FIXED POINT THEOREMS FOR NONLINEAR NON-SELF MAPPINGS IN HILBERT SPACES AND APPLICATIONS

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Dedicated to Professor Hari M. Srivastava

Abstract. Recently, Kawasaki and Takahashi [8] defined a broad class of nonlinear mappings, called widely more generalized hybrid, in a Hilbert space which contains generalized hybrid mappings [10] and strict pseudo-contractive mappings [2]. They proved fixed point theorems for such mappings. In this paper, we prove fixed point theorems for widely more generalized hybrid non-self mappings in a Hilbert space by using an idea of Hojo, Takahashi and Yao [4], and Kawasaki and Takahashi fixed point theorems [8]. Using these fixed point theorems for non-self mappings, we proved Browder and Petryshyn fixed point theorem [2] for strict pseudo-contractive non-self mappings and Kocourek, Takahashi and Yao fixed point theorem [10] for super hybrid non-self mappings. In particular, we solve a fixed point problem.

1. Introduction

Let $\mathbb{R}$ be the real line and let $[0, \frac{\pi}{2}]$ be a bounded, closed and convex subset of $\mathbb{R}$. Consider a mapping $T : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by

$$Tx = (1 + \frac{1}{2}x)\cos x - \frac{1}{2}x^2$$

for all $x \in [0, \frac{\pi}{2}]$. Such a mapping T has a unique fixed point $z \in [0, \frac{\pi}{2}]$ such that $\cos z = z$. What kind of fixed point theorems can we use to find such a unique fixed point z of T?

Let H be a real Hilbert space and let C be a non-empty subset of H. Kocourek, Takahashi and Yao [10] introduced a class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [12] and hybrid mappings [17]. A mapping $T : C \to H$ is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|x - Ty\|^2 + \|y - Tx\|^2$$

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for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [1] for generalized hybrid mappings; see also Kohsaka and Takahashi [11] and Iemoto and Takahashi [5]. Very recently, Kawasaki and Takahashi [8] introduced a more broad class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping T from C into H is called widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx-Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. In particular, an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [10] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is strict pseudo-contractive in the sense of Browder and Petryshyn [2] if $\alpha = 1, \beta = \gamma = 0, \delta = -1, \varepsilon = \zeta = 0, \eta = -k$, where $0 \leq k < 1$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive in general even if it has a fixed point. In [8], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [1] for such widely more generalized hybrid mappings in a Hilbert space. In particular, they proved directly Browder and Petryshyn fixed point theorem [2] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao fixed point theorem [10] for super hybrid mappings by using their fixed point theorems. However, we can not use Kawasaki and Takahashi fixed point theorems to solve the above problem. For a nice synthesis on metric fixed point theory, see Kirk [9].

In this paper, motivated by such a problem, we prove fixed point theorems for widely more generalized hybrid non-self mappings in a Hilbert space by using an idea of Hojo, Takahashi and Yao [4], and Kawasaki and Takahashi fixed point theorems [8]. Using these fixed point theorems for non-self mappings, we proved Browder and Petryshyn fixed point theorem [2] for strict pseudo-contractive non-self mappings and Kocourek, Takahashi and Yao fixed point theorem [10] for super hybrid non-self mappings. In particular, we solve the above problem by using one of our fixed point theorems.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. From [16], we know the following basic equality: For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore, we know that for $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

Let C be a non-empty, closed and convex subset of H and let T be a mapping from C into H. Then, we denote by $F(T)$ the set of fixed points of T. A mapping

$S : C \to H$ is called super hybrid [10, 20] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}(2.3)\quad &\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\le \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ &\qquad + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2\end{aligned}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-super hybrid mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is (α, β)-generalized hybrid. Thus the class of super hybrid mappings contains generalized hybrid mappings. The following theorem was proved in [20]; see also [10].

