AUTOMATIC CONTINUITY AND $C_0(\Omega)$-LINEARITY OF LINEAR MAPS BETWEEN $C_0(\Omega)$-MODULES

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Abstract. Let Ω be a locally compact Hausdorff space. We show that any local $\mathbb{C}$-linear map (where "local" is a weaker notion than $C_0(\Omega)$-linearity) between Banach $C_0(\Omega)$-modules are "nearly $C_0(\Omega)$-linear" and "nearly bounded". As an application, a local $\mathbb{C}$-linear map θ between Hilbert $C_0(\Omega)$-modules is automatically $C_0(\Omega)$-linear. If, in addition, Ω contains no isolated point, then any $C_0(\Omega)$-linear map between Hilbert $C_0(\Omega)$-modules is automatically bounded. Another application is that if a sequence of maps $\{\theta_n\}$ between two Banach spaces "preserve c_0-sequences" (or "preserve ultra-c_0-sequences"), then θ_n is bounded for large enough n and they have a common bound. Moreover, we will show that if θ is a bijective "biseparating" linear map from a "full" essential Banach $C_0(\Omega)$-module E into a "full" Hilbert $C_0(\Delta)$-module F (where Δ is another locally compact Hausdorff space), then θ is "nearly bounded" (in fact, it is automatically bounded if Δ or Ω contains no isolated point) and there exists a homeomorphism $\sigma : \Delta \to \Omega$ such that $\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma$ $(e \in E, \varphi \in C_0(\Omega))$.

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1. Introduction

A linear map θ between the spaces of continuous sections of two bundle spaces over the same locally compact Hausdorff base space Ω is said to be *local* if for any continuous section f, one has $\operatorname{supp}\theta(f) \subseteq \operatorname{supp} f$, or equivalently, for each $g \in C_0(\Omega)$,

$$fg = 0 \quad \Longrightarrow \quad \theta(f)g = 0.$$

Consequently, local property is weaker than $C_0(\Omega)$-linearity. In the case when the domain and the range bundles are over different base spaces, a more general notion is defined; namely, *disjointness preserving*, or *separating* (see Section 5).

Local and disjointness preserving linear maps are found in many researches in analysis. For example, a theorem of Peetre [19] states that local linear maps of the space of smooth functions defined on a manifold modelled on $\mathbb{R}^n$ are exactly linear differential

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operators (see, e.g., [17]). This is further extended to the case of vector-valued differentiable functions defined on a finite dimensional manifold by Kantrowitz and Neumann [16] and Araujo [3].

In the topological setting, similar results have been obtained. Local linear maps of the space of continuous functions over a locally compact Hausdorff space are multiplication operators, while disjointness preserving (separating) linear maps between two such spaces over possibly different base spaces are weighted composition operators (see, e.g., [1, 5, 18, 14, 12, 15]). Among many interesting questions arising from these two notions, quite a few efforts has been put on the automatic continuity of such maps. See, e.g., [2, 7, 14, 15] for the scalar case, and [13, 4, 3, 6] for the vector-valued case.

In this paper, we extend this context to local or separating linear maps between spaces of continuous sections of vector bundles. Note that similar to the correspondence developed by Swan [20] between finite dimensional vector bundles over a locally compact Hausdorff space Ω and certain $C_0(\Omega)$-modules, the spaces of continuous sections of "Banach bundles" are certain Banach $C_0(\Omega)$-modules (see, e.g., [10], and Section 2 below).

One of the original motivation behind this work is to investigate up to what extend will a local linear map between two Banach $C_0(\Omega)$-modules being $C_0(\Omega)$-linear. Surprisingly, on top of finding that such maps are "nearly $C_0(\Omega)$-linear", we find that they are also "nearly bounded". In fact, it is well known that there are many unbounded $\mathbb{C}$-linear maps from an infinite dimensional Banach space to another Banach space and so, if S is a finite set, there are many unbounded $C(S)$-module maps from certain Banach $C(S)$-module to another Banach $C(S)$-module. The interesting thing we discovered is that the above is, in many cases, the "only obstruction" to the automatic boundedness of $C_0(\Omega)$-module maps (see Proposition 3.5 as well as Theorems 3.7 and 4.2).

More precisely, if θ is a local $\mathbb{C}$-linear map (not assumed to be bounded) from an essential Banach $C_0(\Omega)$-module E to another such module F, then θ is "nearly $C_0(\Omega)$-linear", in the sense that the induced map $\tilde{\theta} : E \to \tilde{F}$ is a $C_0(\Omega)$-module map (where $\tilde{F}$ is the image of F in the space of C_0-sections on the canonical "(H)-Banach bundle" associated with F; see Section 2). Moreover, θ is "nearly bounded" in the sense that there exists a finite subset $S \subseteq \Omega$ such that

$$\sup_{\omega\in\Omega\setminus S}\ \sup_{\substack{e\in E;\\ \|e\|\leq 1}} \left\|\tilde{\theta}(e)(\omega)\right\| \ < \ \infty.$$

Furthermore, if F is "$C_0(\Omega)$-normed" (in particular, if F is a Hilbert $C_0(\Omega)$-module), then the finite set S consists of isolated points in Ω, and

$$\theta \ = \ \theta_0 \oplus \bigoplus_{\omega\in S} \theta_\omega$$

where $\theta_0 : E_{\Omega\setminus S} \to F_{\Omega\setminus S}$ is a bounded $C_0(\Omega \setminus S)$-linear map (where $E_{\Omega\setminus S}$ and $F_{\Omega\setminus S}$ are the canonical essential Banach $C_0(\Omega \setminus S)$-modules induced from E and F respectively) and θ_ω are (unbounded) $\mathbb{C}$-linear maps (see Theorems 4.2 and 3.7). Consequently, if Ω contains no isolated point and F is $C_0(\Omega)$-normed, then θ is automatically bounded. As another application, if X and Y are two Banach spaces and if $\theta_k : X \to Y$ is a sequence of $\mathbb{C}$-linear maps (not assumed to be bounded) such that for any $(x_n) \in c_0(X)$, we have $(\theta_n(x_n)) \in c_0(Y)$, then there exists n_0 with

$$\sup_{n\geq n_0} \|\theta_n\| \; < \; \infty.$$

On the other hand, we will also study $\mathbb{C}$-linear maps between two Banach modules over two different base spaces. In this case, we will consider "separating" maps instead of local maps. More precisely, if Ω and Δ are two locally compact Hausdorff spaces, E is a "full" essential Banach $C_0(\Omega)$-module (see Remark 3.2(b)), and F is a "full" Banach $C_0(\Delta)$-normed module, then for any bijective linear map $\theta : E \to F$ (not assumed to be bounded) with both θ and θ^{-1} being separating, there exists a homeomorphism $\sigma : \Delta \to \Omega$ such that $\theta(e\cdot\varphi) = \theta(e)\cdot\varphi\circ\sigma$ $(e \in E, \varphi \in C_0(\Omega))$, and there exists a finite set S consisting of isolated points of Δ such that the restriction of θ from $E_{\Omega\setminus\sigma(S)}$ to $F_{\Delta\setminus S}$ is bounded.

This paper is organised as follows. In Section 2, we will first collect some basic facts about the correspondence between Banach bundles and Banach $C_0(\Omega)$-modules. In Section 3, we will show two technical lemmas concerning "near $C_0(\Omega)$-linearity" and "near boundedness" of certain mappings. Section 4 is devoted to automatic $C_0(\Omega)$-linearity and automatic boundedness of local linear mappings, while Section 5 is devoted to the automatic boundedness of bijective biseparating linear mappings between Banach modules over different base spaces. Finally, as an attempt to a further generalisation, we show in the Appendix that for an arbitrary C*-algebra A, every *bounded* local linear map from a Banach A-module into a Hilbert A-module is A-linear. The boundedness assumption can be removed in the case when A is finite dimension (Corollary 4.9).

2. Preliminaries and Notations

Let us first set some terminologies and recall (mainly from [10]) some basic terminologies and results concerning Banach modules and Banach bundles.

