounded set $\Omega\subset\mathcal{BC}$, there exists a function $\alpha_\Omega\in L^1_+[0,T]$ such that

  $$\|F(t,\psi)\|_E:=\sup\{\|z\|_E : z\in F(t,\psi)\}\le\alpha_\Omega(t)$$

  for a.e. $t\in[0,T]$ $\psi\in\Omega$;
- (F4) there exists a function $k\in L^1_+[0,T]$ such that for each nonempty bounded set $\Omega\subset\mathcal{BC}$

  $$\chi(F(t,\Omega))\le k(t)\varphi(\Omega)$$

  for a.e. $t\in[0,T]$, where $\chi$ is the Hausdorff MNC in $E$ and $\varphi(\Omega)$ is the modulus of fiber noncompactness of the set $\Omega$.

By the symbol $\mathcal{C}((-\infty;T];E_0)$ we wll denote the space of bounded continuous functions $x:(-\infty;T]\to E_0$ endowed with the norm

$$\|x\|_{\mathcal{C}}=\|x_0\|_{\mathcal{B}}+\|x\mid_{[0;T]}\|_C,$$

where the last norm is the usual sup-norm in the space $C([0;T];E_0)$.

For operators from boundary condition (3) we assume that:

$(\mathcal{Q})$ $\mathcal{Q}:\mathcal{C}((-\infty;T];E_0)\to\mathcal{BC}$ is a linear bounded operator;

$(\mathcal{S})$ the multimap $\mathcal{S}:\mathcal{C}((-\infty;T];E_0)\to Kv(\mathcal{BC})$ is completely continuous, i.e., it is u.s.c. and transforms bounded sets into relatively compact ones.

From conditions $(F1)-(F3)$ and $(\mathcal{B}1)$ it follows that the superposition multioperator $\mathcal{P}_F:\mathcal{C}((-\infty;T];E_0)\to P(L^1([0,T];E))$, given by

$$\mathcal{P}_F(x)=\{f\in L^1([0,T];E): f(t)\in F(t,x_t)\ \text{ a.e. } t\in[0,T]\}$$

is well-defined (see, e.g., [4], [12]).

**Definition 17.** *The linear operator $G:L^1([0,T];E)\to\mathcal{C}((-\infty;T];E_0)$, defined as*

$$Gf(t)=\begin{cases}\int\limits_0^t U(t-s)\,f(s)\,ds, & t\in[0;T];\\ 0, & t\in(-\infty;0]\end{cases}$$

*is called the* ***Cauchy operator****.*

Following [12], one may verify that the Cauchy operator has the next properties.

**Theorem 18.** *For every semicompact sequence $\{f_n\}_{n=1}^{\infty}$ in the space $L^1([0,T];E)$ the sequence $\{Gf_n\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{C}((-\infty,T];E_0)$.*

**Theorem 19.** *The composition $G\circ\mathcal{P}_F:\mathcal{C}((-\infty,T];E_0)\to Kv(\mathcal{C}((-\infty,T];E_0))$ is u.s.c. with compact convex values.*

Denote by $\mathcal{C}_0$ the subspace of $\mathcal{C}((-\infty;T];E_0)$, consisting of functions of the form

$$x(t)=U(t)\,x(0)\,,\ \ t\in[0,T]$$

and denote by $\mathcal{Q}_0$ the restriction of $\mathcal{Q}$ to $\mathcal{C}_0$.

Our main assumption on boundary operators $\mathcal{Q}$ and $\mathcal{S}$ will be the following.

$(\mathcal{QS})$ There exists a continuous linear operator $\Lambda:\mathcal{BC}\to\mathcal{C}_0$ such that $(I-\mathcal{Q}_0\Lambda)(z-\mathcal{Q}Gf)=0$ for each $x\in\mathcal{C}((-\infty,T];E_0)$, $z\in\mathcal{S}(x)$ and $f\in\mathcal{P}_F(x)$.

