Ishikawa Iteration with Errors for Approximating Fixed Points of Strictly Pseudocontractive Mappings of Browder-Petryshyn Type

L. C. Zeng[1], G. M. Lee[2] and N. C. Wong[3]

[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China. This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai.

[2]Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea

[3]Corresponding author, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 804. This research was partially supported by grant from the National Science Council of Taiwan.

Abstract. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping in the sense of F. E. Browder and W. V. Petryshyn [1]. Let $\{u_n\}$ be a bounded sequence in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in [0,1] satisfying some restrictions. Let $\{x_n\}$ be the bounded sequence in K generated from any given $x_1 \in K$ by the Ishikawa iteration method with errors: $y_n = (1 - \beta_n)x_n + \beta_n T x_n, x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n, n \geq 1$. It is shown in this paper that if T is compact or demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Key Words: Ishikawa iteration method with errors, strictly pseudocontractive mappings of Browder-Petryshyn type, fixed point, $q-$uniformly smooth Banach Space.

MR(2000) Subject Classification: 49H09, 47H10, 47H17.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and dual E^*. Let $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing between E and E^*, and let $J_q : E \to 2^{E^*} (q > 1)$ denote the generalized duality mapping defined as the following: for each $x \in E$,

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\}.$$

In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see [6]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single-valued. In the sequel we shall denote the single-valued generalized duality mapping by j_q.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{1.1}$$

Remark 1.1. Without loss of generality we may assume $\lambda \in (0, 1)$. If I denotes the identity operator, then (1.1) can be rewritten in the form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^2. \tag{1.2}$$

In Hilbert space, (1.1) (and hence (1.2)) is equivalent to the following inequality:

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad k = (1-\lambda) < 1.$$

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(i) compact if for any bounded sequence $\{x_n\}$ in $D(T)$, there exists a strongly convergent subsequence of $\{Tx_n\}$;

(ii) demicompact if for any bounded sequence $\{x_n\}$ in $D(T)$, whenever $\{x_n - Tx_n\}$ is strongly convergent, there exists a strongly convergent subsequence of $\{x_n\}$.

In 1974, Rhoades [4] proved the following strong convergence theorem using the Mann iteration method.

Theorem 1.1. Let H be a real Hilbert space and K be a nonempty compact convex subset of H. Let $T : K \to K$ be a strictly pseudocontractive mapping and let $\{\alpha_n\}$ be a real sequence satisfying the following conditions:

(i) $\alpha_0 = 1$; (ii) $0 < \alpha_n < 1, \ \ \forall n \geq 1$; (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iv) $\lim_{n\to\infty}\alpha_n = \alpha < 1$.
Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by the Mann iteration method

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha T x_n, \quad \forall n \geq 1,$$

converges strongly to a fixed point of T.

Recently, Osilike and Udomene [3] improved, unified and developed Theorem 1.1 and Browder and Petryshyn's corresponding result [1] in the following aspects: (1) Hilbert spaces are extended to the setting of $q-$uniformly smooth Banach spaces. (2) The Mann iteration method is extended to the case of Ishikawa iteration method.

Theorem 1.2 [3, Corollary 2]. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, $T : K \to K$ be a demicompact strictly pseudocontractive mapping with a nonempty fixed-point set, i.e.,$F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0,1] satisfying the following conditions:
(i) $0 < a \leq \alpha_n^{q-1} \leq b < (q\lambda^{q-1}/c_q)(1 - \beta_n), \ \ \forall n \geq 1$ and for some constants $a, b \in (0, 1)$;
(ii) $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, where $\tau = \min\{1, (q - 1)\}$.
Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1. \end{cases}$$

converges strongly to a fixed point of T.

Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex (not necessarily bounded) subset of E, and $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K,$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in [0,1] satisfying certain restrictions. Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method with errors

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n, \quad n \geq 1. \end{cases}$$

It is shown in this paper that if T is compact or demicompact then $\{x_n\}$ converges strongly to a fixed point of T. Our result improves, extends and develops Osilike and Udomene [3, Corollary 2] in the following aspects: (1) The Ishikawa iteration method is extended to the case of Ishikawa iteration method with errors. (2) The stronger condition (ii) in [3, Corollary 2] is removed and replaced by a weaker condition which is convenient to verify. In addition, our result also improves and generalizes corresponding results in [1] and [4], respectively.

2. Preliminaries

In this section, we give some preliminaries whih will be used in the rest of this paper.

