Implicit Predictor-Corrector Iteration Process for Finitely Many Asymptotically (Quasi-)Nonexpansive Mappings

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Abstract. In this paper, we study an implicit predictor-corrector iteration process for finitely many asymptotically quasi-nonexpansive self-mappings on a nonempty closed convex subset of a Banach space E. We derive a necessary and sufficient condition for the strong convergence of this iteration process to a common fixed point of these mappings. In the case E is a uniformly convex Banach space and the mappings are asymptotically nonexpansive, we verify the weak (resp. strong) convergence of this iteration process to a common fixed point of these mappings if Opial's condition is satisfied (resp. one of these mappings is semi-compact). Our results improve and extend earlier and recent ones in the literature.

Key Words. Asymptotically nonexpansive mappings, Implicit predictor-corrector iteration processes, Common fixed points, Opial's condition, Semi-compactness, Demi-closed principle.

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1 Introduction and Preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|$, let C be a nonempty subset of E, and let $T : C \to C$. The set $F(T) = \{x \in C : Tx = x\}$ consists of all fixed points of T.

Definition 1.1. T is said to be

(1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(2) *asymptotically nonexpansive* [4] if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \ n \geq 1;$$

(3) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|, \quad \forall x \in C, \ p \in F(T), \ n \geq 1;$$

(4) *semi-compact* [2] if for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a strongly convergent subsequence of $\{x_n\}$.

The class of asymptotically nonexpansive mappings, as a natural extension of that of nonexpansive mappings, was introduced by Goebel and Kirk [4] in 1972. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Furthermore, the study of iterative construction for fixed points of asymptotically nonexpansive mappings began in 1978. Bose [1] first proved that if the uniformly convex Banach space E satisfies Opial's condition [6] then $\{T^n x\}$ converges weakly to a fixed point of T, provided T is asymptotically regular at x, i.e., $\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0$. A Banach space E is said to satisfy *Opial's condition* [6] if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, one has

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \text{for all } y \in E,\ y \neq x.$$

It is well known that every Hilbert space satisfies Opial's condition (see, for example, [6]).

In 2001, Xu and Ori [9] first introduced an implicit iteration process for N nonexpansive mappings in a Hilbert space and proved the following weak convergence theorem.

Theorem 1.2 ([9]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings on C such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined* implicity *by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{n(\mathrm{mod}N)} x_n, \quad n \geq 1,$$

converges weakly to a common fixed point of mappings $\{T_j\}_{j=1}^N$.

Later, Sun [8] introduced and studied another implicit iteration process

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{n(\mathrm{mod}N)}^{l_n+1} x_n, \quad n \geq 1,$$

for N asymptotically quasi-nonexpansive self-mappings $\{T_j\}_{j=1}^N$ on a nonempty bounded closed convex subset C of a Banach space E, where $\{\alpha_n\}$ is a sequence in $(0,1)$, x_0 is an initial point in C, and $n = l_n N + n(\mathrm{mod}N)$. Moreover, he proved that the sequence $\{x_n\}$ defined by his iteration process converges strongly to a common fixed point of $\{T_j\}_{j=1}^N$ under suitable conditions.

At the same time, Zhou and Chang [10] introduced and studied the following implicit iteration process

$$x_n = \alpha_n x_{n-1} + \beta_n T^n_{n(\mathrm{mod}N)} x_n + \gamma_n u_n, \quad n \geq 1,$$

for N asymptotically nonexpansive self-mappings $\{T_j\}_{j=1}^N$ on a nonempty closed convex subset C of a Banach space E, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, x_0 is an initial point in C, and $\{u_n\}$ is a bounded sequence in C. Moreover, they proved that the sequence $\{x_n\}$ defined by their iteration process converges weakly to a common fixed point of $\{T_j\}_{j=1}^N$ under suitable conditions.

