THE BRONOLOGICALLY SURJECTIVE HULL OF AN OPERATOR IDEAL ON LOCALLY CONVEX SPACES

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Abstract. We provided an answer to an open problem of A. Pietsch by giving a direct construction of the bornologically surjective hull $\mathfrak{A}^{\mathrm{bsur}}$ of an operator ideal $\mathfrak{A}$ on *LCS*'s. Discussion of some extension problems of operator ideals were given.

1. Introduction and notations

In his classic [4], A. Pietsch asked for a direct construction of the injective hull $\mathfrak{A}^{\mathrm{inj}}$ and the surjective hull $\mathfrak{A}^{\mathrm{sur}}$ of an operator ideal $\mathfrak{A}$ on *LCS*'s (locally convex spaces) which should be similar to the ones about operator ideals on Banach spaces. L. Franco and Piñeiro [1] answered the problem about injective hulls. In this paper, we shall provide a direct construction of the bornologically surjective hull $\mathfrak{A}^{\mathrm{bsur}}$ of $\mathfrak{A}$ after introducing the notion of bornological surjectivity. We shall discuss the solvability of the original problem about surjective hulls. By the way, the concept of bornological surjectivity was proved to be more interesting and suitable for applications in [6, 7, 8, 9].

Throughout this paper, $\mathfrak{A}$ always denotes an operator ideal on either the class $\mathbb{L}$ of *LCS*'s or $\mathbb{B}$ of Banach spaces in the sense of A. Pietsch [4]. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the underlying scalar field. $X, Y, X_0, Y_0, \dots$ denotes *LCS*'s and $E, F, E_0, F_0, \dots$ denotes Banach spaces. Let N be a normed space, $\cup_N$ always denotes the norm *closed* unit ball of N. $\mathfrak{L}(X, Y)$ denotes the family of all continuous (linear) operator between X and Y. An *injection* means a relatively open and one-to-one continuous operator and a (topological) *surjection* means an open continuous operator. Q in $\mathcal{L}(X, Y)$ is said to be a *bornological surjection* if for every bounded set B in Y there is a bounded set A in X such that $QA = B$. In other words, a topological surjection induces the topology of the range space and a bornological surjection induces the bornology of the range space, cf. [2] for more information. It is easy to see that any bornological surjection from an LCS X onto an infrabarrelled LCS Y is a (topological) surjection, and any surjection from a normed space onto a normed space is a bornological surjection. It is also true that any surjection from a Fréchet space onto a Fréchet–Montel space is a bornological surjection (cf. Wong [8, p. 45]). Let N be an infinite–dimensional normed space and N_σ be the LCS $(N, \sigma(N, N'))$. The canonical map $I : N \to N_\sigma$ is a bornological surjection but not a surjection. See also [5, ex. 4.9 and 4.20], in which a surjection from a Fréchet space onto a Fréchet space is not a bronological surjection. However, in the case of normed spaces there is no difference between these two concepts.

Let $\mathcal{C}$ be either $\mathbb{L}$ or $\mathbb{B}$. An operator ideal $\mathfrak{A}$ on $\mathcal{C}$ is said to be *bornologically surjective* if whenever T is a continuous operator from X into Y and Q is a bornological surjection from X_0 onto X such that $TQ \in \mathfrak{A}(X_0, Y)$, we have $T \in \mathfrak{A}(X, Y)$, where $X, X_0, Y \in \mathcal{C}$. The bornologically surjective hull $\mathfrak{A}^{\mathrm{bsur}}$ of $\mathfrak{A}$ is the intersection of all bornological surjective operator ideals containing $\mathfrak{A}$. Clearly, $\mathfrak{A}^{\mathrm{bsur}}$ is the smallest bornologically surjective operator ideal containing $\mathfrak{A}$. If $\mathcal{C} = \mathbb{B}$, we have $\mathfrak{A}^{\mathrm{bsur}} = \mathfrak{A}^{\mathrm{sur}}$. But, if $\mathcal{C} = \mathbb{L}$ then they are, in general, different objects. The ideal $\mathfrak{L}$ of all continuous operators between LCS's and the ideal $\mathfrak{F}$ of all continuous operators between LCS's of finite rank are both simultanuously surjective and bornologically surjective. However, we have