¿From (1.2) we have

$$\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\| \geq \lambda\|Tx-Ty\| - \lambda\|x-y\|,$$

so that

$$\|Tx-Ty\| \leq L\|x-y\|, \quad \forall x,y \in K, \text{where}\, L=(1+\lambda)/\lambda.$$

Since $\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\|$, we have

$$\begin{aligned} \langle x-Tx-(y-Ty), j_q(x-y)\rangle &= \|x-y\|^{q-2}\langle x-Tx-(y-Ty), j_q(x-y)\rangle \\ &\geq \lambda\|x-y\|^{q-2}\|x-Tx-(y-Ty)\|^2 \\ &\geq \lambda^{q-1}\|x-Tx-(y-Ty)\|^q. \qquad (2.1) \end{aligned}$$

Recall that the modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}.$$

E is uniformly smooth if and only if $\lim_{\tau\to 0^+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c_q > 0$ such that $\rho_E(\tau) < c_q\tau^q$. Hilbert spaces, L_p, l_p spaces, $1 < p < \infty$, and the Sobolev spaces, $W^p_m, 1 < p < \infty$, are $q-$uniformly smooth. Hilbert spaces are 2-uniformly smooth while if $1 < p < 2$, then L_p, l_p and W^p_m is $p-$uniformly smooth; if $p \geq 2$, then L_p, l_p and W^p_m are 2-uniformly smooth.

Theorem 2.1 [6, p. 1130]. Let $q > 1$ and E be a real Banach space. Then the following are equivalent:

(1) E is $q-$uniformly smooth.

(2) There exists a constant $c_q > 0$ such that for all $x, y \in E$

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. (2.2)$$

(3) There exists a constant d_q such that for all $x, y \in E$ and $t \in [0,1]$

$$\|(1-t)x+ty\|^q \geq (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q, (2.3)$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Furthermore, Xu and Roach [7, Remark 5] proved that if E is $q-$uniformly smooth $(q > 1)$, then for all $x, y \in E$, there exists a constant $L_* > 0$ such that

$$\|j_q(x) - j_q(y)\| \leq L_*\|x-y\|^{q-1}. (2.4)$$

Lemma 2.1 [5, p. 303]. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} < a_n + b_n, \quad \forall n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, Let λ be the constant appearing in (1.1), L be the Lipschitz constant of T, and c_q, d_q, $w_q(t)$, and L_* be the constants appearing in equations (2.2)-(2.4), respectively.

Lemma 3.1. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty convex subset of E, $T : K \to K$ be strictly pseudocontractive with $F(T) \neq \emptyset$. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in K, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma\}_{n=1}^{\infty}$ be real sequences in [0,1] with $\alpha_n + \gamma_n \leq 1, \quad \forall n \geq 1$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the following Ishikawa iteration method with errors:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n. \end{cases} \tag{3.1}$$

Then,

$$\begin{aligned} \|x_{n+1} - x^*\|^q \leq\ & (1 + 2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q + a_n\beta_n^{q-1}qL_*(1+L)^{q+1} \\ & + a_n\beta_n q\lambda^{q-1}(1+L^2)^q)\|x_n - x^*\|^q \\ & - a_n(q\lambda^{q-1} - c_q a_n^{q-1})\|x_n - Ty_n\|^q \\ & + q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q, \end{aligned} \tag{3.2}$$

where $a_n = \alpha_n + \gamma_n$, and $e_n = \gamma_n(u_n - Ty_n), \quad \forall n \geq 1$.

Proof. For each $n \geq 1$, set $a_n = \alpha_n + \gamma_n$ and $e_n = \gamma_n(u_n - Ty_n)$. Then it follows from (3.1) that for each $n \geq 1$,

$$x_{n+1} = (1-a_n)x_n + a_n T y_n + e_n.$$

Let x^* be an arbitrary fixed point of T. Then from (2.2) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^q &= \|(1-a_n)x_n + a_nTy_n + e_n - x^*\|^q \\ &\leq \|(1-a_n)x_n + a_nTy_n - x^*\|^q + q\langle e_n, j_q(x_{n+1} - e_n - x^*)\rangle + c_q\|e_n\|^q \\ &\leq \|(1-a_n)x_n + a_nTy_n - x^*\|^q + q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} \|(1-a_n)x_n + a_nTy_n - x^*\|^q &= \|x_n - x^* - a_n(x_n - Ty_n)\|^q \\ &\leq \|x_n - x^*\|^q - qa_n\langle x_n - Ty_n, j_q(x_n - x^*)\rangle \\ &\quad + a_n^q c_q\|x_n - Ty_n\|^q, \end{aligned} \tag{3.4}$$