Notation 2.1. *In this article,* Ω *and* Δ *are two locally compact Hausdorff spaces,* E *is an essential Banach* $C_0(\Omega)$*-module,* F *is an essential Banach* $C_0(\Delta)$*-module, and* $\theta : E \to F$ *is a* $\mathbb{C}$*-linear map (not assumed to be bounded). Furthermore,* Ω_∞ *and* Δ_∞ *are the one-point compactifications of* Ω *and* Δ *respectively. We denote by* $\mathfrak{N}_\Omega(\omega)$ *the set of all compact neighbourhoods of an element* ω *in* Ω*, and by* $\mathrm{Int}_\Omega(S)$ *the set of all interior points of a subset* S *in* Ω*. Moreover, if* $U, V \subseteq \Omega$ *such that the closure of* V *is a compact subset of* $\mathrm{Int}_\Omega(U)$*, we denote by* $\mathfrak{U}_\Omega(V, U)$ *the collection of all* $\lambda \in C_c(\Omega)$ *with* $0 \leq \lambda \leq 1$*,* $\lambda \equiv 1$ *on* V *and the support of* λ *lies inside* $\mathrm{Int}_\Omega(U)$*.*

Definition 2.2. *Let Ξ be a Hausdorff space and $p : \Xi \to \Omega$ be a surjective continuous open map. Suppose that for each $\omega \in \Omega$,*

1). there exists a complex Banach space structure on $\Xi_\omega := p^{-1}(\omega)$ such that its norm topology coincides with the topology on Ξ_ω (as a topological subspace of Ξ);

2). $\{W(\epsilon, U) : \epsilon > 0, U \in \mathcal{N}_\Omega(\omega)\}$ forms a neighbourhood basis for the zero element $0_\omega \in \Xi_\omega$ where $W(\epsilon, U) := \{\xi \in p^{-1}(U) : \|\xi\| < \epsilon\}$;

3). the maps $\mathbb{C} \times \Xi \to \Xi$ and $\{(\xi, \eta) \in \Xi \times \Xi : p(\xi) = p(\eta)\} \to \Xi$ given respectively, by the scalar multiplications and the additions are continuous.

Then (Ξ, Ω, p) (or simply, Ξ) is called an (H)-Banach bundle *(respectively, an* (F)-Banach bundle*) over Ω if $\xi \mapsto \|\xi\|$ is an upper-semicontinuous (respectively, continuous) map from Ξ into $\mathbb{R}_+$. In this case, Ω is called the* base space *of Ξ, the map p is called the* canonical projection *and Ξ_ω is called the* fibre *over $\omega \in \Omega$.*

If Ξ is an (H)-Banach bundle over Ω and $\Omega_0 \subseteq \Omega$ is an open set, then

$$\Xi_{\Omega_0} \ := \ p^{-1}(\Omega_0)$$

is an (H)-Banach bundle over Ω_0 and is called the *restriction of* Ξ *to* Ω_0. If Ξ is an (F)-Banach bundle, then so is Ξ_{Ω_0}.

Definition 2.3. *If Λ is another (H)-Banach bundle over Δ, a map $\rho : \Xi \to \Lambda$ is said to be* bounded *if $\sup_{\substack{\xi \in \Xi; \\ \|\xi\| \leq 1}} \|\rho(\xi)\| < \infty$. Moreover, ρ is called a* fibrewise linear map *(respectively,* Banach bundle map*) if ρ restricted to a (respectively, bounded) linear map ρ_ω from Ξ_ω to a fibre in Λ.*

For any map $e : \Omega \to \Xi$, we denote

$$|e|(\omega) \ := \ \|e(\omega)\| \qquad (\omega \in \Omega).$$

Such an e is called a C_0*-section* on Ξ if e is continuous, $p(e(\omega)) = \omega$ ($\omega \in \Omega$), and for any $\epsilon > 0$, there exists a compact set $C \subseteq \Omega$ such that $|e|(\omega) < \epsilon$ ($\omega \in \Omega \setminus C$). We put

$$\Gamma_0(\Xi) \ := \ \{e : \Omega \to \Xi \mid e \text{ is a } C_0\text{-section on } \Xi\}.$$

Note that $|e|$ is always upper semi-continuous for every $e \in \Gamma_0(\Xi)$ and Ξ is an (F)-Banach bundle if and only if all such $|e|$ are continuous.

Next, we recall some terminologies and properties concerning an essential Banach (right) $C_0(\Omega)$-module E (regarded as a unital Banach $C(\Omega_\infty)$-module). For any $\omega \in \Omega_\infty$ and $S \subseteq \Omega_\infty$, we denote

$$K_S \ := \ \{\varphi \in C(\Omega_\infty) : \varphi(S) = \{0\}\}, \quad K_S^E \ := \ \overline{E \cdot K_S} \quad \text{and} \quad I_\omega^E \ := \bigcup_{V \in \mathcal{N}_{\Omega_\infty}(\omega)} K_V^E.$$

For simplicity, we set $K^E_\omega := K^E_{\{\omega\}}$. By [10, p.37], there exists an (H)-Banach bundle $\check{\Xi}^E$ over Ω_∞ with $\check{\Xi}^E_\omega = E/K^E_\omega$. Since $\check{\Xi}^E_\infty = \{0\}$, if we set $\Xi^E := p^{-1}(\Omega)$, then $\Gamma_0(\Xi^E) \cong \Gamma_0(\check{\Xi}^E)$ under the canonical identification. Furthermore, there exists a contraction

$$\sim : \ E \ \longrightarrow \ \Gamma_0(\Xi^E)$$

such that $\tilde{e}(\omega) = e + K^E_\omega$. We put $\tilde{E}$ to be the image of $\sim$.

On the other hand, if θ is as in Notation 2.1, we define

$$\tilde{\theta} : E \to \tilde{F} \qquad \text{by} \qquad \tilde{\theta}(e) = \widetilde{\theta(e)} \quad (e \in E).$$

Definition 2.4. *Let E be an essential Banach $C_0(\Omega)$-module.*
(a) E is called a Banach $C_0(\Omega)$-convex module *if for any $\varphi, \psi \in C(\Omega_\infty)_+$ with $\varphi + \psi = 1$, one has $\|x\varphi + y\psi\| \leq \max\{\|x\|, \|y\|\}$.*
(b) E is called a Banach $C_0(\Omega)$-normed module *if there exists a map $|\cdot| : E \to C_0(\Omega)_+$ such that for any $x, y \in X$ and $a \in A$,*
i). $|x + y| \leq |x| + |y|$;
ii). $|xa| = |x||a|$;
iii). $\|x\| = \||x|\|$.

Recall that every Hilbert $C_0(\Omega)$-module is $C_0(\Omega)$-normed, and every Banach $C_0(\Omega)$-normed module is $C_0(\Omega)$-convex. On the other hand, an essential Banach $C_0(\Omega)$-module E is $C_0(\Omega)$-convex if and only if $\sim$ is an isometric isomorphism onto $\Gamma_0(\Xi^E)$ (see e.g. [10, Theorem 2.5]). In this case, we will not distinguish E and $\Gamma_0(\Xi^E)$. Furthermore, E is $C_0(\Omega)$-normed if and only if E is $C_0(\Omega)$-convex and Ξ^E is an (F)-Banach bundle (see e.g. [10, p.48]).

For any open subset $\Omega_0 \subseteq \Omega$, we set $E_{\Omega_0} := K^E_{\Omega\setminus\Omega_0}$ and $\tilde{E}_{\Omega_0} := \Gamma_0(\Xi^E_{\Omega_0})$. One can regard $K^E_{\Omega\setminus\Omega_0}$ as an essential Banach $C_0(\Omega_0)$-module under the identification $C_0(\Omega_0) \cong K_{\Omega\setminus\Omega_0}$. Note that if E is $C_0(\Omega)$-convex, then $\tilde{E}_{\Omega_0} = E_{\Omega_0}$.