Theorem 2.1 ([20]). *Let C be a non-empty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, S is (α, β, γ)-super hybrid if and only if T is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$. In particular, let C be a nonempty, closed and convex subset of H and let α, β and γ be real numbers with $\gamma \geq 0$. If a mapping $S : C \to C$ is (α, β, γ)-super hybrid, then the mapping $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ is an (α, β)-generalized hybrid mapping of C into itself.*

In [10], Kocourek, Takahashi and Yao also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

Theorem 2.2 ([10]). *Let C be a non-empty, bounded, closed and convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping. Then, S has a fixed point in C. In particular, if $S : C \to C$ be an (α, β)-generalized hybrid mapping, then S has a fixed point in C.*

A super hybrid mapping is not quasi-nonexpansive in general even if it has a fixed point. There exists a class of nonlinear mappings in a Hilbert space defined by Kawasaki and Takahashi [7] which covers contractive mappings and generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x{-}Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \le 0\end{aligned}$$

for any $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid. Kawasaki and Takahashi [7] proved the following fixed point theorem.

Theorem 2.3 ([7]). *Let H be a Hilbert space, let C be a non-empt,y closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *and* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$*;*
(2) $\varepsilon + \alpha + \gamma > 0$*, or* $\zeta + \alpha + \beta > 0$.

Then, T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Very recently, Kawasaki and Takahashi [8] also proved the following fixed point theorem which will be used in the proofs of our main theorems in this paper.

Theorem 2.4 ([8])**.** *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\alpha\|Tx-Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then, T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

In particular, we have the following theorem from Theorem 2.4.

Theorem 2.5. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then, T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

3. Fixed Point Theorems for Non-Self Mappings

In this section, using the fixed point theorem (Theorem 2.5), we first prove the following fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.1. *Let C be a non-empty, bounded, closed and convex subset of a Hilbert space H and let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\alpha + \beta + \zeta + \eta \geq 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\alpha + \gamma + \varepsilon + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

Assume that there exists a positive number $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then, T has a fixed point in C. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

Proof. We give the proof for the case of (1). By the assumption, we have that for any $x \in C$, there exist $y \in C$ and t with $0 < t \leq m$ such that $Tx = x + t(y - x)$. From this, we have $Tx = ty + (1 - t)x$ and hence

$$y = \frac{1}{t}Tx + \frac{t-1}{t}x.$$

Define $Ux \in C$ as follows:

$$Ux = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}y = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}\left(\frac{1}{t}Tx + \frac{t-1}{t}x\right) = \frac{1}{m}Tx + \frac{m-1}{m}x.$$

Taking $\lambda > 0$ with $m = 1 + \lambda$, we have that

$$Ux = \frac{1}{1+\lambda}Tx + \frac{\lambda}{1+\lambda}x$$

and hence

$$T = (1+\lambda)U - \lambda I. \tag{3.1}$$

Since $T : C \to H$ is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we have from (3.1) and (2.1) that for any $x, y \in C$,

$$\begin{aligned}
&\alpha\|(1+\lambda)Ux - \lambda x - ((1+\lambda)Uy - \lambda y)\,\|^2 \\
&\quad + \beta\|x - ((1+\lambda)Uy - \lambda y)\,\|^2 + \gamma\|(1+\lambda)Ux - \lambda x - y\|^2 + \delta\|x-y\|^2 \\
&\quad + \varepsilon\|x - ((1+\lambda)Ux - \lambda x)\,\|^2 + \zeta\|(1+\lambda)Uy - \lambda y - y\|^2 \\
&\quad + \eta\|x - ((1+\lambda)Ux - \lambda x) - (y - ((1+\lambda)Uy - \lambda y))\,\|^2 \\
&= \alpha\|(1+\lambda)(Ux - Uy) - \lambda(x-y)\|^2 \\
&\quad + \beta\|(1+\lambda)(x - Uy) - \lambda(x-y)\|^2 + \gamma\|(1+\lambda)(Ux - y) - \lambda(x-y)\|^2 \\
&\quad + \delta\|x-y\|^2 + \varepsilon\|(1+\lambda)(x - Ux)\|^2 + \zeta\|(1+\lambda)(y - Uy)\|^2 \\
&\quad + \eta\|(1+\lambda)(x - Ux) - (1+\lambda)(y - Uy)\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 - \alpha\lambda\|x-y\|^2 + \alpha\lambda(1+\lambda)\|x - y - (Ux - Uy)\|^2 \\
&\quad + \beta(1+\lambda)\|x - Uy\|^2 - \beta\lambda\|x-y\|^2 + \beta\lambda(1+\lambda)\|y - Uy\|^2 \\
&\quad + \gamma(1+\lambda)\|Ux - y\|^2 - \gamma\lambda\|x-y\|^2 + \gamma\lambda(1+\lambda)\|x - Ux\|^2 + \delta\|x-y\|^2 \\
&\quad + \varepsilon(1+\lambda)^2\|x - Ux\|^2 + \zeta(1+\lambda)^2\|y - Uy\|^2 \\
&\quad + \eta(1+\lambda)^2\|x - Ux - (y - Uy)\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 + \beta(1+\lambda)\|x - Uy\|^2 + \gamma(1+\lambda)\|Ux - y\|^2 \\
&\quad + (-\alpha\lambda - \beta\lambda - \gamma\lambda + \delta)\|x-y\|^2 \\
&\quad + (\gamma\lambda + \varepsilon\lambda + \varepsilon)(1+\lambda)\|x - Ux\|^2 + (\beta\lambda + \zeta\lambda + \zeta)(1+\lambda)\|y - Uy\|^2 \\
&\quad + (\alpha\lambda + \eta\lambda + \eta)(1+\lambda)\|x - y - (Ux - Uy)\|^2 \leq 0.
\end{aligned}$$