$$\begin{aligned}\langle x_n - Ty_n, j_q(x_n - x^*)\rangle &= \langle x_n - y_n, j_q(x_n - x^*)\rangle + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &= \beta_n\langle x_n - Tx_n - (x^* - Tx^*), j_q(x_n - x^*)\rangle + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &\geq \beta_n\lambda^{q-1}\|x_n - Tx_n - (x^* - Tx^*)\|^q + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &= \beta_n\lambda^{q-1}\|x_n - Tx_n\|^q + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle, \qquad (3.5)\end{aligned}$$

and

$$\begin{aligned}\langle y_n - Ty_n, j_q(x_n - x^*)\rangle &= \langle y_n - Ty_n - (x^* - Tx^*), j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\ &\quad + \langle y_n - Ty_n - (x^* - Tx^*), j_q(y_n - x^*)\rangle \\ &\geq \lambda^{q-1}\|y_n - Ty_n - (x^* - Tx^*)\|^q \\ &\quad + \langle y_n - Ty_n - (x^* - Tx^*), j_q(x_n - x^*) - j_q(y_n - x^*)\rangle. \qquad (3.6)\end{aligned}$$

Furthermore, using (2.3), we have

$$\begin{aligned}\|y_n - Ty_n\|^q &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\|^q \\ &\geq (1-\beta_n)\|x_n - Ty_n\|^q + \beta_n\|Tx_n - Ty_n\|^q - \omega_q(\beta_n)d_q\|x_n - Tx_n\|^q. \qquad (3.7)\end{aligned}$$

Thus, from (3.4)-(3.7) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq \|x_n - x^*\| - qa_n\{\beta_n\lambda^{q-1}\|x_n - Tx_n\|^q + \lambda^{q-1}(1-\beta_n)\|x_n - Ty_n\|^q \\ &\quad + \lambda^{q-1}\beta_n\|Tx_n - Ty_n\|^q - \lambda^{q-1}\omega_q(\beta_n)d_q\|x_n - Tx_n\|^q \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - y^*)\rangle\} \\ &\quad + a_n^q c_q\|x_n - Ty_n\|^q + q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q \\ &\leq \|x_n - x^*\|^q - a_n(q\lambda^{q-1}(1-\beta_n) - a_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ &\quad + qd_q\lambda^{q-1}a_n\omega_q(\beta_n)\|x_n - Tx_n\|^q \\ &\quad + qa_n\|y_n - Ty_n\|\|j_q(x_n - x^*) - j_q(y_n - x^*)\| \\ &\quad + q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q.\end{aligned}$$

On the other hand, observe that

$$\omega_q(\beta_n) = \beta_n(1-\beta_n)^q + \beta_n^q(1-\beta_n) \leq 2\beta_n,$$

$$\|x_n - Tx_n\| \leq (1+L)\|x_n - x^*\|,$$

$$\begin{aligned}\|j_q(x_n - x^*) - j_q(y_n - x^*)\| &\leq L_*\beta_n^{q-1}\|x_n - Tx_n\| \quad \text{(using (2.4))} \\ &\leq L_*(1+L)^{q-1}\beta_n^{q-1}\|x_n - x^*\|^{q-1},\end{aligned}$$

and

$$\begin{aligned}\|y_n - Ty_n\| &\leq (1+L)\|y_n - x^*\| \\ &\leq (1+L)[(1-\beta_n)\|x_n - x^*\| + \beta_n L\|x_n - x^*\|] \\ &\leq (1+L)^2\|x_n - x^*\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le (1+2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q+a_n\beta_n^{q-1}qL_*(1+L)^{q+1})\|x_n-x^*\|^q\\ &\quad -a_n(q\lambda^{q-1}(1-\beta_n)-a_n^{q-1}c_q)\|x_n-Ty_n\|^q\\ &\quad +q\|e_n\|\|x_{n+1}-e_n-x^*\|^{q-1}+c_q\|e_n\|^q. \qquad (3.8)\end{aligned}$$