Remark 2.5. *(a) Let E be a Banach $C_0(\Omega)$-convex module and 0_ω is the zero element in the fibre Ξ^E_ω ($\omega \in \Omega$). It is well-known that $\omega \mapsto 0_\omega$ is a continuous map from Ω into Ξ^E. Thus, if $\{\omega_i\}_{i\in I}$ is a net in Ω converging to $\omega_0 \in \Omega$ and $e \in \bigcap_{i\in I} K^E_{\omega_i}$, then $e \in K^E_{\omega_0}$. Consequently, if $e \notin K^E_\omega$, there exists $U \in \mathcal{N}_\Omega(\omega)$ such that $e \notin K^E_\alpha$ for any $\alpha \in U$.*
(b) Let $\Omega = \{\omega_1, \omega_2, ...\}$ be a countable compact Hausdorff space and E be a Banach $C(\Omega)$-module. Then

$$\bigcap_{\omega\in\Omega} K^E_\omega \ = \ \{0\},$$

or equivalently, the map $\sim$ is injective. In fact, consider any $e \in \bigcap_{\omega\in\Omega} K^E_\omega$ and any $\epsilon > 0$. For $k \in \mathbb{N}$, there exists $\bar{\varphi}_k \in K_{\{\omega_k\}}$ with $\|e - e\bar{\varphi}_k\| < \epsilon/2^{k+1}$. Thus, there exists $\varphi_k \in C(\Omega)$ with φ_k vanishing on an open neighbourhood V_k of ω_k and $\|e - e\varphi_k\| < \epsilon/2^k$. Now, *consider a finite subcover* $\{V_1, ..., V_n\}$ *for* Ω *and a continuous partition of unity* $\{\psi_1, ...\psi_n\}$ *subordinated to* $\{V_1, ..., V_n\}$*. Then* $\|e\| = \|e - e\sum_{k=1}^n \varphi_k\psi_k\| \leq \sum_{k=1}^n \|e - e\varphi_k\| < \epsilon$*.*

(c) For any $\omega \in \Omega$ *and* $e \in K^E_\omega$*, there exists a net* $\{e_V\}_{V\in\mathcal{N}_\Omega(\omega)}$ *such that* $e_V \in K^E_V$ *and* $\|e - e_V\| \to 0$*.*

3. Some technical results

In this section, we will give two technical lemmas (3.3 and 3.6) which are major ingredients for all the results in this paper. Before that, let us give another automatic continuity type lemma that is needed for these two essential lemmas.

Lemma 3.1. $\mathfrak{Z}_\theta := \{\nu \in \Delta : \tilde\theta(e)(\nu) = 0 \text{ for all } e \in E\}$ *is a closed subset (where* $\tilde\theta$ *is as in Section 2). Moreover, if* $\sigma : \Delta_\theta \to \Omega_\infty$ *(where* $\Delta_\theta := \Delta \setminus \mathfrak{Z}_\theta$*) is a map such that* $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ $(\nu \in \Delta_\theta)$*, then* σ *is continuous.*

Proof: It follows from Remark 2.5(a) that $\mathfrak{Z}_\theta$ is closed. Suppose on the contrary, that there exists a net $\{\nu_i\}_{i\in I}$ in Δ_θ that converges to $\nu_0 \in \Delta_\theta$ but $\sigma(\nu_i) \nrightarrow \sigma(\nu_0)$. Then there are $U, W \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu_0))$ with $\{i \in I : \sigma(\nu_i) \notin \mathrm{Int}_\Omega(W)\}$ being cofinal and $U \subseteq \mathrm{Int}_{\Omega_\infty}(W)$. As Ω_∞ is compact, by passing to a subnet if necessary, we can assume that $\{\sigma(\nu_i)\}$ converges to an element $\omega \in \Omega_\infty$ such that there exists $V \in \mathcal{N}_{\Omega_\infty}(\omega)$ with $V \cap U = \emptyset$. Pick any $e \in E$ and $\varphi \in \mathcal{U}_{\Omega_\infty}(V, \Omega_\infty \setminus U)$. Since $\sigma(\nu_i) \to \omega$, we see that $e(1 - \varphi) \in I^E_{\sigma(\nu_i)}$ eventually and so,

$$\tilde\theta(e(1 - \varphi))(\nu_i) \;=\; 0 \quad \text{eventually}$$

(by the hypothesis). By Remark 2.5(a), we see that $\tilde\theta(e(1 - \varphi))(\nu_0) = 0$. On the other hand, we have $\theta(e\varphi) \in K^F_{\nu_0}$ (because $e\varphi \in I^E_{\sigma(\nu_0)}$) and so $\theta(e) \in K^F_{\nu_0}$ which gives the contradiction that $\nu_0 \in \mathfrak{Z}_\theta$. □

Remark 3.2. *(a) Note that for any* $\nu \in \mathfrak{Z}_\theta$*, one has*

$$\theta(I^E_\omega) \subseteq K^F_\nu \qquad (\omega \in \Omega). \tag{3.1}$$

Consequently, if we extend σ *in Lemma 3.1 by setting* $\sigma(\nu)$ *arbitrarily for each* $\nu \in \mathfrak{Z}_\theta$*, then* $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ $(\nu \in \Delta)$ *but one should not expect such* σ *to be continuous.*

(b) θ *is said to be* full *if* $\mathfrak{Z}_\theta = \emptyset$*. Moreover,* E *is said to be* full *if* $\mathrm{id} : E \to E$ *is full (or equivalently,* $E \neq K^E_\omega$ *for any* $\omega \in \Omega$*).*

(c) One can use our proof for Lemma 3.1 to give the following (probably known) result:

> *Suppose that* $\sigma : \Delta \to \Omega$ *is a map and* $\Phi : C_0(\Omega) \to C_b(\Delta)$ *is a* $\mathbb{C}$*-linear map such that* $\Phi(\lambda \cdot \psi) = \Phi(\lambda) \cdot (\psi \circ \sigma)$ $(\lambda, \psi \in C_0(\Omega))$*, and for any* $\nu \in \Delta$*, there exists* $\lambda \in C_0(\Omega)$ *with* $\Phi(\lambda)(\nu) \neq 0$*. Then* σ *is continuous.*

Lemma 3.3. *Let $\sigma : \Delta_\theta \to \Omega$ be a map such that $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ $(\nu \in \Delta_\theta)$.*

(a) If $\mathfrak{U}_\theta := \left\{\nu \in \Delta : \sup_{\|e\|\leq 1} \|\tilde{\theta}(e)(\nu)\| = \infty\right\}$, then $\mathfrak{U}_\theta \subseteq \Delta_\theta$,

$$\sup_{\nu\in\Delta\setminus\mathfrak{U}_\theta;\|e\|\leq 1} \|\tilde{\theta}(e)(\nu)\| \ < \ \infty$$

(we use the convention that $\sup\emptyset = 0$) and $\sigma(\mathfrak{U}_\theta)$ is a finite set.

(b) If $\mathfrak{N}_{\theta,\sigma} := \left\{\nu \in \Delta_\theta : \theta(K^E_{\sigma(\nu)}) \nsubseteq K^F_\nu\right\}$, then $\mathfrak{N}_{\theta,\sigma} \subseteq \mathfrak{U}_\theta$ and $\sigma(\mathfrak{N}_{\theta,\sigma})$ consists of non-isolated points in Ω.

(c) If, in addition, σ is an injection sending isolated points in Δ_θ to isolated points in Ω, then $\tilde{\theta}(e \cdot \varphi) = \tilde{\theta}(e) \cdot \varphi \circ \sigma$ $(e \in E, \varphi \in C_0(\Omega))$.

Proof: (a) The first conclusion is clear. We put Y to be the c_0-direct sum $\bigoplus^{c_0}_{\nu\in\Delta} \Xi^F_\nu$. As $e \mapsto \tilde{\theta}(e)(\nu)$ can be regarded a bounded $\mathbb{C}$-linear map from E into Y when $\nu \in \Delta\setminus\mathfrak{U}_\theta$ and $\|\tilde{\theta}(e)(\nu)\| \leq \|\theta(e)\|$, the uniform boundedness principle will give the second conclusion.