This implies that U is widely more generalized hybrid. Since $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\alpha + \beta + \zeta + \eta \geq 0$ and $\zeta + \eta \geq 0$, we obtain that

$$\alpha(1+\lambda) + \beta(1+\lambda) + \gamma(1+\lambda) - \alpha\lambda - \beta\lambda - \gamma\lambda + \delta = \alpha + \beta + \gamma + \delta \geq 0,$$

$$\begin{aligned}
\alpha(1+\lambda) + &\gamma(1+\lambda) + (\gamma\lambda + \varepsilon\lambda + \varepsilon)(1+\lambda) + (\alpha\lambda + \eta\lambda + \eta)(1+\lambda) \\
&= (1+\lambda)(\alpha + \gamma + \varepsilon + \eta + \lambda(\gamma + \varepsilon + \alpha + \eta)) \\
&= (1+\lambda)^2(\alpha + \gamma + \varepsilon + \eta) > 0,
\end{aligned}$$

$$\begin{aligned}
(\beta\lambda + \zeta\lambda + \zeta)&(1+\lambda) + (\alpha\lambda + \eta\lambda + \eta)(1+\lambda) \\
&= ((\alpha + \beta + \zeta + \eta)\lambda + \zeta + \eta)\,(1+\lambda) \geq 0.
\end{aligned}$$

By Theorem 2.5, we obtain that $F(U) \neq \emptyset$. Therefore, we have from $F(U) = F(T)$ that $F(T) \neq \emptyset$. Suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of

T. We have that

$$\begin{aligned}
&\alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\
&\qquad + \varepsilon\|p_1 - Tp_1\|^2 + \zeta\|p_2 - Tp_2\|^2 + \eta\|(p_1 - Tp_1) - (p_2 - Tp_2)\|^2 \\
&= (\alpha + \beta + \gamma + \delta)\|p_1 - p_2\|^2 \leq 0
\end{aligned}$$

and hence $p_1 = p_2$. Therefore, a fixed point of T is unique.

Similarly, we can obtain the desired result for the case when $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\alpha + \gamma + \varepsilon + \eta \geq 0$ and $\varepsilon + \eta \geq 0$. This completes the proof. □

The following theorem is a useful extension of Theorem 3.1.

Theorem 3.2. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\alpha + \beta + \zeta + \eta \geq 0$ *and* $[0, 1) \cap \{\lambda \mid (\alpha + \beta)\lambda + \zeta + \eta \geq 0\} \neq \emptyset$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\alpha + \gamma + \varepsilon + \eta \geq 0$ *and* $[0, 1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$.

Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then, T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

Proof. Let $\lambda \in [0, 1) \cap \{\lambda \mid (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$ and define $S = (1 - \lambda)T + \lambda I$. Then S is a mapping from C into H. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned}
&\alpha \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 \\
&\qquad + \beta \left\| x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 + \gamma \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - y \right\|^2 \\
&\qquad + \delta\|x - y\|^2 \\
&\qquad + \varepsilon \left\| x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) \right\|^2 + \zeta \left\| y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right) \right\|^2 \\
&\qquad + \eta \left\| \left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right)\right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right) \right\|^2 \\
&= \alpha \left\| \frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right\|^2 \\
&\qquad + \beta \left\| \frac{1}{1-\lambda}(x - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right\|^2 \\
&\qquad + \gamma \left\| \frac{1}{1-\lambda}(Sx - y) - \frac{\lambda}{1-\lambda}(x - y) \right\|^2 + \delta\|x - y\|^2 \\
&\qquad + \varepsilon \left\| \frac{1}{1-\lambda}(x - Sx) \right\|^2 + \zeta \left\| \frac{1}{1-\lambda}(y - Sy) \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\eta\left\|\frac{1}{1-\lambda}(x-Sx)-\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
=&\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2\\
&+\frac{\gamma}{1-\lambda}\|Sx-y\|^2+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\leq 0.
\end{aligned}
$$

Therefore S is an $\left(\frac{\alpha}{1-\lambda},\frac{\beta}{1-\lambda},\frac{\gamma}{1-\lambda},-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta,\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2},\frac{\zeta+\beta\lambda}{(1-\lambda)^2},\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping. Furthermore, we obtain that

$$
\begin{gathered}
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta=\alpha+\beta+\gamma+\delta\geq 0,\\
\frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda}+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}=\frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2}>0,\\
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}=\frac{\alpha+\beta+\zeta+\eta}{(1-\lambda)^2}\geq 0,\\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}=\frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2}\geq 0.
\end{gathered}
$$

Furthermore, from the assumption, there exists $m>1$ such that for any $x\in C$,

$$
\begin{aligned}
Sx&=(1-\lambda)Tx+\lambda x\\
&=(1-\lambda)\left(x+t(y-x)\right)+\lambda x\\
&=t(1-\lambda)(y-x)+x,
\end{aligned}
$$

where $y\in C$ and $0<t\leq m$. ¿From $0\leq\lambda<1$, we have $0<t(1-\lambda)\leq m$. Putting $s=t(1-\lambda)$, we have that there exists $m>1$ such that for any $x\in C$,

$$Sx=x+s(y-x)$$

for some $y\in C$ and s with $0<s\leq m$. Therefore, we obtain from Theorem 3.1 that $F(S)\neq\emptyset$. Since $F(S)=F(T)$, we obtain that $F(T)\neq\emptyset$.

Next, suppose that $\alpha+\beta+\gamma+\delta>0$. Let p_1 and p_2 be fixed points of T. As in the proof of Theorem 3.1, we have $p_1=p_2$. Therefore a fixed point of T is unique.

In the case of $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\beta+\zeta+\eta>0$, $\alpha+\gamma+\varepsilon+\eta\geq 0$ and $[0,1)\cap\{\lambda\mid(\alpha+\gamma)\lambda+\varepsilon+\eta\geq 0\}\neq\emptyset$, we can obtain the desired result by replacing the variables x and y. □

Remark 1. We can also prove Theorems 3.1 and 3.2 by using the condition

$$-\beta-\delta+\varepsilon+\eta>0,\quad \textit{or}\quad -\gamma-\delta+\varepsilon+\eta>0$$

instead of the condition

$$\alpha+\gamma+\varepsilon+\eta>0,\quad \textit{or}\quad \alpha+\beta+\zeta+\eta>0,$$

respectively. In fact, in the case of the condition $-\beta-\delta+\varepsilon+\eta>0$, we obtain from $\alpha+\beta+\gamma+\delta\geq 0$ that

$$0<-\beta-\delta+\varepsilon+\eta\leq\alpha+\gamma+\varepsilon+\eta.$$

Thus we obtain the desired results by Theorems 3.1 and 3.2. Similary, in the case of $-\gamma-\delta+\varepsilon+\eta > 0$, we can obtain the results by using the case of $\alpha+\beta+\zeta+\eta > 0$.