As indicated in [10], if $T_1, T_2, ..., T_N : C \to C$ are N asymptotically nonexpansive mappings, then there exists a sequence, called *common Lipschitz constants*, $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $i = 1, 2, ..., N$,

$$\|T_i^n x - T_i^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \ n \geq 1.$$

A similar situation occurs when $T_1, T_2, \ldots, T_N$ are asymptotically quasi-nonexpansive. By convention, we write $T_n := T_{n(\mathrm{mod}N)}$, for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, ..., N\}$. In other words, if $n = l_n N + q$ for some unique integers $l_n \geq 0$ and $1 \leq q \leq N$, then we set $T_n = T_q$.

In this paper, we introduce the following implicit predictor-corrector iteration process with an auxiliary finite family of asymptotically quasi-nonexpansive self-mappings on C.

Definition 1.3 (Basic set up). Let C be a nonempty closed convex subset of a Banach space E, and $\{T_1, T_2, ..., T_N\}$ and $\{\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_{\hat{N}}\}$ be two families of asymptotically quasi-nonexpansive mappings from C into C with common Lipschitz constants $\{k_n\}$ and $\{\hat{k}_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ and $\sum_{n=1}^{\infty}(\hat{k}_n - 1) < +\infty$, respectively. Let $\{x_n\}$ be an iterative sequence in C generated from an arbitrary $x_0 \in C$ by three steps:

Auxiliary step. With x_{n-1} $(n \geq 1)$ established, y_n is computed implicitly by

$$y_n = \hat{\alpha}_n x_{n-1} + \hat{\beta}_n \hat{T}_n^{\hat{l}_n} y_n + \hat{\gamma}_n \hat{u}_n; \tag{1.1a}$$

Predictor step. With y_n obtained in the auxiliary step, z_n is computed implicitly by

$$z_n = \bar{\alpha}_n y_n + \bar{\beta}_n T_n^{l_n} z_n + \bar{\gamma}_n \bar{u}_n; \tag{1.1b}$$

Corrector step. With z_n obtained in the predictor step, x_n is computed explicitly by

$$x_n = \alpha_n y_n + \beta_n T_n^{l_n} z_n + \gamma_n u_n, \tag{1.1c}$$

Here, $T_n := T_{n(\text{mod}N)}$ and $\hat{T}_n := \hat{T}_{n(\text{mod}\hat{N})}$ for $n = 1, 2, \ldots$. On the other hand, $\{u_n\}_{n=1}^{\infty}$, $\{\hat{u}_n\}_{n=1}^{\infty}$, $\{\bar{u}_n\}_{n=1}^{\infty}$ are three bounded sequences in C; and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\hat{\alpha}_n\}_{n=1}^{\infty}$, $\{\bar{\alpha}_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\hat{\beta}_n\}_{n=1}^{\infty}$, $\{\bar{\beta}_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\hat{\gamma}_n\}_{n=1}^{\infty}$, $\{\bar{\gamma}_n\}_{n=1}^{\infty}$ are nine real sequences in $[0, 1]$ such that

$$\begin{cases} \alpha_n + \beta_n + \gamma_n = 1 \ \ (\forall n \geq 1), & \sum_{n=1}^{\infty} \gamma_n < +\infty, \\ \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1 \ \ (\forall n \geq 1), & \sum_{n=1}^{\infty} \hat{\gamma}_n < +\infty, \\ \bar{\alpha}_n + \bar{\beta}_n + \bar{\gamma}_n = 1 \ \ (\forall n \geq 1), & \sum_{n=1}^{\infty} \bar{\gamma}_n < +\infty, \\ 0 < \hat{\beta}_n, \bar{\beta}_n \leq c < K^{-1} \ \ (\forall n \geq 1), & K = \max\{\sup_{n\geq 1} k_n, \sup_{n\geq 1} \hat{k}_n\} \geq 1. \end{cases} \tag{1.2}$$