Assume now that $\sigma(\mathfrak{U}_\theta)$ is infinite. For $n = 1$, we can find $\nu_1 \in \Delta$ as well as $e_1 \in E$ with $\|e_1\| \leq 1$ and $\|\tilde{\theta}(e_1)(\nu_1)\| > 1$. Inductively, we can find $\nu_n \in \Delta$ and $e_n \in E$ such that

$$\sigma(\nu_n) \ \neq \ \sigma(\nu_k) \ (k = 1, ..., n-1), \quad \|e_n\| \leq 1 \quad \text{and} \quad \|\tilde{\theta}(e_n)(\nu_n)\| > n^3.$$

There exist $n_1 \in \mathbb{N}$ and $U_1 \in \mathfrak{N}_\Omega(\sigma(\nu_{n_1}))$ such that $\{n \in \mathbb{N} : n > n_1 \text{ and } \sigma(\nu_n) \notin U_1\}$ is infinite. Inductively, we can find a subsequence $\{\nu_{n_k}\}$ and $U_k \in \mathfrak{N}_\Omega(\sigma(\nu_{n_k}))$ $(k \in \mathbb{N})$ such that $U_k \cap U_l = \emptyset$ for distinct $k, l \in \mathbb{N}$. Without loss of generality, one can assume that $n_k = k$. Pick $V_n \in \mathfrak{N}_\Omega(\sigma(\nu_n))$ such that V_n is subset of $\mathrm{Int}_\Omega(U_n)$. Consider $\lambda_n \in \mathfrak{U}_\Omega(V_n, U_n)$ $(n \in \mathbb{N})$. Notice that $\|e_n\lambda_n^2\| \leq 1$ and $e := \sum_{k=1}^\infty \frac{e_k\lambda_k^2}{k^2} \in E$. Take any $n \in \mathbb{N}$. Since

$$n^2 e - e_n\lambda_n^2 \ = \ n^2 \left(\sum_{k\neq n} \frac{e_k\lambda_k}{k^2}\right)\left(\sum_{k\neq n} \lambda_k\right) \ \in \ K^E_{U_n},$$

we have $n^2\tilde{\theta}(e)(\nu_n) = \tilde{\theta}(e_n\lambda_n^2)(\nu_n)$ (by the hypothesis). On the other hand, as $e_n - e_n\lambda_n^2 = e_n(1-\lambda_n^2) \in K^E_{V_n}$, we have,

$$\left\|\tilde{\theta}(e)\right\| \ \geq \ \left\|\tilde{\theta}(e)(\nu_n)\right\| \ = \ \frac{1}{n^2}\left\|\tilde{\theta}(e_n\lambda_n^2)(\nu_n)\right\| \ = \ \frac{1}{n^2}\left\|\tilde{\theta}(e_n)(\nu_n)\right\| \ > \ n$$

which contradicts the finiteness of $\|\tilde{\theta}(e)\|$.

(b) Consider $\nu \in \Delta \setminus \mathfrak{U}_\theta$. Then $\kappa := \sup_{\|e\|\leq 1} \left\|\tilde{\theta}(e)(\nu)\right\| < \infty$. Take any $e \in K^E_{\sigma(\nu)}$. Pick $e_V \in K^E_V$ $(V \in \mathfrak{N}_\Omega(\sigma(\nu)))$ with $\|e_V - e\| \to 0$. Since $\theta(e_V) \in K^F_\nu$,

$$\left\|\tilde{\theta}(e)(\nu)\right\| \ = \ \left\|\tilde{\theta}(e - e_V)(\nu)\right\| \ \leq \ \kappa \left\|e - e_V\right\|.$$

This shows that $\nu \in \Delta\setminus\mathfrak{N}_{\theta,\sigma}$. The second statement is clear because if $\sigma(\nu)$ is an isolated point in Ω, then $\{\sigma(\nu)\} \in \mathfrak{N}_\Omega(\sigma(\nu))$ and so, $\theta(K^E_{\sigma(\nu)}) \subseteq K^F_\nu$.

(c) For any $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$ and $e \in E$, we have $e\varphi - e\varphi(\sigma(\nu)) = e(\varphi - \varphi(\sigma(\nu))1) \in K^E_{\sigma(\nu)}$. Thus,

$$\tilde{\theta}(e\varphi)(\nu) \;=\; \tilde{\theta}(e)(\nu)\varphi(\sigma(\nu)) \qquad (e \in E, \nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}). \tag{3.2}$$

In particular, (3.2) is true when $\nu \in \Delta \setminus \mathfrak{U}_\theta$ (by part (b)) or when $\nu \in \mathfrak{U}_\theta$ is an isolated point of Δ_θ (by the hypothesis as well as part (b)). Suppose that $\nu \in \mathfrak{U}_\theta$ is a non-isolated point of Δ_θ. As σ is injective, part (a) implies that $\mathfrak{U}_\theta$ is a finite set. Hence, there exists a net $\{\nu_i\}$ in $\Delta_\theta \setminus \mathfrak{U}_\theta$ converging to ν. Now, by Lemma 3.1,

$$\tilde{\theta}(e\varphi)(\nu) \;=\; \lim \tilde{\theta}(e\varphi)(\nu_i) \;=\; \lim \tilde{\theta}(e)(\nu_i)\varphi(\sigma(\nu_i)) \;=\; \tilde{\theta}(e)(\nu)\varphi(\sigma(\nu)).$$

□

Remark 3.4. *Note that since $\mathfrak{Z}_\theta$ is closed, isolated points in Δ_θ are the same as isolated points of Δ. Moreover, for any $\nu \in \mathfrak{Z}_\theta$, we have $\sup_{\|e\|\leq 1} \|\tilde{\theta}(e)(\nu)\| = 0$ and (3.1) holds. Therefore, Lemma 3.3 remains valid if we replace all the Δ_θ with Δ (in fact, the current form is stronger as any injection on Δ restricted to an injection on Δ_θ). The same is true for all the remaining results in this section.*

If σ is injective, then $\mathfrak{U}_\theta$ is finite and we have our first nearly automatically bounded result which states that if θ is a "module map through an injection $\sigma : \Delta \to \Omega$" (one can relax this slightly to an injection on Δ_θ), then θ is "bounded after taking away finite number of points from Δ".

Proposition 3.5. *Let Ω and Δ be two locally compact Hausdorff spaces. Let E and F be an essential Banach $C_0(\Omega)$-module and an essential Banach $C_0(\Delta)$-module respectively, and let $\theta : E \to F$ be a $\mathbb{C}$-linear map (not assumed to be bounded). Suppose that $\sigma : \Delta_\theta \to \Omega$ is an injection such that $\theta(e \cdot \varphi)(\nu) = \theta(e)(\nu)\varphi(\sigma(\nu))$ ($e \in E, \varphi \in C_0(\Omega), \nu \in \Delta_\theta$). Then there exists a finite subset $T \subseteq \Delta$ such that*

$$\sup_{\substack{\nu\in\Delta\setminus T \\ e\in E, \|e\|\leq 1}} \|\tilde{\theta}(e)(\nu)\| \;<\; \infty.$$

Lemma 3.6. *Let $\sigma : \Delta_\theta \to \Omega$ be a map such that $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta_\theta$). Suppose, in addition, that F is a Banach $C_0(\Delta)$-normed module.*

(a) $\mathfrak{N}_{\theta,\sigma}$ is an open subset of Δ.

(b) If σ is injective, then $\mathfrak{U}_\theta$ is a finite set consisting of isolated points of Δ. If, in addition, $\mathfrak{U}_\theta \neq \Delta$, then $F = F_{\Delta\setminus\mathfrak{U}_\theta} \oplus \bigoplus_{\nu\in\mathfrak{U}_\theta} \Xi^F_\nu$ and

$$\theta_0 := P_{\theta,\sigma} \circ \theta|_{E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)}} : E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)} \to F_{\Delta\setminus\mathfrak{U}_\theta}$$

is a bounded linear map (where $P_{\theta,\sigma} : F \to F_{\Delta\setminus\mathfrak{U}_\theta}$ is the canonical projection) such that

$$\theta_0(e \cdot \varphi) = \theta_0(e) \cdot \varphi \circ \sigma \qquad (e \in E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)}, \varphi \in C_0(\Omega \setminus \sigma(\mathfrak{U}_\theta))) \tag{3.3}$$

(the value of σ on $\mathfrak{Z}_\theta$ can be set arbitrarily).

Proof: Note, first of all, that as F is $C_0(\Omega)$-convex, one can regard $\tilde{\theta} = \theta$.