Note that

$$\|Ty_n-x^*\| \le L\|y_n-x^*\| \le L((1-\beta_n)\|x_n-x^*\|+\beta_n\|Tx_n-x^*\|) \le L^2\|x_n-x^*\|,$$

and

$$\|x_n-Ty_n\| \le \|x_n-x^*\|+\|Ty_n-x^*\| \le (1+L^2)\|x_n-x^*\|.$$

Therefore, from (3.8) we get

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le (1+2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q+a_n\beta_n^{q-1}qL_*(1+L)^{q+1}\\ &\quad +a_n\beta_nq\lambda^{q-1}(1+L^2)^q)\|x_n-x^*\|^q\\ &\quad -a_n(q\lambda^{q-1}-a_n^{q-1}c_q)\|x_n-Ty_n\|^q\\ &\quad +q\|e_n\|\|x_{n+1}-e_n-x^*\|^{q-1}+c_q\|e_n\|^q.\end{aligned}$$

Lemma 3.2. Let $q>1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty convex subset of E, and $T:K\to K$ be strictly pseudocontractive with $F(T)\neq\emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in [0,1] satisfying the following conditions:

(i) $\alpha_n+\gamma_n\le 1, \quad \forall n\ge 1$;

(ii) $\overline{\lim}_{n\to\infty}\alpha_n<\lambda(q/c_q)^{1/(q-1)}$, $\overline{\lim}_{n\to\infty}\beta_n<1/L$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(iii) $\sum_{n=1}^{\infty}\gamma_n<\infty$ and $\sum_{n=1}^{\infty}\alpha_n\beta_n^{\tau}<\infty$, where $\tau=\min\{1,(q-1)\}$.

Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1\in K$ by the Ishikawa iteration method (3.1) with errors. Then,

(a) for each $x^*\in F(T), \lim_{n\to\infty}\|x_n-x^*\|$ exists;

(b) there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i}-Tx_{n_i}\|=0$.

Proof. From Lemma 3.1, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le (1+\delta_n)\|x_n-x^*\|^q-a_n(q\lambda^{q-1}-c_qa_n^{q-1})\|x_n-Ty_n\|^q\\ &\quad +q\|e_n\|\|x_{n+1}-e_n-x^*\|^{q-1}+c_q\|e_n\|^q, \qquad (3.9)\end{aligned}$$

where $a_n=\alpha_n+\gamma_n, e_n=\gamma_n(u_n-Ty_n)$, and

$$\delta_n=2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q+a_n\beta_n^{q-1}qL_*(1+L)^{q+1}+a_n\beta_nq\lambda^{q-1}(1+L^2)^q, \quad \forall n\ge 1.$$

Since $\|x_n - Ty_n\| \leq (1+L^2)\|x_n - x^*\|$, it follows from the boundedness of $\{x_n\}$ that $\{Ty_n\}$ is bounded. Hence, we know that $\{u_n - Ty_n\}$ is bounded. Note that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Thus, we infer that

$$\sum_{n=1}^{\infty} \|e_n\| = \sum_{n=1}^{\infty} \|\gamma_n(u_n - Ty_n)\| < \infty,$$

which hence implies that

$$\sum_{n=1}^{\infty} \|e_n\|^q < \infty.$$

Since $\{e_n\}$ and $\{x_n\}$ are both bounded, there exists a number $M > 0$ such that

$$\|x_n - x^*\| \leq M \quad \text{and} \quad \|x_{n+1} - e_n - x^*\| \leq M, \quad \forall n \geq 1$$

Hence, from (3.9) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^q \quad &\leq \quad \|x_n - x^*\|^q - a_n(q\lambda^{q-1} - a_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ &\quad + \delta_n M^q + q\|e_n\|M^{q-1} + c_q\|e_n\|^q. \qquad (3.10) \end{aligned}$$

Since $\overline{\lim}_{n\to\infty}\alpha_n < \lambda(q/c_q)^{1/(q-1)}$, we have $\overline{\lim}_{n\to\infty}a_n < \lambda(q/c_q)^{1/(q-1)}$. So, for any given $\varepsilon > 0$, there exists an integer $N_0 \geq 1$ such that $\sup_{n\geq N_0} a_n < \lambda(q/c_q)^{1/(q-1)}$. Let $b = \sup_{n\geq N_0} a_n$. Then for all $n \geq N_0$, we have $a_n \leq b < \lambda(q/c_q)^{1/(q-1)}$. Obviously, it is easy to see that

$$q\lambda^{q-1} - a_n^{q-1}c_q \geq q\lambda^{q-1} - b^{q-1}c_q = c_q(\lambda^{q-1}(q/c_q) - b^{q-1}) > 0.$$