4. Fixed point theorems for well-known mappings

Using Theorem 3.1, we first show the following fixed point theorem for generalized hybrid non-self mappings in a Hilbert space; see also Kocourek, Takahashi and Yao [10].

Theorem 4.1. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a generalized hybrid mapping from C into H, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Suppose $\alpha - \beta \geq 0$ and assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then, T has a fixed point.

Proof. An (α, β)-generalized hybrid mapping T from C into H is an $(\alpha, 1-\alpha, -\beta, -(1-\beta), 0, 0, 0)$-widely more generalized hybrid mapping. Furthermore, $\alpha + (1-\alpha) - \beta - (1-\beta) = 0$, $\alpha + (1-\alpha) + 0 + 0 = 1 > 0$, $\alpha - \beta + 0 + 0 = \alpha - \beta \geq 0$ and $0 + 0 = 0$, that is, it satisfies the condition (2) in Theorem 3.1. Furthermore, since there exists $m \geq 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$, we obtain the desired result from Theorem 3.1. □

Using Theorem 3.1, we can also show the following fixed point theorem for widely generalized hybrid non-self mappings in a Hilbert space; see Kawasaki and Takahashi [7].

Theorem 4.2. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon > 0$ *and* $\alpha + \beta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ *and* $\alpha + \gamma \geq 0$*.*

Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and $t \in \mathbb{R}$ with $0 < t \leq m$. Then, T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

Proof. Since T is $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid, we obtain that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0$$

for any $x, y \in C$. In the case of $\alpha + \gamma + \varepsilon > 0$, from

$$\varepsilon\|x - Tx\|^2 \leq \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\},$$

we obtain that

$$\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2+\varepsilon\|x-Tx\|^2 \le 0,$$

that is, it is an $(\alpha,\beta,\gamma,\delta,\varepsilon,0,0)$-widely more generalized hybrid mapping. Furthermore, we have that $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\gamma+\varepsilon+0=\alpha+\gamma+\varepsilon > 0$, $\alpha+\beta+0+0=\alpha+\beta\ge 0$ and $0+0=0$, that is, it satisfies the condition (1) in Theorem 3.1. Furthermore, since there exists $m \ge 1$ such that for any $x \in C$,

$$Tx = x + t(y-x)$$

for some $y \in C$ and t with $0 < t \le m$, we obtain the desired result from Theorem 3.1. In the case of $\alpha+\beta+\gamma+\delta\ge 0$, $\alpha+\beta+\zeta > 0$ and $\alpha+\gamma\ge 0$, we can obtain the desired result by replacing the variables x and y. □

We know that an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping with $\alpha=1$, $\beta=\gamma=\varepsilon=\zeta=0$, $\delta=-1$ and $\eta=-k\in(-1,0]$ is a strict pseudo-contractive mapping in the sense of Browder and Petryshyn [2]. We also define the following mapping: $T: C\to H$ is called a generalized strict pseudo-contractive mapping if there exist $r,k\in\mathbb{R}$ with $0\le r\le 1$ and $0\le k<1$ such that

$$\|Tx-Ty\|^2 \le r\|x-y\|^2 + k\|(x-Tx)-(y-Ty)\|^2$$

for any $x,y\in C$. Using Theorem 3.2, we can show the following fixed point theorem for generalized strict pseudo-contractive non-self mappings in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a generalized strict pseudo-contractive mapping from C into H, that is, there exist $r,k\in\mathbb{R}$ with $0\le r\le 1$ and $0\le k<1$ such that*

$$\|Tx-Ty\|^2 \le r\|x-y\|^2 + k\|(x-Tx)-(y-Ty)\|^2$$

for all $x,y\in C$. Assume that there exists $m>1$ such that for any $x\in C$,

$$Tx = x + t(y-x)$$

for some $y\in C$ and $t\in\mathbb{R}$ with $0<t\le m$. Then, T has a fixed point. In particular, if $0\le r<1$, then T has a unique fixed point.