Remark 1.4. Since $0 < \hat{\beta}_n, \bar{\beta}_n \leq c < K^{-1}$, it is clear that the mappings $y \mapsto \hat{\alpha}_n x_{n-1} + \hat{\beta}_n \hat{T}_n^{\hat{l}_n} y + \hat{\gamma}_n \hat{u}_n$ and $z \mapsto \bar{\alpha}_n y_n + \bar{\beta}_n T_n^{l_n} z + \bar{\gamma}_n \bar{u}_n$ are two contractions from the nonempty closed convex set C into itself. Thus, by the Banach Contraction Principle there exist the unique points $y_n, z_n \in C$ such that (1.1a) and (1.1b) hold, respectively. Therefore, the sequence $\{x_n\}$ is well defined.

Our aim is to consider and study the strong and weak convergence of the above implicit predictor-corrector iteration process. To this end, we need the following lemmas.

Lemma 1.5. *Let* $\{b_n\}, \{\bar{b}_n\}, \{\hat{b}_n\}$ *be three nonnegative real sequences with finite sums. Then* $\sum_{n=1}^{\infty} \lambda_n < +\infty$*, where* $\lambda_n = (1 + b_n)(1 + \bar{b}_n)(1 + \hat{b}_n) - 1$ *for each* ≥ 1.

Lemma 1.6 ([10]). *Let* $\{a_n\}, \{\lambda_n\}, \{\mu_n\}$ *be three nonnegative real sequences such that* $\sum_{n=1}^{\infty} \lambda_n < +\infty$*,* $\sum_{n=1}^{\infty} \mu_n < +\infty$*, and*

$$a_{n+1} \leq (1 + \lambda_n)a_n + \mu_n, \quad \forall n \geq 1.$$

Then $\lim_{n\to\infty} a_n$ *exists.*

Lemma 1.7 ([7]). *Let* E *be a uniformly convex Banach space,* $\{t_n\} \subset [b, c] \subset (0, 1)$*, and* $\{x_n\}, \{y_n\} \subset E$*. If* $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d < +\infty$*,* $\limsup_{n\to\infty} \|x_n\| \leq d$*, and* $\limsup_{n\to\infty} \|y_n\| \leq d$*, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.8 (Demi-closed principle [3]). *Let* E *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* E*, and* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $F(T) \neq \emptyset$*. Then* $I-T$ *is demiclosed at zero, that is, for any sequence* $\{x_n\} \subset C$*,*

$$\left.\begin{array}{r} x_n \to q \in C \text{ weakly} \\ (I - T)x_n \to 0 \text{ strongly} \end{array}\right\} \quad \Longrightarrow \quad (I - T)q = 0.$$

2 Main Results

Lemma 2.1. *Let C be a nonempty closed convex subset of a Banach space E, and $\{T_i\}_{i=1}^N$ and $\{\hat{T}_j\}_{j=1}^{\hat{N}}$ be two finite families of asymptotically quasi-nonexpansive self-mappings on C such that $\bigcap_{i=1}^N F(T_i) \cap \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j) \neq \emptyset$. If $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are the iterative sequences defined by (1.1a), (1.1b) and (1.1c), then for each $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j)$, there hold*

$$\lim_{n\to\infty} \|x_n - p\| = d, \quad \limsup_{n\to\infty} \|y_n - p\| \leq d, \quad \text{and} \quad \limsup_{n\to\infty} \|z_n - p\| \leq d.$$

Proof. Since $\{u_n\}_{n=1}^\infty, \{\hat{u}_n\}_{n=1}^\infty, \{\bar{u}_n\}_{n=1}^\infty$ are three bounded sequences in C, for any given $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j)$ we have

$$M := \max\{\sup_{n\geq 1} \|u_n - p\|, \sup_{n\geq 1} \|\hat{u}_n - p\|, \sup_{n\geq 1} \|\bar{u}_n - p\|\} < +\infty.$$