Consequently, (3.10) reduces to

$$\begin{aligned} \|x_{n+1} - x^*\|^q \quad &\leq \quad \|x_n - x^*\|^q - a_n(q\lambda^{q-1} - b^{q-1}c_q)\|x_n - Ty_n\|^q \\ &\quad + \delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q, \quad \forall n \geq N_0, (3.11) \end{aligned}$$

which hence implies that

$$\|x_{n+1} - x^*\|^q \leq \|x_n - x^*\|^q + \delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q.$$

Since

$$\sum_{n=1}^{\infty} \|e_n\| < \infty, \sum_{n=1}^{\infty} \|e_n\|^q < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \beta_n^{\tau} < \infty,$$

we conclude that

$$\sum_{n=1}^{\infty} (\delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q) < \infty.$$

Hence, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

On the other hand, from (3.11) we deduce that for all $n \geq N_0$

$$\begin{aligned} (q\lambda^{q-1} - b^{q-1}c_q)a_n\|x_n - Ty_n\|^q \quad &\leq \quad \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + \delta_n M^q \\ &\quad + \|e_n\|qM^{q-1} + c_q\|e_n\|^q \end{aligned}$$

from which it follows

$$\begin{aligned}(q\lambda^{q-1}-b^{q-1}c_q)\textstyle\sum_{j=N_0}^{n}a_j\|x_j-Ty_j\|^q &\le \|x_{N_0}-x^*\|^q-\|x_{n+1}-x^*\|^q\\ &\quad +\textstyle\sum_{j=N_0}^{n}(\delta_jM^q+\|e_j\|qM^{q-1}+c_q\|e_j\|^q)\\ &\le \|x_{N_0}-x^*\|^q+\textstyle\sum_{j=1}^{\infty}(\delta_jM^q+\|e_j\|qM^{q-1}+c_q\|e_j\|^q)\\ &< \infty.\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty}a_n\|x_n-Ty_n\|^q<\infty$.

Next, we claim that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\|x_{n_i}-Tx_{n_i}\|=0.$$

Indeed, since $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}a_n=\infty$ and we have $\underline{\lim}_{n\to\infty}\|x_n-Ty_n\|=0$. If it is false, then $\underline{\lim}_{n\to\infty}\|x_n-Ty_n\|=\delta>0$. Hence, there exists an integer $N_1>1$ such that $\inf_{n\ge N_1}\|x_n-Ty_n\|>\delta/2$. This implies that

$$\infty=(\frac{\delta}{2})^q\sum_{n=N_1}^{\infty}a_n\le\sum_{n=1}^{\infty}a_n\|x_n-Ty_n\|^q<\infty,$$

which leads to a contradiction. Thus, $\underline{\lim}_{n\to\infty}\|x_n-Ty_n\|=0$. Since

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-Ty_n\|+\|Ty_n-Tx_n\|\\ &\le \|x_n-Ty_n\|+L\|y_n-x_n\|\\ &\le \|x_n-Ty_n\|+L\beta_n\|x_n-Tx_n\|,\end{aligned}$$

we have

$$(1-L\beta_n)\|x_n-Tx_n\|\le\|x_n-Ty_n\|.$$

So, we derive

$$L(\frac{1}{L}-\overline{\lim}_{n\to\infty}\beta_n)\cdot\underline{\lim}_{n\to\infty}\|x_n-Tx_n\|\le\underline{\lim}_{n\to\infty}\|x_n-Ty_n\|=0.$$

Note that $\overline{\lim}_{n\to\infty}\beta_n<1/L$. Hence we have $\underline{\lim}_{n\to\infty}\|x_n-Tx_n\|=0$. This shows that there exists a subsequences $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\|x_{n_i}-Tx_{n_i}\|=0.$$

Now we can state and prove our main results in this paper.

Theorem 3.1. Let $q>1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, and $T:K\to K$ be compact and strictly pseudocontactive with $F(T)\neq\emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in [0,1] satisfying the following conditions:

(i) $\alpha_n + \gamma_n \leq 1, \quad \forall n \geq 1$;
(ii) $\overline{\lim}_{n\to\infty}\alpha_n < \lambda(q/c_q)^{1/(q-1)}, \overline{\lim}_{n\to\infty}\beta_n < 1/L$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\alpha_n\beta_n^{\tau} < \infty$, where $\tau = \min\{1, (q-1)\}$.
Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method (3.1) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T.
Proof. From Lemma 3.2, it follows that for each $x^* \in F(T), \lim_{n\to\infty}\|x_n - x^*\|$ exists, and that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Since $\{x_{n_i}\}$ is bounded and T is compact, so, $\{Tx_{n_i}\}$ has a strongly convergent subsequence. Without loss of generality, we may assume that $\{Tx_{n_i}\}$ converges strongly to some $p \in K$. Observe that

$$\|x_{n_i} - p\| \leq \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - p\| \to 0 \ \ (i \to \infty).$$