Proof. A generalized strict pseudo-contractive mapping T from C into H is a $(1,0,0,-r,0,0,-k)$-widely more generalized hybrid mapping. Furthermore, $1+0+0+(-r)\ge 0$, $1+0+0+(-k)=1-k>0$, $1+0+0+(-k)=1-k>0$ and $[0,1)\cap\{\lambda \mid (1+0)\lambda+0-k\ge 0\}=[k,1)\ne\emptyset$, that is, it satisfies the condition (1) in Theorem 3.2. Furthermore, since there exists $m\ge 1$ such that for any $x\in C$,

$$Tx = x + t(y-x)$$

for some $y\in C$ and t with $0<t\le m$, we obtain the desired result from Theorem 3.2. In particular, if $0\le r<1$, then $1+0+0+(-r)>0$. We have from Theorem 3.2 that T has a unique fixed point. □

Let us consider the problem in Introduction. A mapping $T:[0,\frac{\pi}{2}]\to\mathbb{R}$ was defined as follows:

$$Tx = (1+\frac{1}{2}x)\cos x - \frac{1}{2}x^2 \tag{4.1}$$

for all $x \in [0, \frac{\pi}{2}]$. We have that

$$Tx = (1 + \frac{1}{2}x)\cos x - \frac{1}{2}x^2$$
$$\iff \frac{1}{1+\frac{1}{2}x}Tx + \frac{\frac{1}{2}x}{1+\frac{1}{2}x}x = \cos x.$$

Thus we have that for any $x \in [0, \frac{\pi}{2}]$,

$$\frac{1+\frac{1}{2}x}{1+\pi}\left(\frac{1}{1+\frac{1}{2}x}Tx + \frac{\frac{1}{2}x}{1+\frac{1}{2}x}\right) + \left(1 - \frac{1+\frac{1}{2}x}{1+\pi}\right)x$$
$$= \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \left(1 - \frac{1+\frac{1}{2}x}{1+\pi}\right)x$$

and hence

$$\frac{1}{1+\pi}Tx + \frac{\pi}{1+\pi}x = \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi-\frac{1}{2}x}{1+\pi}x.$$

Using this, we also have from (2.1) that for any $x, y \in [0, \frac{\pi}{2}]$,

$$\left|\frac{1}{1+\pi}Tx + \frac{\pi}{1+\pi}x - \left(\frac{1}{1+\pi}Ty + \frac{\pi}{1+\pi}y\right)\right|^2$$
$$= \left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi-\frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi-\frac{1}{2}y}{1+\pi}y\right)\right|^2$$

and hence

$$\frac{1}{1+\pi}|Tx - Ty|^2 + \frac{\pi}{1+\pi}|x-y|^2 - \frac{\pi}{(1+\pi)^2}|x - y - (Tx - Ty)|^2 \tag{4.2}$$
$$= \left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi-\frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi-\frac{1}{2}y}{1+\pi}y\right)\right|^2.$$

Define a function $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ as follows:

$$f(x) = \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi-\frac{1}{2}x}{1+\pi}x$$

for all $x \in [0, \frac{\pi}{2}]$. Then we have

$$f^{'}(x) = \frac{\frac{1}{2}}{1+\pi}\cos x - \frac{1+\frac{1}{2}x}{1+\pi}\sin x + \frac{\pi}{1+\pi} - \frac{x}{1+\pi}$$

and

$$f^{''}(x) = -\frac{1}{1+\pi}\sin x - \frac{1+\frac{1}{2}x}{1+\pi}\cos x - \frac{1}{1+\pi}.$$

Since

$$f^{'}(0) = \frac{\frac{1}{2}+\pi}{1+\pi}, \qquad f^{'}(\frac{\pi}{2}) = \frac{-1+\frac{1}{4}\pi}{1+\pi}$$

and $f^{''}(x) < 0$ for all $x \in [0, \frac{\pi}{2}]$, we have from the mean value theorem that there exists a positive number r with $0 < r < 1$ such that

$$\left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi-\frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi-\frac{1}{2}y}{1+\pi}y\right)\right|^2 \leq r|x-y|^2$$

for all $x, y \in [0, \frac{\pi}{2}]$. Therefore, we have from (4.2) that

$$\frac{1}{1+\pi}|Tx - Ty\|^2 + \frac{\pi}{1+\pi}|x-y|^2 \le r|x-y|^2 + \frac{\pi}{(1+\pi)^2}|x-y-(Tx-Ty)|^2$$