Note that $1 - \bar{\beta}_n k_{l_n} \geq 1 - cK > 0$ and $1 - \hat{\beta}_n \hat{k}_{\hat{l}_n} \geq 1 - cK > 0$. Put

$$L = \frac{1}{1-cK}, \quad b_n = \beta_n(k_{l_n} - 1), \quad \bar{b}_n = \frac{1-\bar{\beta}_n}{1-\bar{\beta}_n k_{l_n}} - 1, \quad \text{and} \quad \hat{b}_n = \frac{1-\hat{\beta}_n}{1-\hat{\beta}_n \hat{k}_{\hat{l}_n}} - 1.$$

Then we have

$$\begin{cases} 0 \leq b_n = \beta_n(k_{l_n} - 1) \leq k_{l_n} - 1, \quad \text{and} \quad 1 + b_n \leq K, \\ 0 \leq \bar{b}_n = \frac{\bar{\beta}_n(k_{l_n}-1)}{1-\bar{\beta}_n k_{l_n}} \leq L(k_{l_n} - 1), \quad \text{and} \quad 1 + \bar{b}_n \leq L, \\ 0 \leq \hat{b}_n = \frac{\hat{\beta}_n(\hat{k}_{\hat{l}_n}-1)}{1-\hat{\beta}_n \hat{k}_{\hat{l}_n}} \leq L(\hat{k}_{\hat{l}_n} - 1), \quad \text{and} \quad 1 + \hat{b}_n \leq L. \end{cases} \tag{2.1}$$

Observe that

$$\begin{aligned} \|y_n - p\| &= \|\hat{\alpha}_n(x_{n-1} - p) + \hat{\beta}_n(\hat{T}_n^{\hat{l}_n} y_n - p) + \hat{\gamma}_n(\hat{u}_n - p)\| \\ &\leq \hat{\alpha}_n\|x_{n-1} - p\| + \hat{\beta}_n \hat{k}_{\hat{l}_n}\|y_n - p\| + \hat{\gamma}_n\|\hat{u}_n - p\|. \end{aligned}$$

It follows

$$\begin{aligned} \|y_n - p\| &\leq \tfrac{\hat{\alpha}_n}{1-\hat{\beta}_n \hat{k}_{\hat{l}_n}}\|x_{n-1} - p\| + \tfrac{\hat{\gamma}_n}{1-\hat{\beta}_n \hat{k}_{\hat{l}_n}}\|\hat{u}_n - p\| \\ &\leq \tfrac{1-\hat{\beta}_n}{1-\hat{\beta}_n \hat{k}_{\hat{l}_n}}\|x_{n-1} - p\| + LM\hat{\gamma}_n \\ &= (1 + \hat{b}_n)\|x_{n-1} - p\| + LM\hat{\gamma}_n. \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}\|z_n - p\| &= \|\bar\alpha_n(y_n - p) + \bar\beta_n(T_n^{l_n} z_n - p) + \bar\gamma_n(\bar u_n - p)\| \\ &\le \bar\alpha_n\|y_n - p\| + \bar\beta_n k_{l_n}\|z_n - p\| + \bar\gamma_n\|\bar u_n - p\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\|z_n - p\| &\le \tfrac{\bar\alpha_n}{1-\bar\beta_n k_{l_n}}\|y_n - p\| + \tfrac{\bar\gamma_n}{1-\bar\beta_n k_{l_n}}\|\bar u_n - p\| \\ &\le \tfrac{1-\bar\beta_n}{1-\bar\beta_n k_{l_n}}\|y_n - p\| + LM\bar\gamma_n \\ &= (1+\bar b_n)\|y_n - p\| + LM\bar\gamma_n.\end{aligned} \tag{2.3}$$