(a) As Δ_θ is open in Δ and $\mathfrak{N}_{\theta,\sigma} \subseteq \mathfrak{U}_\theta \subseteq \Delta_\theta$, it suffices to show that $\mathfrak{N}_{\theta,\sigma}$ is open in Δ_θ. By Lemma 3.3(a),

$$\kappa \ := \ \sup_{\nu\notin\mathfrak{U}_\theta} \sup_{\|e\|\leq 1} \|\theta(e)(\nu)\| \ < \ \infty.$$

Let $\{\nu_i\}_{i\in I}$ be a net in $\Delta_\theta \setminus \mathfrak{N}_{\theta,\sigma}$ converging to $\nu_0 \in \Delta_\theta$, and e be an arbitrary element in $K^E_{\sigma(\nu_0)}$. By Lemma 3.1, we know that $\sigma(\nu_i) \to \sigma(\nu_0)$. Suppose that $\{\sigma(\nu_i)\}_{i\in I}$ is a finite set. By passing to subnet, we can assume that $\sigma(\nu_i) = \sigma(\nu_0)$ $(i \in I)$. As $e(\sigma(\nu_0)) = 0$ and $\nu_i \notin \mathfrak{N}_{\theta,\sigma}$, we have $\theta(e)(\nu_i) = 0$ which gives $\theta(e)(\nu_0) = 0$ and so $\theta(e) \in K^F_{\nu_0}$. Suppose that $\{\sigma(\nu_i)\}_{i\in I}$ is infinite. If there exists $i_0 \in I$ such that $\nu_j \in \mathfrak{U}_\theta$ for every $j \geq i_0$, then we can assume that $\{\sigma(\nu_i)\}_{i\in I} \subseteq \sigma(\mathfrak{U}_\theta)$ which is a finite set, and the above implies that $\theta(e) \in K^F_{\nu_0}$. Otherwise, $\{i \in I : \nu_i \notin \mathfrak{U}_\theta\}$ is cofinal, and by passing to a subnet, we can assume that $\nu_i \notin \mathfrak{U}_\theta$ $(i \in I)$. For any $\epsilon > 0$, pick $V \in \mathcal{N}_\Omega(\sigma(\nu_0))$ and $e_V \in K^E_V$ with $\|e_V - e\| < \epsilon$. When i is large enough, $\sigma(\nu_i) \in V$ and $e_V(\sigma(\nu_i)) = 0$. Thus,

$$\|\theta(e)(\nu_i)\| \ = \ \|\theta(e - e_V)(\nu_i)\| \ \leq \ \kappa\epsilon.$$

By the continuity of the norm function on Ξ^F, we have $\|\theta(e)(\nu_0)\| \leq \kappa\epsilon$ which implies that $\theta(e)(\nu_0) = 0$.

(b) By the hypothesis and Lemma 3.3(a), $\mathfrak{U}_\theta$ is finite. Without loss of generality, we assume that $\Delta \neq \mathfrak{U}_\theta$. Let

$$\kappa \ := \ \sup_{\nu\in\Delta\setminus\mathfrak{U}_\theta} \sup_{\|e\|\leq 1} \|\theta(e)(\nu)\| \ < \ \infty. \tag{3.4}$$

Assume on the contrary that there is $\nu_0 \in \mathfrak{U}_\theta$ which is not an isolated point in Δ. As $\mathfrak{U}_\theta$ is finite, there is a net $\{\nu_i\}$ in $\Delta \setminus \mathfrak{U}_\theta$ such that $\nu_i \to \nu_0$. By the definition of $\mathfrak{U}_\theta$, there is $e \in E$ with $\|e\| \leq 1$ and $\|\theta(e)(\nu_0)\| > \kappa + 1$ and this will contradicts the continuity of $|\theta(e)|$ (because of (3.4)). Now, as $\mathfrak{U}_\theta$ is a finite set consisting of isolated points in Δ and F is the space of C_0-sections on Ξ^F, we see that

$$F \ = \ K^F_{\mathfrak{U}_\theta} \oplus \bigoplus_{\nu\in\mathfrak{U}_\theta} \Xi^F_\nu.$$

By Lemma 3.3(b) and the argument of Lemma 3.3(c) (more precisely, (3.2)), we see θ_0 will satisfy (3.3). On the other hand, the boundedness θ_0 follows from (3.4). □

Note that in both Lemmas 3.3(c) and 3.6(b), one can replace the injectivity of σ with the condition that $\sigma^{-1}(\omega)$ is at most finite for any $\omega \in \Omega$.

The following is our second nearly automatic boundedness result that applies, in particular, when F is a Hilbert $C_0(\Delta)$-module.

Theorem 3.7. *Let Ω and Δ be two locally compact Hausdorff spaces. Let E be an essential Banach $C_0(\Omega)$-module, let F be an essential Banach $C_0(\Delta)$-normed module, and let $\theta : E \to F$ be a $\mathbb{C}$-linear map (not assumed to be bounded). Suppose that $\sigma : \Delta_\theta \to \Omega$ is an injection such that $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ $(\nu \in \Delta)$.*

(a) If Δ contains no isolated point, then θ is bounded.

(b) If σ sends isolated points in Δ_θ to isolated points in Ω, then $\mathfrak{N}_{\theta,\sigma} = \emptyset$ and there exists a finite set T consisting of isolated points of Δ, a bounded linear map $\theta_0 : E_{\Omega\setminus\sigma(T)} \to F_{\Delta\setminus T}$ as well as linear maps $\theta_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ $(\nu \in T)$ such that $E = E_{\Omega\setminus\sigma(T)} \oplus \bigoplus_{\nu\in T} \Xi^E_{\sigma(\nu)}$,

$$F = F_{\Delta\setminus T} \oplus \bigoplus_{\nu\in T} \Xi^F_\nu \quad \text{and} \quad \theta = \theta_0 \oplus \bigoplus_{\nu\in T} \theta_\nu.$$

Proof: (a) This follows directly from Lemma 3.6(b).

(b) The first conclusion follows from Lemma 3.3(c) and the second conclusion follows from Lemma 3.6(b) (note that we have a sharper conclusion because $\mathfrak{N}_{\theta,\sigma} = \emptyset$). □

4. Applications to local linear mappings

In the section, we assume that $\Delta = \Omega$ and $\sigma = \mathrm{id}$. More precisely, we consider the case when the $\mathbb{C}$-linear map θ is a *local map* in the sense that $\theta(e) \cdot \varphi = 0$ whenever $e \in E$ and $\varphi \in C_0(\Omega)$ satisfying $e \cdot \varphi = 0$. It is obvious that any $C_0(\Omega)$-module map is local.

Remark 4.1. *Suppose that θ is local. Let $U, V \subseteq \Omega$ be open sets with the closure of V being a compact subset of U, and consider $\lambda \in \mathfrak{U}_\Omega(V, U)$. Pick any $e \in K^E_U$. For any $\epsilon > 0$, there exists $\varphi \in K_U$ with $\|e - e\varphi\| < \epsilon$. Thus, $e\lambda = 0$ which implies that $\theta(e)\lambda = 0$ and $\theta(e) = \theta(e)(1 - \lambda) \in K^F_V$. This shows that $\sigma = \mathrm{id}$ will satisfy the hypothesis in all the results in Section 3.*

The following theorem (which follows directly from the results in Section 3 as well as Remark 4.1) is our main result concerning local linear maps.

Theorem 4.2. *Let Ω be a locally compact Hausdorff space. Suppose that E and F are essential Banach $C_0(\Omega)$-modules, and $\theta : E \to F$ is a local $\mathbb{C}$-linear map (not assumed to be bounded).*

(a) $\tilde{\theta}$ is a $C_0(\Omega)$-module map and the conclusion of Proposition 3.5 holds.

(b) If, in addition, F is $C_0(\Omega)$-normed, then θ is a $C_0(\Omega)$-module map and the conclusions of Theorem 3.7 hold.

It is natural to ask if one can relax the assumption of F being $C_0(\Omega)$-normed to $C_0(\Omega)$-convex in the second statement of Theorem 4.2 (i.e. whether every $C_0(\Omega)$-module map from an essential Banach $C_0(\Omega)$-module to an essential Banach $C_0(\Omega)$-convex module is automatically bounded provided that Ω contains no isolated point). Unfortunately, it is not the case as can be seen by the following simple example.