To present an example of the realization of condition $(\mathcal{QS})$, consider the linear operator $r:\mathcal{BC}\to\mathcal{C}_0$ defined in the following way:

$$(r\psi)(t)=\begin{cases}\psi(t), & t\in(-\infty,0];\\ U(t)\,\psi(0)\,, & t\in[0;T].\end{cases}$$

Notice that from condition $(\mathcal{B}2)$ it follows that the operator $r$ is continuous.

Assume that

($\widetilde{\mathcal{Q}}$) The linear continuous operator $\widetilde{\mathcal{Q}} : \mathcal{BC} \to \mathcal{BC}$ defined as $\widetilde{\mathcal{Q}}\psi = \mathcal{Q}(r\psi)$ has the continuous inverse $\widetilde{\mathcal{Q}}^{-1}$.

It is easy to see that under condition ($\widetilde{\mathcal{Q}}$) the operator $\Lambda$ may be presented in the following form:

$$\Lambda\psi = r[\widetilde{\mathcal{Q}}^{-1}(\psi)]. \tag{7}$$

Supposing that condition $(\mathcal{QS})$ is fulfilled, consider the multioperator

$$\Gamma : \mathcal{C}((-\infty;T];E_0) \to Kv(\mathcal{C}((-\infty;T];E_0))$$

defined in the following way:

$$\Gamma(x) = \Lambda\mathcal{S}(x) + (I - \Lambda\mathcal{Q})G\mathcal{P}_F(x).$$

From Theorem 19 and the conditions posed on the operators $\mathcal{Q}$, $\mathcal{S}$, and $\Lambda$ it follows that the multioperator $\Gamma$ is u.s.c. and has convex compact values. Also, it is easy to see that $\Gamma$ is a bounded operator, i.e., it takes bounded sets into bounded ones. Describe its subsequent properties.

**Theorem 20.** *Fixed points of the multioperator $\Gamma$ are mild solutions of problem (4)-(5) and hence they define mild solutions of problem (2)-(3).*

*Proof.* Let $x \in \Gamma(x)$. It means that there exist $z \in \mathcal{S}(x)$, $f \in \mathcal{P}_F(x)$ such that

$$x = \Lambda z + (I - \Lambda\mathcal{Q})Gf.$$

Since the function $x$ may be represented in the form

$$x = \Lambda(z - \mathcal{Q}Gf) + Gf$$

we obtain that $x$ satisfies integral equation (6).

Let us verify the fulfilment of the boundary condition. Using condition $(\mathcal{QS})$ we get

$$\mathcal{Q}x = \mathcal{Q}_0\Lambda z + \mathcal{Q}\,(I - \Lambda\mathcal{Q})\,Gf = z - (z - \mathcal{Q}_0\Lambda z) + \mathcal{Q}Gf + \mathcal{Q}_0\Lambda\mathcal{Q}Gf$$
$$= z - (I - \mathcal{Q}_0\Lambda)\,(z - \mathcal{Q}Gf) = z \in \mathcal{S}x.$$

□

Consider the MNC $\nu$ on the space $\mathcal{C}((-\infty;T];E_0)$ with values in the cone $\mathbb{R}^2_+$:

$$\nu(\Omega) = (\varphi_{\mathcal{C}}(\Omega), \mathrm{mod}_C(\Omega)),$$

where $\varphi_{\mathcal{C}}$ is the modulus of fiber noncompactness in the space $\mathcal{C}((-\infty;T];E_0)$. Notice that

$$\varphi_{\mathcal{C}}(\Omega) = \sup_{0\le t\le T} \varphi_{\mathcal{BC}}(\Omega_t),$$

where $\Omega_t \subset \mathcal{BC}$, $\Omega_t = \{x_t : x \in \Omega\}$ and, for $t \in [0,T]$ :

$$\varphi_{\mathcal{BC}}(\Omega_t) = \sup_{-\infty\le\tau\le 0} \chi(\Omega_t(\tau)) = \sup_{-\infty\le\tau\le 0} \chi(\Omega(t+\tau)) = \sup_{-\infty\le\tau\le t} \chi(\Omega(\tau)),$$

where $\chi$ is the Hausdorff MNC in $E_0$.