Hence, we know that $\{x_{n_i}\}$ converges strongly to $p \in K$. Obviously, according to the Lipschitz continuity of T, it is easy to see that

$$p = \lim_{i\to\infty}x_{n_i} = \lim_{n\to}Tx_{n_i} = Tp,$$

that is, $p \in F(T)$. Therefore, we have

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{i\to\infty}\|x_{n_i} - p\| = 0,$$

which hence implies that $\{x_n\}$ converges strongly to $p \in F(T)$.

Remark 3.1. If K is a compact subset of E, then it follows immediately from Theorem 3.1 that $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 3.2. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, and $T : K \to K$ be demicompact and strictly pseudocontractive with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem 3.1. Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method (3.1) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T.
Proof. From Lemma 3.2, it follows that for each $x^* \in F(T)$, $\lim_{n\to\infty}\|x_n - x^*\|$ exists, and that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Since $\{x_{n_i}\}$ is bounded and $\{x_{n_i} - Tx_{n_i}\}$ is strongly convergent, it follows from the demicompactness of T that there exists a subsequence of $\{x_{n_i}\}$ which converges strongly to some $p \in K$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges strongly to $p \in K$. Hence, taking into account that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$ and the Lipschitz continuity of T, we derive $p \in F(T)$. Observe that

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{i\to\infty}\|x_{n_i} - p\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to $p \in F(T)$.

Remark 3.2. If we take $\beta_n = 0 \ \ \forall n \geq 1$ in Lemmas 3.1, 3.2 and Theorems 3.1, 3.2, respectively, then we can obtain the results corresponding to Mann iteration method with errors

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T x_n + \gamma_n u_n, \quad \forall n \geq 1.$$

In addition, if we take $\gamma_n = 0 \ \ \forall n \geq 1$ in (3.1), then under the lack of the assumption that $\{x_n\}$ is bounded, Lemmas 3.1, 3.2 and Theorems 3.1, 3.2 are still valid. Indeed, if $\gamma_n = 0 \ \ \forall n \geq 1$, then it follows from (3.9) that for all $n \geq N_0$

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq (1+\delta_n)\|x_n - x^*\|^q \\ &\leq (1+\delta_n)(1+\delta_{n-1})\dots(1+\delta_{N_0})\|x_{N_0} - x*\|^q \\ &\leq e^{\sum_{j=1}^{\infty}\delta_j}\|x_{N_0} - x^*\|^q \\ &< \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, Theorems 3.1 and 3.2 generalize Theorems 1.1 and 1.2, respectively.

Remark 3.3. It is well known that in the sense of Xu [2], the Ishikawa iteration method with errors is defined as the following: for an arbitrary $x_1 \in K$, the sequence $\{x_n\}$ is generated by the iterative scheme

$$\begin{cases} y_n = (1 - \beta_n - \theta_n)x_n + \beta_n T x_n + \theta_n v_n, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n, \ n \geq 1, \end{cases} \quad (3.12)$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K, and $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}, \{\gamma_n\}$ are real sequences in [0,1] satisfying the restrictions: $\alpha_n + \gamma_n \leq 1$, $\beta_n + \theta_n \leq 1$, $\forall n \leq 1$. Naturally, we put forth the following open question.

Open Question: Can the Ishikawa iteration method (3.12) with errors in the sense of Xu [2] be extended to Theorems 3.1 and 3.2, respectively?

References

1. Browder, F. E. and Petryshyn, W. V., Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl., 20, 197-228, 1967.

2. Xu, Y. G., Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl., 224, 91-101, 1998.

3. Osilike, M. O. and Udomene, A., Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, J. Math. Anal. Appl., 256, 431-445, 2001.

4. Rhoades, B. E., Fixed point iteratios using matrices, Trans .Amer. Math. Soc., 196, 161-176, 1974.

5. Tan, K. K. and Xu, H. K., Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl., 178, 301-308, 1993.

6. Xu, H. K., Inequalities in Banach spaces with applications, Nonlinear Anal. TMA, 16, 1127-1138, 1991.

7. Xu, Z. B. and Roach, G. F., Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl.,157, 189-210, 1991.