Example 4.3. *Let $E := C([0,1]) \oplus^\infty X$ and $F := C([0,1]) \oplus^\infty Y$, where X and Y are two infinite dimensional Banach spaces. Then E is an essential Banach $C([0,1])$-convex module under the multiplication: $(e,x)\cdot\varphi = (e\cdot\varphi, x\varphi(0))$ $(e,\varphi \in C([0,1]); x \in X)$. In the same way, F is an essential Banach $C([0,1])$-convex module. Suppose that $R : X \to Y$ is an unbounded linear map and $\theta : E \to F$ is given by $\theta(e,x) = (e, R(x))$ $(e \in C([0,1]); x \in X)$. Then θ is a $C([0,1])$-module map which is not bounded (as its restriction on X is R). In this case, we have $\mathfrak{U}_\theta = \{0\}$.*

Corollary 4.4. *Let Ω be a locally compact Hausdorff. Any local $\mathbb{C}$-linear θ from an essential Banach $C_0(\Omega)$-module into a Hilbert $C_0(\Omega)$-module is a $C_0(\Omega)$-module map. Moreover, if Ω contains no isolated point, then any such θ is automatically bounded.*

Remark 4.5. *Let $L_{C_0(\Omega)}(E; C_0(\Omega))$ (respectively, $\mathcal{B}_{C_0(\Omega)}(E; C_0(\Omega))$) be the "algebraic dual" (respectively, "topological dual") of E, i.e. the collection of all (respectively, all bounded) $C_0(\Omega)$-module maps from E into $C_0(\Omega)$. An application of Corollary 4.4 is that the algebraic dual and the topological dual of E coincide in many cases:*

> *If Ω is a locally compact Hausdorff space having no isolated point and E is an essential Banach $C_0(\Omega)$-module, then $\mathcal{B}_{C_0(\Omega)}(E; C_0(\Omega)) = L_{C_0(\Omega)}(E; C_0(\Omega))$.*

Corollary 4.6. *Let Ξ and Λ be respectively an (H)-Banach bundle and an (F)-Banach bundle over the same base space Ω. If $\rho : \Xi \to \Lambda$ is a fibrewise linear map (without any boundedness nor continuity assumption) such that $\rho \circ e \in \Gamma_0(\Lambda)$ for every $e \in \Gamma_0(\Xi)$, then there exists a finite subset $S \subseteq \Omega$ consisting of isolated points such that ρ restricts to a bounded Banach bundle map $\rho_0 : \Xi_{\Omega\setminus S} \to \Lambda_{\Omega\setminus S}$.*

Let X be a Banach space. We denote by $\ell^\infty(X)$ and $c_0(X)$ the set of all bounded sequences and the set of all c_0-sequences in X, respectively. We also recall that $\ell^\infty \cong C(\beta\mathbb{N})$ where $\beta\mathbb{N}$ is the Stone-Cech compactification of $\mathbb{N}$ (which can be identified with the collection of all ultrafilters on $\mathbb{N}$).

Proposition 4.7. *Let X and Y be Banach spaces, and let $\theta_k : X \to Y$ $(k \in \mathbb{N} \cup \{\infty\})$ be linear maps (not assumed to be bounded). For any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X, we put $\theta(\{x_k\}_{k\in\mathbb{N}}) := \{\theta_k(x_k)\}_{k\in\mathbb{N}}$.*

(a) If $\theta(c_0(X)) \subseteq c_0(Y)$, then there exists $n_0 \in \mathbb{N}$ such that $\sup_{n\geq n_0} \|\theta_n\| < \infty$.

(b) If $\lim_{k\to\infty} \theta_k(x_k) = \theta_\infty(x)$ *for any* $\{x_k\}_{k\in\mathbb{N}} \in \ell^\infty(X)$ *with* $\lim_{k\to\infty} x_k = x$*, then* θ_∞ *is bounded, and there is* $n_0 \in \mathbb{N}$ *with* $\sup_{n\geq n_0} \|\theta_n\| < \infty$*.*
(c) Suppose that $\theta(\ell^\infty(X)) \subseteq \ell^\infty(Y)$ *and* $\lim_{\mathcal{F}} \theta_n(x_k) = 0$ *for every* $\{x_k\}_{k\in\mathbb{N}} \in \ell^\infty(X)$ *and every ultrafilter* $\mathcal{F}$ *on* $\mathbb{N}$ *with* $\lim_{\mathcal{F}} x_k = 0$*. Then there exist* $\mathcal{F}_1, ..., \mathcal{F}_n \in \beta\mathbb{N}$ *with* $\sup_{\mathcal{F}\neq\mathcal{F}_1,...,\mathcal{F}_n} \|\theta_{\mathcal{F}}\| < \infty$ *(where* $\theta_{\mathcal{F}} : \Xi_{\mathcal{F}}^{\ell^\infty(X)} \to \Xi_{\mathcal{F}}^{\ell^\infty(Y)}$ *is the induced map). In particular,* $\sup_{n\geq n_0} \|\theta_n\| < \infty$ *for some* $n_0 \in \mathbb{N}$*.*

Proof: (a) Let $E = c_0(X)$ and $F = c_0(Y)$. Then θ is a $C_0(\mathbb{N})$-module map and we can apply Theorem 4.2.
(b) Let $E = C(\mathbb{N}_\infty, X)$ and $F = C(\mathbb{N}_\infty, Y)$. Then $\theta \oplus \theta_\infty$ is a well defined $C(\mathbb{N}_\infty)$-module map from E into F and Theorem 4.2 implies this part.
(c) Let $E = \ell^\infty(X)$ and $F = \ell^\infty(Y)$. Then E and F are unital Banach $C(\beta\mathbb{N})$-modules. For any ultrafilter $\mathcal{F} \in \beta\mathbb{N}$, one has

$$K_{\mathcal{F}}^E = \{(x_n) \in E : \lim_{\mathcal{F}} x_n = 0\} \quad \text{and} \quad K_{\mathcal{F}}^F = \{(y_n) \in F : \lim_{\mathcal{F}} y_n = 0\}.$$

The first hypothesis shows that $\theta(E) \subseteq F$ and the second one tells us that $\theta(K_{\mathcal{F}}^E) \subseteq K_{\mathcal{F}}^F$. On the other hand, if $n \in \mathbb{N}$ and $\mathcal{F}_n = \{U \subseteq \mathbb{N} : n \in U\}$, then

$$K_{\mathcal{F}_n}^E = \{(x_k) \in \ell^\infty(X) : x_n = 0\}$$

and so, $\theta_{\mathcal{F}_n} = \theta_n$. Now, the conclusion follows from Theorem 4.2. □

Remark 4.8. *Note that if* $\mathcal{F}$ *is a free ultrafilter on* $\mathbb{N}$*, then* $\Xi_{\mathcal{F}}^{\ell^\infty(X)}$ *and* $\Xi_{\mathcal{F}}^{\ell^\infty(Y)}$ *can be identified with the ultrapowers* $X^{\mathcal{F}}$ *and* $Y^{\mathcal{F}}$ *of* X *and* Y *(over* $\mathcal{F}$*) respectively. One can interpret Proposition 4.7(c) as follows:*

> *If the sequence* $\{\theta_n\}$ *as in Proposition 4.7 induces canonically a map* $\theta : \ell^\infty(X) \to \ell^\infty(Y)$ *as well as a map* $\theta_{\mathcal{F}} : X^{\mathcal{F}} \to Y^{\mathcal{F}}$ *for every free ultrafilter* $\mathcal{F}$ *(none of them assumed to be bounded), then for all but finite number of ultrafilters* $\mathcal{F}$*, the map* $\theta_{\mathcal{F}}$ *is bounded and they have a common bound (in particular, all but finite number of* θ_n *are bounded and they have a common bound).*

It can be shown easily that the converse of the above is also true (but we left it to the readers to check the details):

> *If the sequence* $\{\theta_n\}$ *is as in Proposition 4.7 and there exists* $n_0 \in \mathbb{N}$ *with* $\sup_{n\geq n_0} \|\theta_n\| < \infty$*, then* $\{\theta_n\}$ *induces canonically a map from* $\ell^\infty(X)$ *to* $\ell^\infty(Y)$ *as well as a map from* $X^{\mathcal{F}}$ *to* $Y^{\mathcal{F}}$ *for every free ultrafilter* $\mathcal{F}$*.*

Another important point in Theorem 4.2 is the automatic $C_0(\Omega)$-linearity. In fact, it can be shown that for every C^*-algebra A, any bounded local linear map from a Banach right A-module into a Hilbert A-module is automatically A-linear (see Proposition A.1 in the Appendix). Theorem 4.2 tells us that if A is commutative, then one can have the same conclusion without the boundedness assumption and the range space can be relax to a Banach A-convex module. Another application of this theorem is that if A is a finite dimensional C^*-algebras, then every local linear map between any two Banach right A-modules is A-linear.