Denote by $\widetilde{\mathcal{C}}$ the subspace of $\mathcal{C}((-\infty;T];E_0)$ consisting of functions vanishing on $(-\infty;0]$. It is clear that $\widetilde{\mathcal{C}}$ is isomorphic to the space $C([0,T];E_0)$.

**Theorem 21.** *Let the following conditions hold:*

(H1) *there exists* $b \geq 0$ *such that*

$$\varphi_{\mathcal{BC}}(\mathcal{Q}\Omega) \leq b\varphi_{\mathcal{C}}(\Omega)$$

*for each bounded set* $\Omega \subset \widetilde{\mathcal{C}}$;

(H2) *for each relatively compact sequence* $\{z_n\} \subset \widetilde{\mathcal{C}}$, *the sequence* $\{\Lambda\mathcal{Q}z_n\}$ *is equicontinuous;*

(H3) *there exists a function* $h \in L^1_+([0,T])$ *such that*

$$\|U\,(t)\,\|^{(\chi)} \leq h(t);$$

(H4) $(1 + \|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})}b) \sup\limits_{0\leq t\leq T} \int\limits_0^t h(t-s)k(s)ds = \mu < 1$, *where* $k(\cdot)$ *is the function from condition* $(F4)$.

*Then the multioperator* $\Gamma$ *is* $\nu$*-condensing on bounded subsets of the space* $\mathcal{C}((-\infty;T];E_0)$.

*Proof.* Assume that $\Omega$ is a bounded subset of $\mathcal{C}((-\infty;T];E_0)$ for which we have

$$\nu(\Gamma\Omega) \geq \nu(\Omega).$$

Let us show that the set $\Omega$ is relatively compact.

From the above inequality it follows that

$$\varphi_{\mathcal{C}}(\Gamma\Omega) \geq \varphi_{\mathcal{C}}(\Omega).$$

Taking arbitrary $t \in [0;T]$ and $\tau \in [-\infty, t]$, let us estimate $\chi(\Gamma\Omega(\tau))$. Since the set $\Lambda\mathcal{S}(\Omega)$ is relatively compact, it is sufficient to estimate the value

$$\chi((I - \Lambda\mathcal{Q})G\mathcal{P}_F(\Omega)(\tau)).$$

We obtain

$$\chi(\Lambda\mathcal{Q}G\mathcal{P}_F(\Omega)(\tau)) \leq \varphi_{\mathcal{C}}(\Lambda\mathcal{Q}G\mathcal{P}_F(\Omega)) \leq \|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})}\varphi_{\mathcal{BC}}(\mathcal{Q}G\mathcal{P}_F(\Omega))$$
$$\leq \|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})}b\varphi_{\mathcal{C}}(G\mathcal{P}_F(\Omega)) = \|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})}b \sup_{0\leq t\leq T} \chi(G\mathcal{P}_F(\Omega)(t)).$$

To estimate $\chi(G\mathcal{P}_F(\Omega)(t))$, notice that for $0 \leq s \leq t$, we have

$$\chi(U\,(t-s)\,F(s,\Omega_s)) \leq \|U\,(t-s)\,\|^{(\chi)}\chi(F(s,\Omega_s)) \leq$$
$$\leq h(t-s)k(s)\varphi_{\mathcal{BC}}(\Omega_s) \leq h(t-s)k(s)\varphi_{\mathcal{C}}(\Omega).$$

Then, applying the theorem on $\chi$-estimation of a multivalued integral (see [12], Theorem 4.2.3) we obtain

$$\chi(G\mathcal{P}_F(\Omega)(t)) \leq \int\limits_0^t h(t-s)k(s)ds \cdot \varphi_{\mathcal{C}}(\Omega).$$

Using now the algebraic semiadditivity of the MNC $\chi$, we have

$$\chi((I - \Lambda\mathcal{Q})G\mathcal{P}_F\Omega(\tau)) \leq (1 + \|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})}b) \sup_{0\leq t\leq T} \int\limits_0^t h(t-s)k(s)ds \cdot \varphi_{\mathcal{C}}(\Omega)$$
$$= \mu \cdot \varphi_{\mathcal{C}}(\Omega).$$