Therefore,

$$\begin{aligned}\|x_n - p\| &= \|\alpha_n(y_n - p) + \beta_n(T_n^{l_n} z_n - p) + \gamma_n(u_n - p)\| \\ &\le \alpha_n\|y_n - p\| + \beta_n k_{l_n}\|z_n - p\| + \gamma_n\|u_n - p\| \\ &\le (1-\beta_n)\|y_n - p\| + \beta_n k_{l_n}[(1+\bar b_n)\|y_n - p\| + LM\bar\gamma_n] + \gamma_n M \\ &\le (1+\beta_n(k_{l_n}-1))(1+\bar b_n)\|y_n - p\| + M[KL\bar\gamma_n + \gamma_n] \\ &\le (1+b_n)(1+\bar b_n)\|y_n - p\| + KLM[\bar\gamma_n + \gamma_n] \\ &\le (1+b_n)(1+\bar b_n)[(1+\hat b_n)\|x_{n-1} - p\| + LM\hat\gamma_n] + KLM[\bar\gamma_n + \gamma_n] \\ &\le (1+b_n)(1+\bar b_n)(1+\hat b_n)\|x_{n-1} - p\| + KL^2M\hat\gamma_n + KLM[\bar\gamma_n + \gamma_n] \\ &\le (1+b_n)(1+\bar b_n)(1+\hat b_n)\|x_{n-1} - p\| + KL^2M[\gamma_n + \bar\gamma_n + \hat\gamma_n] \\ &= (1+\lambda_n)\|x_{n-1} - p\| + \mu_n,\end{aligned} \tag{2.4}$$

where $\lambda_n = (1+b_n)(1+\bar b_n)(1+\hat b_n) - 1$, and $\mu_n = KL^2M[\gamma_n + \bar\gamma_n + \hat\gamma_n]$.

Since $\sum_{n=1}^{\infty}(k_{l_n} - 1) < +\infty$ and $\sum_{n=1}^{\infty}(\hat k_{\hat l_n} - 1) < +\infty$, it follows from (2.1) that $\sum_{n=1}^{\infty} b_n < +\infty$, $\sum_{n=1}^{\infty} \bar b_n < +\infty$, and $\sum_{n=1}^{\infty} \hat b_n < +\infty$. Hence, we derive $\sum_{n=1}^{\infty} \lambda_n < +\infty$ by Lemma 1.5. Note that $\sum_{n=1}^{\infty} \gamma_n < +\infty$, $\sum_{n=1}^{\infty} \bar\gamma_n < +\infty$, and $\sum_{n=1}^{\infty} \hat\gamma_n < +\infty$. This provides $\sum_{n=1}^{\infty} \mu_n < +\infty$. By Lemma 1.6, $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = d$.

Since $\lim_{n\to\infty} \hat b_n = \lim_{n\to\infty} \hat\gamma_n = 0$, from (2.2) we obtain

$$\limsup_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}(1+\hat b_n)\|x_{n-1} - p\| + LM\limsup_{n\to\infty}\hat\gamma_n \le d.$$

Further, since $\lim_{n\to\infty} \bar b_n = \lim_{n\to\infty} \bar\gamma_n = 0$, from (2.3) we obtain

$$\limsup_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}(1+\bar b_n)\|y_n - p\| + LM\limsup_{n\to\infty}\bar\gamma_n \le d.$$

□

Theorem 2.2. *Let C be a nonempty closed convex subset of a Banach space E. Let $\{T_i\}_{i=1}^N$ and $\{\hat{T}_j\}_{j=1}^{\hat{N}}$ be two finite families of asymptotically quasi-nonexpansive self-mappings on C such that $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence defined by (1.1a), (1.1b) and (1.1c). Then $\{x_n\}$ converges strongly to an element of F if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Proof. The necessity is obvious. For the sufficiency, we assume $\liminf_{n\to\infty} d(x_n, F) = 0$. Let p be any given element in F. Then from (2.4) we obtain

$$\|x_n - p\| \le (1 + \lambda_n)\|x_{n-1} - p\| + \mu_n, \tag{2.5}$$

where $\sum_{n=1}^{\infty} \lambda_n < +\infty$ and $\sum_{n=1}^{\infty} \mu_n < +\infty$. Taking the infimum over all $p \in F$, we get