Corollary 4.9. *Let A be a finite dimensional C^*-algebra. Suppose that E and F are unital Banach right A-modules. If $\theta : E \to F$ is a local $\mathbb{C}$-linear map in the sense of Proposition A.1 (not assumed to be bounded), then θ is an A-module map.*

Proof: Pick any $x \in E$ and $a \in A_{sa}$. Let $A_a := C^*(a, 1)$. By Remark 2.5(b), both E and F are unital Banach A_a-convex modules. Thus, Theorem 4.2 tell us that θ is a A_a-module map. In particular, $\theta(xa) = \theta(x)a$. □

Remark 4.10. *(a) Suppose that A is any unital C^*-algebra and F is a unital Banach right A-convex module in the sense $\|xa + y(1-a)\| \leq \max\{\|x\|, \|y\|\}$ for any $x, y \in F$ and $a \in A_+$ with $a \leq 1$. Then, by the argument of Corollary 4.9, any local linear map from any unital Banach right A-module into F is automatically A-linear.*

(b) If one can show that for any compact subset $\Omega \subseteq \mathbb{R}$ and any essential Banach $C(\Omega)$-module F, the map $\sim: F \to \tilde{F}$ is injective, then using the argument of Corollary 4.9, one can show that for any C^*-algebra A, any local linear map between any two Banach right A-modules is an A-module map (without assuming that θ is bounded). However, we do not know if it is true.

5. Applications to separating mappings

In this section, we consider Ω and Δ to be possibly different spaces. In this case, one cannot define local property any more but one has a weaker natural property called separating. More precisely, θ is said to be *separating* if

$$|\tilde{\theta}(e)| \cdot |\tilde{\theta}(g)| = 0, \quad \text{whenever } e, g \in E \text{ satisfying } |\tilde{e}| \cdot |\tilde{g}| = 0.$$

In the case when $E = C_0(\Omega)$ and $F = C_0(\Delta)$, this coincides with the well-known notion of disjointness preserving (see e.g. [1, 5, 18, 14, 12, 15]).

Lemma 5.1. *If θ is separating, there is a continuous map $\sigma : \Delta_\theta \to \Omega_\infty$ such that $\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu$ $(\nu \in \Delta_\theta)$.*

Proof: Let

$$S_\nu := \{\omega \in \Omega_\infty : \theta(I^E_\omega) \subseteq I^F_\nu\} \quad (\nu \in \Delta_\theta).$$

Assume that there is $\nu \in \Delta_\theta$ with $S_\nu = \emptyset$. Then for each $\omega \in \Omega_\infty$, there exist $U_\omega \in \mathcal{N}_{\Omega_\infty}(\omega)$ and $e_\omega \in K^E_{U_\omega}$ with $\theta(e_\omega) \notin I^F_\nu$. Let $\{U_{\omega_i}\}_{i=1}^n$ be a finite subcover of $\{U_\omega\}_{\omega \in \Omega_\infty}$ and $\{\varphi_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_{\omega_i}\}_{i=1}^n$. Take any $g \in E$. From

$|\widetilde{g\varphi_i}||\widetilde{e_{\omega_i}}| = 0$, we have $|\tilde{\theta}(g\varphi_i)||\tilde{\theta}(e_{\omega_i})| = 0$ which implies that $\tilde{\theta}(g\varphi_i)(\nu) = 0$ (because of Remark 2.5(a) and the fact that $\theta(e_{\omega_i}) \notin I_\nu^F$). Consequently,

$$\tilde{\theta}(g)(\nu) \;=\; \sum_{i=1}^n \tilde{\theta}(g\varphi_i)(\nu) \;=\; 0,$$

and we obtain the contradiction that $\nu \in \mathfrak{Z}_\theta$. Assume that there is $\nu \in \Delta_\theta$ with S_ν containing two distinct points ω_1 and ω_2. Let $U, V \in \mathcal{N}_{\Omega_\infty}(\omega_1)$ such that $V \subseteq \mathrm{Int}_{\Omega_\infty}(U)$ and $\omega_2 \notin U$. Take $\varphi \in \mathcal{U}_{\Omega_\infty}(V, U)$. For any $e \in E$, we have $e(1-\varphi) \in I_{\omega_1}^E$ and $e\varphi \in I_{\omega_2}^E$ which implies that

$$\theta(e) \;=\; \theta(e(1-\varphi)) + \theta(e\varphi) \;\in\; I_\nu^F.$$

This gives the contradiction that $\nu \in \mathfrak{Z}_\theta$. Therefore, we can define $\sigma(\nu)$ to be the only point in S_ν, and it is clear that $\theta(I_{\sigma(\nu)}^E) \subseteq I_\nu^F$. The continuity of σ follows from Lemma 3.1. □

Corollary 5.2. *Let Ξ be an (H)-Banach bundle over Ω, let Λ be an (F)-Banach bundle over Δ, and let $\rho : \Xi \to \Lambda$ be a map (not assumed to be bounded nor continuous). Suppose that $\sigma : \Delta \to \Omega$ is an injection sending isolated points in Δ to isolated points in Ω such that $e \mapsto \rho \circ e \circ \sigma$ defines a linear map $\theta : \Gamma_0(\Xi) \to \Gamma_0(\Lambda)$. Then there exists a finite set T consisting of isolated points of Δ such that the restriction of ρ induces a bounded Banach bundle map $\rho_0 : \Xi_{\Omega\setminus\sigma(T)} \to \Lambda_{\Delta\setminus T}$. Moreover, σ is continuous on $\Delta \setminus \mathfrak{Z}_{\rho,\sigma}$ where $\mathfrak{Z}_{\rho,\sigma} := \{\nu \in \Delta : \rho(e(\sigma(\nu))) = 0$ for all $e \in E\}$.*

Proof: The first conclusion follows from Theorem 3.7. To see the second conclusion, we note that θ is separating and we can apply Lemma 5.1 (note that $\mathfrak{Z}_{\rho,\sigma} = \mathfrak{Z}_\theta$). □

Theorem 5.3. *Let Ω and Δ be two locally compact Hausdorff spaces, and let E be a full (see Remark 3.2(b)) essential Banach $C_0(\Omega)$ module and F be a full essential Banach $C_0(\Delta)$-normed module. Suppose that $\theta : E \to F$ is a bijective $\mathbb{C}$-linear map (not assumed to be bounded) such that it is* biseparating *in the sense that both θ and θ^{-1} are separating.*

(a) There exists a homeomorphism $\sigma : \Delta \to \Omega$ such that

$$\theta(e \cdot \varphi) \;=\; \theta(e) \cdot \varphi \circ \sigma \qquad (e \in E; \varphi \in C_0(\Omega)).$$

(b) There exists isolated points $\nu_1, ..., \nu_n \in \Delta$ such that the restriction of θ induces a Banach space isomorphism $\theta_0 : E_{\Omega_\theta} \to F_{\Delta_\theta}$, where $\Delta_\theta := \Delta\setminus\{\nu_1, ..., \nu_n\}$ and $\Omega_\theta := \sigma(\Delta_\theta)$.

Proof: (a) If $e \in E$ with $\tilde{e} = 0$, then $\theta(e) = \tilde{\theta}(e) = 0$ (as θ is separating and F is $C_0(\Delta)$-convex) and $e = 0$ (as θ is injective). Hence, one can regard $\widetilde{\theta^{-1}} = \theta^{-1}$ as well.