Then

$$\varphi_{\mathcal{C}}(\Gamma\Omega) = \sup_{0\leq t\leq T}\ \sup_{-\infty\leq\tau\leq t} \chi(\Gamma\Omega(\tau)) \leq \mu \cdot \varphi_{\mathcal{C}}(\Omega).$$

We obtain

$$\varphi_{\mathcal{C}}(\Omega) \le \varphi_{\mathcal{C}}(\Gamma\Omega) \le \mu \cdot \varphi_{\mathcal{C}}(\Omega),$$

and therefore

$$\varphi_{\mathcal{C}}(\Omega) = 0. \tag{8}$$

Let us demonstrate now that the set $\Omega$ is equicontinuous. Notice that the relation

$$\mathrm{mod}_C(\Omega) \le \mathrm{mod}_C(\Gamma\Omega)$$

implies that it is sufficient to prove the equicontinuity of the set $\Gamma\Omega$. In turn, it is equivalent to the equicontinuity of each sequence

$$\{g_n\} \subset (I - \Lambda\mathcal{Q})G\mathcal{P}_F(\Omega).$$

For any such sequence $\{g_n\}$, consider sequences $\{x_n\} \subset \Omega$ and $\{f_n\}$, $f_n \in \mathcal{P}_F(x_n)$ such that

$$g_n = (I - \Lambda\mathcal{Q})Gf_n, \qquad n = 1, 2, ...$$

From condition $(F3)$ it follows that the sequence of functions $\{f_n\}$ is integrably bounded. Equality (8) implies that sequence $\{x_n\}$ satisfies the relation

$$\chi(\{x_n(t)\}) = 0, \qquad \forall t \in [0, T]$$

and then, from condition $(F4)$ we obtain that

$$\chi(\{f_n(t)\}) = 0 \qquad \text{a.e. } t \in [0, T],$$

i.e., the sequence $\{f_n\}$is semicompact. From Theorem 18 it follows that the sequence $\{Gf_n\} \subset \widetilde{\mathcal{C}}$ is relatively compact and hence equicontinuous. Applying condition $(H2)$ we obtain that the sequence $\{g_n\}$ is equicontinuous.

From the Arzela–Ascoli theorem (see, e.g., [13]) it follows that the set $\Omega$ is relatively compact w.r.t. the topology of uniform convergence on compact subsets of $(-\infty; 0]$. But then Theorem 15 yields the relative compactness of the set $\Omega$ in the space $\mathcal{C}((-\infty; T]; E_0)$ also. □

**Remark 22.** *Notice that condition $H(4)$ obviously holds in each of the following cases: $(i)$ $k \equiv 0$, i.e., the multimap $F$ is completely continuous in the second argument; $(ii)$ $h \equiv 0$, i.e., the semigroup $U$ is compact. In each of these cases the multioperator $\Gamma$ is completely continuous.*

So, the properties of the multioperator $\Gamma$ open the possibility to apply the topological degree theory for its study. We can formulate the following general principle for the existence of mild solutions of problem (2)-(3).

**Theorem 23.** *Under above conditions, let an open bounded set $\Omega \subset \mathcal{C}((-\infty; T]; E_0)$ does not have mild solutions of problem (4)-(5) on its boundary $\partial\Omega$ and let*

$$deg(i - \Gamma, \overline{\Omega}) \ne 0.$$

*Then the set of mild solutions of problem (2)-(3) is non empty.*

As the example of application of this principle consider he following assertion.