$$d(x_n, F) \le (1 + \lambda_n)d(x_{n-1}, F) + \mu_n.$$

Hence, $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Put $\delta_n = \lambda_n\|x_{n-1} - p\| + \mu_n$. Then $\sum_{n=1}^{\infty} \delta_n < +\infty$, and (2.5) can be rewritten as

$$\|x_n - p\| \le \|x_{n-1} - p\| + \delta_n.$$

For arbitrary $\varepsilon > 0$, choose N_0 such that $d(x_{N_0}, F) < \varepsilon/4$ and $\sum_{j=N_0}^{\infty} \delta_j < \varepsilon/4$. Consequently, for all $n, m \ge N_0$ we have

$$\begin{aligned} \|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ &\le \|x_{N_0} - p\| + \textstyle\sum_{j=N_0+1}^{n} \delta_j + \|x_{N_0} - p\| + \sum_{j=N_0+1}^{m} \delta_j \\ &\le 2\|x_{N_0} - p\| + 2\textstyle\sum_{j=N_0}^{\infty} \delta_j. \end{aligned}$$

Taking the infimum over all $p \in F$, we obtain

$$\|x_n - x_m\| \le 2d(x_{N_0}, F) + 2\sum_{j=N_0}^{\infty} \delta_j \le \frac{2\varepsilon}{4} + \frac{2\varepsilon}{4} = \varepsilon.$$

This shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Let $\lim_{n\to\infty} x_n = u$. It is easy to verify that F is closed. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we must have that $u \in F$. $\square$

As a consequence of Lemma 2.1, the iterated sequence $\{x_n\}$ is bounded. If the underlying space E is reflexive then we can expect that its weak cluster points provide common fixed points of $T_1, T_2, \ldots, T_N$. This leads to the following

Theorem 2.3. *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $\{T_i\}_{i=1}^{N}$ (resp. $\{\hat{T}_j\}_{j=1}^{\hat{N}}$) be a finite family of asymptotically nonexpansive (resp. asymptotically quasi-nonexpansive) self-mappings on C such that $\bigcap_{j=1}^{\hat{N}} F(\hat{T}_j) \cap \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\lim_{n\to\infty} \hat{\beta}_n = 0$ and $\{\beta_n\}_{n=1}^{\infty} \subset [b, c] \subset (0, K^{-1})$, where K is as in (1.2). Then every weak cluster point of the bounded iterative sequence $\{x_n\}$ defined by (1.1a), (1.1b) and (1.1c) belongs to $\bigcap_{i=1}^{N} F(T_i)$.*

Proof. Let $p \in \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j) \cap \bigcap_{i=1}^{N} F(T_i)$. By Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_n - p\| = d, \quad \limsup_{n\to\infty} \|y_n - p\| \le d, \quad \text{and} \quad \limsup_{n\to\infty} \|z_n - p\| \le d.$$

Obviously, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded sequences in C.

Observe that

$$\|x_n - p\| = \|(1-\beta_n)[y_n - p + \gamma_n(u_n - y_n)] + \beta_n[T_n^{l_n} z_n - p + \gamma_n(u_n - y_n)]\| \to d,$$

as $n \to \infty$. Since $\lim_{n\to\infty} \gamma_n = 0$ and $\{u_n\}$ is bounded, we have

$$\limsup_{n\to\infty} \|y_n - p + \gamma_n(u_n - y_n)\| \le \limsup_{n\to\infty}[\|y_n - p\| + \gamma_n\|u_n - y_n\|] \le d,$$

and

$$\limsup_{n\to\infty} \|T_n^{l_n} z_n - p + \gamma_n(u_n - y_n)\| \le \limsup_{n\to\infty}[k_{l_n}\|z_n - p\| + \gamma_n\|u_n - y_n\|] \le d.$$