The fullness of E and F as well as the surjectivity of θ and θ^{-1} ensure that $\mathfrak{Z}_\theta = \emptyset$ and $\mathfrak{Z}_{\theta^{-1}} = \emptyset$. Therefore, by Lemma 5.1, we have continuous maps

$$\tau : \Omega \to \Delta_\infty \quad \text{and} \quad \sigma : \Delta \to \Omega_\infty$$

such that $\theta^{-1}(I^F_{\tau(\omega)}) \subseteq I^E_\omega$ ($\omega \in \Omega$) and $\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu$ ($\nu \in \Delta$). Consequently, for any $\nu \in \Delta_0 := \sigma^{-1}(\Omega)$ and $\omega \in \Omega_0 := \tau^{-1}(\Delta)$, we have

$$\sigma(\tau(\omega)) = \omega \quad \text{and} \quad \tau(\sigma(\nu)) = \nu$$

(because $I^E_{\sigma(\tau(\omega))} \subseteq I^E_\omega$, $I^F_{\tau(\sigma(\nu))} \subseteq I^F_\nu$, and E and F are full). Assume that there exists $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$ ($\mathfrak{N}_{\theta,\sigma}$ as in Lemma 3.3(b)) with $\sigma(\nu) = \infty$. Then $F = \theta(K^E_\infty) \subseteq K^F_\nu$ which contradicts the fullness of F. Thus,

$$\Delta \setminus \mathfrak{N}_{\theta,\sigma} \ \subseteq \ \Delta_0.$$

On the other hand, as $\Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$ is a finite set (by Lemma 3.3(a)&(b) and the fact that σ is injective in Δ_0) that is open in Δ (by Lemma 3.6(a)), we see that $\Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$ consists of isolated points of Δ, and so, $\sigma(\Delta_0 \cap \mathfrak{N}_{\theta,\sigma})$ consists of isolated points of Ω_0 (as σ restricts to a homeomorphism from Δ_0 to Ω_0). We want to show that

$$\Delta_0 \cap \mathfrak{N}_{\theta,\sigma} \ = \ \emptyset.$$

Suppose on the contrary that there is $\nu \in \Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$. We know that $\sigma(\nu)$ ($\neq \infty$) is a non-isolated point of Ω_∞ (by Lemma 3.3(b)). Therefore, there exists a net $\{\omega_i\}_{i\in I}$ in $\Omega \setminus \{\sigma(\nu)\}$ converging to $\sigma(\nu)$. If $\{i \in I : \omega_i \in \Omega_0\}$ is cofinal, then there is a net in $\Omega_0 \setminus \{\sigma(\nu)\}$ converging to $\sigma(\nu)$ which contradicts the fact that $\sigma(\nu)$ is an isolated point in Ω_0. Otherwise, $\omega_i \in \tau^{-1}(\infty)$ eventually which gives the contradiction that $\nu = \infty$ (note that $\tau(\omega_i) \to \nu$ as $\nu \in \Delta_0$). Consequently,

$$\Delta \setminus \mathfrak{N}_{\theta,\sigma} \ = \ \Delta_0.$$

Assume that $\mathfrak{N}_{\theta,\sigma} \neq \emptyset$ and $\nu \in \mathfrak{N}_{\theta,\sigma}$. Since $\mathfrak{N}_{\theta,\sigma}$ is an open subset of Δ (by Lemma 3.6(a)), there exists $V \in \mathfrak{N}_\Delta(\nu)$ such that $V \subseteq \mathfrak{N}_{\theta,\sigma}$. Take any $f \in F$ with $f(\nu) \neq 0$ (by the fullness of F) and f vanishes outside V. Thus, $f \in I^F_\infty$ (as V is compact) and so, $\theta^{-1}(f)(\omega) = 0$ for any $\omega \in \tau^{-1}(\infty)$. On the other hand, for any $\omega \in \Omega_0$, one has $\tau(\omega) \in \Delta_0$ and so, $f \in I^F_{\tau(\omega)}$ (as f vanishes on the open set Δ_0 containing $\tau(\omega)$) which implies that $\theta^{-1}(f)(\omega) = 0$. Hence $\theta^{-1}(f) = 0$ which contradicts the injectivity of θ^{-1}. Therefore, $\mathfrak{N}_{\theta,\sigma} = \emptyset$. Now, part (a) follows from Lemma 3.3(c).

(b) This follows directly from Theorem 3.7(b). □

One can apply the above to the case when F is a full Hilbert $C_0(\Delta)$-module. Another direct application of Theorem 5.3 is the following theorem which extends and enriches a result of Chan [8] (by removing the boundedness assumption on θ), as well as results concerning the product bundle cases discussed in [13, 4]. Notice that if $(\Omega, \{\Xi_x\}, E)$ is a continuous fields of Banach spaces over a locally compact Hausdorff space Ω (as defined in [11, 9]), then E is a full essential Banach $C_0(\Omega)$-normed module.

Theorem 5.4. *Let $(\Omega, \{\Xi_x\}, E)$ and $(\Delta, \{\Lambda_y\}, F)$ be continuous fields of Banach spaces over locally compact Hausdorff spaces Ω and Δ respectively. Let $\theta : E \to F$ be a bijective linear map such that both θ and its inverse θ^{-1} are separating. Then there is a homeomorphism $\sigma : \Delta \to \Omega$ and a bijective linear operator $H_\nu : \Xi_{\sigma(\nu)} \to \Lambda_\nu$ such that*

$$\theta(f)(\nu) \;=\; H_\nu(f(\sigma(\nu))) \quad (f \in E, \nu \in \Delta).$$

Moreover, at most finitely many H_ν are unbounded, and this can happen only when ν is an isolated point in Δ. In particular, if Ω (or Δ) contains no isolated point, then θ is automatically bounded.

Appendix A. Bounded local linear maps are A-linear

Proposition A.1. *Let A be a C^*-algebra, and let θ be a bounded linear map from a Banach right A-modules E into a Hilbert A-module F. Then θ is a right A-module map if and only if θ is* local *(in the sense that $\theta(e)a = 0$ whenever $e \in E$ and $a \in A$ with $ea = 0$).*

Proof. Suppose θ is local. Observe, first of all, that E^{**} and F^{**} are unital Banach A^{**}-modules, and the bidual map $\theta^{**} : E^{**} \to F^{**}$ is a bounded weak*-weak* continuous linear map. Fix $x \in E$ and $a \in A_+$. Let

$$\Phi : C(\sigma(a))^{**} \to A^{**}$$

be the map induced by the canonical normal $*$-homomorphism $\Psi : M(A)^{**} \to A^{**}$. Pick $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$ and define $p := \Phi(\chi_{\sigma(a)\cap(\alpha,\beta)})$. Let $\{f_n\}$ and $\{g_n\}$ be two bounded sequences in $C(\sigma(a))_+$ such that

$$f_n g_n = 0, \quad \text{as well as} \quad f_n \uparrow \chi_{\sigma(a)\cap(\alpha,\beta)} \quad \text{and} \quad g_n \downarrow \chi_{\sigma(a)\setminus(\alpha,\beta)} \quad \text{pointwisely.}$$

Note that as $\Psi(A) \subseteq A$, we have $a_n := \Phi(f_n) \in A$ and we can write $b_n := \Phi(g_n)$ as $c_n + \gamma_n 1$ where $c_n \in A$ and $\gamma_n \in \mathbb{C}$. Fix $n \in \mathbb{N}$. Since a_n and c_n commute, there is a locally compact Hausdorff space Ω with $C^*(a_n, c_n) \cong C_0(\Omega)$. By considering $b_n \in C(\Omega_\infty)_+ \cong C^*(1, a_n, c_n)_+$, one can find a net $\{d_i\}_{i\in I}$ in $C_0(\Omega)_+ \subseteq A_+$ such that $d_i \leq b_n$ $(i \in I)$ and $d_i \to b_n$ pointwisely. As $0 \leq d_i \leq b_n$ and $a_n b_n = 0$ in $C(\Omega_\infty)$, we see that $a_n d_i = 0$. Now, the relation $\theta(xa_n)d_i = 0$ and $\theta(xd_i)a_n = 0$ imply that $\theta^{**}(xa_n)b_n = 0$ and $\theta^{**}(xb_n)a_n = 0$. Since the multiplication in the bidual of the linking algebra of F is jointly weak* continuous on bounded subsets, we see that $\theta^{**}(xp)(1 - p) = 0$ and

$\theta^{**}(x(1-p))p = 0$ which implies that $\theta^{**}(xp) = \theta^{**}(x)p$. Finally, there exists $r_k \in \mathbb{R}$ and $\alpha_k, \beta_k \in \mathbb{R}_+$ such that $\alpha_k \leq \beta_k$ and

$$\sup_{t\in\sigma(a)} \left| a(t) - \sum_{k=1}^{M} r_k \chi_{\sigma(a)\cap(\alpha_k,\beta_k)}(t) \right| \to 0.$$

Thus, by the weak* continuity again, we see that $\theta^{**}(xa) = \theta^{**}(x)a$ as required. □

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