**Theorem 24.** *Under above conditions, let us assume, in addition, that*

(H5) *there exists a sequence of functions* $\omega_n \in L^1_+[0;T]$, $n = 1, 2, \ldots$ *such that:*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^T \omega_n(t)dt = 0;$$

*and*

$$\sup_{\|\varphi\|_{\mathcal{B}} \le n} \|F(t,\varphi)\| \le \omega_n(t) \qquad \text{for a.e. } t \in [0;T],$$

(H6) *the following asymptotic condition holds:*

$$\liminf_{\|x\|_{\mathcal{C}}\to\infty} \frac{\|\mathcal{S}(x)\|_{\mathcal{B}}}{\|x\|_{\mathcal{C}}} = 0.$$

*Then the set of mild solutions to problem (2)-(3) is non empty.*

*Proof.* Let us show that there exists a closed ball $B_r \subset \mathcal{C}((-\infty;T];E_0)$ such that $\Gamma(B_r) \subseteq B_r$.

Supposing the contrary and using the boundedness of the multioperator $\Gamma$, we may find the sequence of integers $q_n \to \infty$ and sequences $\{x_n\}$, $\{z_n\}$ in $\mathcal{C}((-\infty;T];E_0)$ such that $z_n \in \Gamma(x_n)$, $\|x_n\|_{\mathcal{C}} \le q_n$, $\|z_n\|_{\mathcal{C}} > q_n$, and $\|x_n\|_{\mathcal{C}} \to \infty$.
Then we obtain

$$\|z_n\|_{\mathcal{C}} \le \|\Lambda\mathcal{S}x_n\|_{\mathcal{C}} + \|Gf_n\|_{\mathcal{C}} + \|\Lambda\mathcal{Q}Gf_n\|_{\mathcal{C}}$$

$$\le \|\Lambda\|\|\mathcal{S}x_n\|_{\mathcal{B}} + (1 + \|\Lambda\mathcal{Q}\|)\|Gf_n\|_{C([0,T];E)},$$

where $f_n \in \mathcal{P}_F(x_n)$.

Using estimate (1), we obtain

$$\|z_n\|_{\mathcal{C}} \le \|\Lambda\|\|\mathcal{S}x_n\|_{\mathcal{B}} + Ce^{\gamma T}(1 + \|\Lambda\mathcal{Q}\|) \int_0^T \|f_n(s)\|ds$$

yielding

$$1 < \frac{\|z_n\|_{\mathcal{C}}}{q_n} \le \|\Lambda\| \frac{\|\mathcal{S}x_n\|_{\mathcal{B}}}{q_n} + Ce^{wT}(1 + \|\Lambda\mathcal{Q}\|) \frac{1}{q_n} \int_0^T \|f_n(s)\|ds$$

$$\le \|\Lambda\| \frac{\|\mathcal{S}x_n\|_{\mathcal{B}}}{\|x_n\|_{\mathcal{C}}} + Ce^{wT}(1 + \|\Lambda\mathcal{Q}\|) \frac{1}{q_n} \int_0^T \|f_n(s)\|ds,$$

contrary to assumptions $(H5)$ and $(H6)$.

It remains to apply the fixed point theorem for condensing multimaps (see, e.g., Theorem 1.2.70 [3] or Corollary 3.3.1 [12]). □

## 4. Some particular cases

4.1. **Condition ($\widetilde{\mathcal{Q}}$).** It is easy to see that the $(\varphi_{\mathcal{BC}}, \varphi_{\mathcal{C}})$-norm of the operator $r$ admits the following estimate :

$$\|r\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})} \leq R = \max\{1, \sup_{0\leq t\leq T} h(t)\}.$$

So, under condition ($\widetilde{\mathcal{Q}}$) we obtain the following estimate for the $(\varphi_{\mathcal{BC}}, \varphi_{\mathcal{C}})$-norm of the operator $\Lambda$ :

$$\|\Lambda\|^{(\varphi_{\mathcal{BC}},\varphi_{\mathcal{C}})} \leq R\|\widetilde{\mathcal{Q}}^{-1}\|^{(\varphi_{\mathcal{BC}})}.$$

It means that in this case condiion $(H4)$ has the form

$$(H4') \qquad (1 + R\|\widetilde{\mathcal{Q}}^{-1}\|^{(\varphi_{\mathcal{BC}})}b) \sup_{0\leq t\leq T} \int_0^t h(t-s)k(s)ds < 1.$$