It follows from Lemma 1.7 that

$$\lim_{n\to\infty} \|T_n^{l_n} z_n - y_n\| = 0.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \|z_n - y_n\| &= \lim_{n\to\infty} \|\bar{\alpha}_n y_n + \bar{\beta}_n T_n^{l_n} z_n + \bar{\gamma}_n \bar{u}_n - y_n\| \\ &= \lim_{n\to\infty} \|\bar{\beta}_n (T_n^{l_n} z_n - y_n) + \bar{\gamma}_n(\bar{u}_n - y_n)\| = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - y_n\| &= \lim_{n\to\infty} \|\alpha_n y_n + \beta_n T_n^{l_n} z_n + \gamma_n u_n - y_n\| \\ &= \lim_{n\to\infty} \|\beta_n (T_n^{l_n} z_n - y_n) + \gamma_n(u_n - y_n)\| = 0.\end{aligned}$$

Moreover,

$$\begin{aligned}\|y_n - x_{n-1}\| &= \|\hat{\alpha}_n x_{n-1} + \hat{\beta}_n \hat{T}_n^{\hat{l}_n} y_n + \hat{\gamma}_n \hat{u}_n - x_{n-1}\| \\ &= \|\hat{\beta}_n (\hat{T}_n^{\hat{l}_n} y_n - x_{n-1}) + \hat{\gamma}_n(\hat{u}_n - x_{n-1})\| \\ &\le \hat{\beta}_n \|\hat{T}_n^{\hat{l}_n} y_n - x_{n-1}\| + \hat{\gamma}_n \|\hat{u}_n - x_{n-1}\| \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

since $\lim_{n\to\infty} \hat{\beta}_n = \lim_{n\to\infty} \hat{\gamma}_n = 0$. As a result, we have

$$\|x_n - x_{n-1}\| \le \|x_n - y_n\| + \|y_n - x_{n-1}\| \to 0, \quad \text{as } n \to \infty.$$

It forces

$$\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0, \quad \text{for each } i = 1, 2, ..., N.$$

On the other hand, we have

$$\begin{aligned}\|x_n - T_n^{l_n} x_n\| &\le \|x_n - y_n\| + \|y_n - T_n^{l_n} z_n\| + \|T_n^{l_n} z_n - T_n^{l_n} x_n\| \\ &\le \|x_n - y_n\| + \|y_n - T_n^{l_n} z_n\| + k_{l_n}\|z_n - x_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

As $n = l_n N + n(\text{mod} N)$ for $n > N$, we get

$$n - N = (l_n - 1)N + n(\text{mod} N),$$

and hence $l_{n-N} = l_n - 1$. Thus, we have

$$T_n^{l_n - 1} = T_{n-N}^{l_{n-N}}.$$

Consequently, we derive

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_n^{l_n} x_n\| + \|T_n^{l_n} x_n - T_n x_n\| \\ &\leq \|x_n - T_n^{l_n} x_n\| + K\|T_n^{l_n - 1} x_n - x_n\| \\ &= \|x_n - T_n^{l_n} x_n\| + K\|T_{n-N}^{l_{n-N}} x_n - x_n\| \\ &\leq \|x_n - T_n^{l_n} x_n\| + K[\|T_{n-N}^{l_{n-N}} x_n - T_{n-N}^{l_{n-N}} x_{n-N}\| \\ &\quad + \|T_{n-N}^{l_{n-N}} x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|] \\ &\leq \|x_n - T_n^{l_n} x_n\| + K[(1+K)\|x_{n-N} - x_n\| \\ &\quad + \|T_{n-N}^{l_{n-N}} x_{n-N} - x_{n-N}\|] \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that for each $j = 1, 2, ..., N$,

$$\begin{aligned} \|x_n - T_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\leq (1+K)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.6}$$