for all $x, y \in [0, \frac{\pi}{2}]$. Furthermore, we have from (4.1) that

$$Tx = (1 + \frac{1}{2}x)(\cos x - x) + x$$

for all $x \in [0, \frac{\pi}{2}]$. Take $m = 1+\pi$ and let $t = 1 + \frac{1}{2}x$ and $y = \cos x$ for all $x \in [0, \frac{\pi}{2}]$. Then we have that

$$Tx = t(y-x) + x, \ y = \cos x \in [0, \frac{\pi}{2}] \text{ and } 0 < \mathrm{t} = 1 + \frac{1}{2}\mathrm{x} \le 1 + \pi.$$

Using Theorem 3.2, we have that T has a unique fixed point $z \in [0, \frac{\pi}{2}]$. We also know that $z = Tz$ is equivalent to $\cos z = z$. In fact,

$$\begin{aligned} z = Tz \iff & z = (1 + \frac{1}{2}z)(\cos z - z) + z \\ \iff & 0 = (1 + \frac{1}{2}z)(\cos z - z) \\ \iff & 0 = \cos z - z. \end{aligned}$$

Using Theorem 3.2, we can also show the following fixed point theorem for super hybrid non-self mappings in a Hilbert space; see [10].

Theorem 4.4. *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a super hybrid mapping from C into H, that is, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that*

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1 - \alpha + \gamma)\|x - Ty\|^2 \\ &\quad \le (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ &\qquad + (\alpha - \beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2 \end{aligned}$$

for all $x, y \in C$. Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \le m$. Suppose that $\alpha - \beta \ge 0$ or $\gamma \ge 0$. Then, T has a fixed point.

Proof. An (α, β, γ)-super hybrid mapping T from C into H is an $(\alpha, 1-\alpha+\gamma, -\beta-(\beta-\alpha)\gamma, -1+\beta+(\beta-\alpha-1)\gamma, -(\alpha-\beta)\gamma, -\gamma, 0)$-widely more generalized hybrid mapping. Furthermore, $\alpha+(1-\alpha+\gamma)+(-\beta-(\beta-\alpha)\gamma)+(-1+\beta+(\beta-\alpha-1)\gamma) = 0$, $\alpha+(1-\alpha+\gamma)+(-\gamma)+0 = 1 > 0$ and $\alpha-\beta-(\beta-\alpha)\gamma-(\alpha-\beta)\gamma+0 = \alpha-\beta \ge 0$, that is, it satisfies the conditions $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\beta+\zeta+\eta > 0$ and $\alpha+\gamma+\varepsilon+\eta \ge 0$ in (2) of Theorem 3.2. Moreover, we have that

$$\begin{aligned} &[0, 1) \cap \{\lambda \mid (\alpha + (-\beta - (\beta - \alpha)\gamma))\lambda + (-(\alpha - \beta)\gamma) + 0 \ge 0\} \\ &\quad = [0, 1) \cap \{\lambda \mid (\alpha - \beta)((1 + \gamma)\lambda - \gamma) \ge 0\}. \end{aligned}$$

If $\alpha - \beta > 0$, then

$$[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda - \gamma) \geq 0\} = [0,1) \cap \{\lambda \mid (1+\gamma)\lambda - \gamma \geq 0\}$$
$$= \begin{cases} [0,1) & \text{if } \gamma < 0, \\ \left[\frac{\gamma}{1+\gamma}, 1\right) & \text{if } \gamma \geq 0, \end{cases}$$
$$\neq \emptyset,$$

that is, it satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. If $\alpha - \beta = 0$, then

$$[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda - \gamma) \geq 0\} = [0,1) \neq \emptyset,$$

that is, it satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. If $\alpha - \beta < 0$ and $\gamma \geq 0$, then

$$[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda - \gamma) \geq 0\}$$
$$= [0,1) \cap \{\lambda \mid (1+\gamma)\lambda - \gamma \leq 0\}$$
$$= \left[0, \frac{\gamma}{1+\gamma}\right] \neq \emptyset,$$

that is, it again satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. Then we obtain the desired result from Theorem 3.2. Similarly, we obtain the desired result from Theorem 3.2 in the case of (1). □

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