4.2. **Cauchy problem.** In this case boundary condition (3) may be written in the following form

$$\mathcal{Q}My = u, \tag{9}$$

or, equivalently,

$$\mathcal{Q}x = u, \tag{10}$$

where $\mathcal{Q}x = x_0$, $u \in \mathcal{BC}$ is a given function. Then, obviously, $\mathcal{S}x \equiv u$, $b = 0$. For each sequence $\{z_n\} \subset \widetilde{\mathcal{C}}$ the sequence $\{\Lambda\mathcal{Q}z_n\}$ is constant and its members equal zero, so condition $(H2)$ is fulfilled. Further, the operator $\widetilde{\mathcal{Q}}$ is identity and condition $(H4)$ takes the following form:

$$(H4'') \qquad \sup_{0\leq t\leq T} \int_0^t h(t-s)k(s)ds < 1.$$

From Theorem 24 we deduce the following result.

**Theorem 25.** *Under conditions $(A)$, $(F1)$, $(F2)$, $(F4)$, $(H3)$, $(H4'')$, and $(H5)$ there exists a mild solution of Cauchy problem (2)– (3).*

4.3. **Periodic problem.** Consider boundary condition

$$\mathcal{Q}My = 0, \tag{11}$$

or, equivalently,

$$\mathcal{Q}x = 0, \tag{12}$$

where $\mathcal{Q}x = x_T - x_0$. Notice that from condition $(\mathcal{B}1)(iii)$ it follows that $\mathcal{Q}$ is a continuous linear operator.

We will assume the following condition:

$(A3)$ the linear operator $U(T) - I$ is invertible on $E_0$.

Taking into account that $\mathcal{S}x \equiv 0$, it is sufficient to construct the operator $\Lambda$ on the subspace $\mathcal{Q}\widetilde{\mathcal{C}} \subset \mathcal{BC}$ proceeding from formula (7). Notice that in our case the subspace $\mathcal{Q}\widetilde{\mathcal{C}}$ consists of continuous functions $\psi : (-\infty, 0] \to E$ vanishing on $(-\infty, -T]$. It is natural enough to suppose that

($\mathcal{Q}\widetilde{\mathcal{C}}$) if a set $\Psi \subset \mathcal{Q}\widetilde{\mathcal{C}}$ is bounded w.r.t. the norm $\|\cdot\|_{\mathcal{B}}$ then it is uniformly bounded.

Now, for a given function $\psi \in \mathcal{Q}\widetilde{\mathcal{C}}$, let us find a function $\xi \in \mathcal{BC}$ such that $\widetilde{\mathcal{Q}}\xi = \psi$, where, as earlier, $\widetilde{\mathcal{Q}}\xi = \mathcal{Q}(r\xi)$. We have

$$(r\xi)_T - (r\xi)_0 = (r\xi)_T - \xi = \psi, \tag{13}$$

implying

$$\xi(0) = (U(T) - I)^{-1}\psi(0),$$

and further, for $\theta \in [-T, 0]$ :

$$\xi(\theta) = U(T+\theta)\,\xi(0) - \psi(\theta) = U(T+\theta)\,(U(T) - I)^{-1}\psi(0) - \psi(\theta). \tag{14}$$

If now $\theta < -T$, then from (13) we obtain

$$(r\xi)_T(\theta) - \xi(\theta) = \xi(T+\theta) - \xi(\theta) = 0,$$

i.e., the function $\xi$ is $T$-periodic on $(-\infty, 0]$ and its values are completely determined by formula (14). Thus we constructed the operator inverse to $\widetilde{\mathcal{Q}}$ on $\mathcal{Q}\widetilde{\mathcal{C}}$.

Further, let a certain set of functions $\Psi \subset \mathcal{Q}\widetilde{\mathcal{C}}$ is bounded w.r.t. $\|\cdot\|_{\mathcal{B}}$. Then, applying property $(\mathcal{Q}\widetilde{\mathcal{C}})$, we see, from formula (14), that the corresponding family of functions $\Xi = \{\xi = \widetilde{\mathcal{Q}}^{-1}\psi : \psi \in \Psi\}$ is uniformly bounded on $(-\infty, 0]$, and therefore, by Theorem 15 (iii) it is bounded in the space $\mathcal{BC}$ also. It means that the operator $\widetilde{\mathcal{Q}}^{-1}$ is continuous on $\mathcal{Q}\widetilde{\mathcal{C}}$.