Note that the closedness and convexity of C imply the weak closedness of C. Let $\tilde{x} \in C$ be any weak cluster point of the bounded sequence $\{x_n\}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \to \tilde{x}$ weakly (see, e.g., [5, p. 313]). Since the pool of mappings $\{T_i : 1 \leq i \leq N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that for some integer $l \in \{1, 2, ..., N\}$, $T_{n_i} = T_l$ for all $i \geq 1$. Then it follows from (2.6) that for each $j = 1, 2, ..., N$,

$$x_{n_i} - T_{l+j} x_{n_i} \to 0, \quad \text{as } i \to \infty,$$

that is, for each $j = 1, 2, ..., N$,

$$x_{n_i} - T_j x_{n_i} \to 0, \quad \text{as } i \to \infty, \tag{2.7}$$

By Lemma 1.8, we can conclude that $\tilde{x} \in \bigcap_{j=1}^{N} F(T_j)$. $\square$

Theorem 2.4. *In addition to the conditions in Theorem 2.3, we assume further that* $\emptyset \neq \bigcap_{i=1}^{N} F(T_i) \subseteq \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j)$.

(a) If E *satisfies Opial's condition, then* $\{x_n\}$ *converges weakly to an element of* $\bigcap_{i=1}^{N} F(T_i)$.

(b) If one of $\{T_i\}_{i=1}^{N}$ *is semi-compact, then* $\{x_n\}$ *converges strongly to an element of* $\bigcap_{i=1}^{N} F(T_i)$.

Proof. We continue the argument in the proof of Theorem 2.3.

For (a), we claim that $\{x_n\}$ is weakly convergent. Were this false, there existed another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \bar{x} \in C$ weakly and $\bar{x} \neq \tilde{x}$. Utilizing the same argument as in Theorem 2.3, we can prove that $\bar{x} \in \bigcap_{j=1}^{N} F(T_j)$. Note that by Lemma 2.1 both $\lim_{n\to\infty} \|x_n - \tilde{x}\|$ and $\lim_{n\to\infty} \|x_n - \bar{x}\|$ exists. It follows from the Opial condition of E that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \tilde{x}\| &= \liminf_{i\to\infty} \|x_{n_i} - \tilde{x}\| \\ &< \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\| = \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - \tilde{x}\| = \lim_{n\to\infty} \|x_n - \tilde{x}\|.\end{aligned}$$

This contradiction indicates that $\bar{x} = \tilde{x}$, and so $\{x_n\}$ converges weakly to $\tilde{x}$.

For (b), by (2.7), we can assume a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ exists such that $x_{n_i} \to \hat{x} \in \bigcap_{i=1}^{N} F(T_i)$ in norm. It then follows from Lemma 2.1 that

$$\lim_{n\to\infty} \|x_n - \hat{x}\| = \lim_{i\to\infty} \|x_{n_i} - \hat{x}\| = 0.$$

This completes the proof. $\square$

References

[1] S. C. Bose, "Weak convergence to the fixed point of an asymptotically nonexpansive map", Proc. Amer. Math. Soc. **68** (1978), 305-308.

[2] S. S. Chang, "On the iterative approximation problem of fixed points for asymptotically nonexpansive type mappings in Banach spaces", Applied Mathematics and Mechanics (English Edition), **22**(1) (2001), 25-34.

[3] S. S. Chang, Y. J. Cho, and H. Y. Zhou, "Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings", J. Korean Math. Soc. **38** (2001), 1245-1260.

[4] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings", Proc. Amer. Math. Soc. **35** (1972), 171-174.

[5] G. Köthe, Topological vector spaces I, Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag, New York, 1969.

[6] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings", Bull. Amer. Math. Soc. **73** (1967), 591-597.

[7] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings", Bull. Austral. Math. Soc. **43** (1991), 153-159.

[8] Z. H. Sun, “Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings”, J. Math. Anal. Appl. **286** (2003), 351-358.

[9] H. K. Xu and R. G. Ori, “An implicit iteration process for nonexpansive mappings”, Numer. Funct. Anal. Optim. **22** (2001), 767-773.

[10] Y. Y. Zhou and S. S. Chang, “Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces”, Numer. Funct. Anal. Optim. **23** (2002), 911-921.