The operator $\Lambda$ on $\mathcal{Q}\widetilde{\mathcal{C}}$ may be presented in the explicit form:

$$(\Lambda\psi)(t) = \begin{cases} \left[U(T+t)\,(U(T) - I))^{-1}\psi(0) - \psi(t)\right]_T, & t \in [-\infty, 0]; \\ U(t)\,(U(T) - I)^{-1}\psi(0), & t \in [0; T], \end{cases}$$

where by $[\cdot]_T$ we denote the $T$-periodic extension to $[-\infty, 0]$ of a function given on $[-T, 0]$.

It is easy to see that condition $(H1)$ is fulfilled with the constant $b = 1$.

Further, let $\{z_n\} \subset \widetilde{\mathcal{C}}$ be a relatively compact sequence. Then the sequence $\{\psi_n\} \subset \mathcal{Q}\widetilde{\mathcal{C}}$, $\psi_n = \mathcal{Q}z_n = (z_n)_T$, is equicontinuous and $\{\psi_n(0)\}$ is a relatively compact subset of $E_0$. But then, from the construction of the operator $\Lambda$, we see that the sequence $\{\Lambda\psi_n\}$ is also equicontinuous and hence condition $(H2)$ is fulfilled.

Now, notice that $(\varphi_{\mathcal{BC}})$-norm of the operator $\widetilde{\mathcal{Q}}^{-1}$ on $\mathcal{Q}\widetilde{\mathcal{C}}$ may be estimated in the following way:

$$\|\widetilde{\mathcal{Q}}^{-1}\|^{(\varphi_{\mathcal{BC}})} \le \sup_{0\le t\le T} h(t) \cdot \|\,(U(T) - I)^{-1}\,\|^{(\chi)} + 1.$$

Then condition $(H4')$ may be written in the following form:

$$(H4''') \qquad [1 + R(\sup_{0\le t\le T} h(t) \cdot \|(U(T) - I)^{-1}\|^{(\chi)} + 1)] \cdot \sup_{0\le t\le T} \int_0^t h(t-s)k(s)ds < 1.$$

The multioperator $\Gamma$ in the periodic problem has the form

$$\Gamma(x) = (I - \Lambda\mathcal{Q})G\mathcal{P}_F(x).$$

To present it in the explicit form, notice that for $f \in \mathcal{P}_F(x)$ we have

$$(\mathcal{Q}Gf)(t) = \int_0^{T+t} U(T+t-s)\,f(s)ds, \qquad t \in [-T, 0].$$

So, $\Gamma(x)$ consists of all functions $z \in \mathcal{C}((-\infty;T];E_0)$ which, for $f \in \mathcal{P}_F(x)$, have the form

$$z(t) = \begin{cases} \left[ \int\limits_0^{T+t} U\left(T+t-s\right) f(s)ds \right. \\ \left. -U\left(T+t\right)\left(U\left(T\right)-I\right)^{-1} \int\limits_0^{T} U\left(T-s\right) f(s)ds \right]_T, & t \in (-\infty, 0]; \\ \int\limits_0^{t} U\left(t-s\right) f(s)ds - U\left(t\right)\left(U\left(T\right)-I\right)^{-1} \int\limits_0^{T} U\left(T-s\right) f(s)ds, \\ & t \in [0,T] \end{cases}$$

(cfr. [11], [12]).

The application of Theorem 24 yields the following assertion.

**Theorem 26.** *Under conditions* $(A)$, $(A3)$, $(F1)$, $(F2)$, $(F4)$, $(H3)$, $(H4''')$, $(H5)$, *and* $(\widetilde{\mathcal{QC}})$ *periodic problem (2), (11) has a mild solution.*

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