Example. Let $\mathfrak{K}_p$ be the ideal of precompact operators between LCS's. $\mathfrak{K}_p$ is surjective but not bornologically surjective. In fact, let E be any reflexive Banach space and consider the canonical maps $E \xrightarrow{i} E_\sigma \xrightarrow{id_{E_\sigma}} E_\sigma$. It is clear that i is a bornological surjection (but not a surjection) and $id_{E_\sigma}\ i$ is precompact. However, id_{E_σ} is not precompact unless $\dim E < \infty$. In particular, we have an example that $\mathfrak{A}^{\text{bsur}} \neq \mathfrak{A}^{\text{sur}}$ (even when $\mathfrak{A}$ is surjective).

Example. Let $\mathfrak{K}_p^{\text{loc}}$ be the ideal of all *locally precompact* operators between LCS's. In other words, $\mathfrak{K}_p^{\text{loc}}$ consists of all such continuous operators between LCS's sending bounded sets to precompact sets. $\mathfrak{K}_p^{\text{loc}}$ is clearly bornologically surjective. Using [5, ex. 4.9], we can represent $E = \ell_1$ as a qoutient space of the locally convex space $X = \underset{x \geq 0}{\oplus} E(B(x))$. Here $x \in E$, $x = (x_n) \geq 0$ means $x_n \geq 0$ for all n, and $B(x) = \{y = (y_n) \in E :\mid y_n \mid \leq x_n, n = 1, 2, \ldots\}$. Since $B(x)$ is precompact in E for every $x \geq 0$, the operator $id_E \circ Q$ belongs to $\mathfrak{K}_p^{\text{loc}}(X, E)$, where Q is the quotient map from X onto E. However, id_E is not locally precompact since E is of infinite dimension. This shows that $\mathfrak{K}_p^{\text{loc}}$ is not surjective. In particular, we have an example that $\mathfrak{A}^{\text{bsur}} \neq \mathfrak{A}^{\text{sur}}$ (even when $\mathfrak{A}$ is bornologically surjective).

A subset B of a LCS X is said to be a *disk* if B is *absolutely convex,* i.e., $\lambda B + \beta B \subset B$ whenever $|\lambda| + |\beta| \leq 1$. A disk B is said to be a σ*–disk,* or *absolutely* σ*–convex* if $\underset{n}{\Sigma} \lambda_n b_n$ converges in X and the sum belongs to B whenever $(\lambda_n) \in \ell_1$ and $b_n \in B$, $n = 1, 2, \ldots$. A bounded disk B is said to be *infracomplete* (or a *Banach disk*) if the normed space $X(B) = \underset{\lambda > 0}{\cup} \lambda B$ equipped with the gauge γ_B of B as its norm is complete, where $\gamma_B(x) = \inf\{|\lambda| : x \in \lambda B\}$, for each x in $X(B)$. Any continuous image of a σ–disk or an infracomplete bounded disk is still a σ–disk or an infracomplete bounded disk, respectively. It is well–known that a bounded disk is infracomplete if B is sequentially complete under some locally convex topology which is compatible with the dual pair (X, X'). In particular, if X is quasi–complete then every closed and bounded disk is infracomplete. We call a LCS X to be *infracomplete* if the *von Neumann bornology* $\mathcal{M}_{\mathrm{von}}(X)$, i.e. the original bornology induced by the topology of X, has a basis consisting of infracomplete subsets of X, or equivalently, σ–disked subsets of X. Hence a quasi–complete LCS is infracomplete. The converse is not true, in general, as $(\ell_1, \sigma(\ell_1, \ell_\infty))$ is sequentially complete (because ℓ_1 is the predual of the W^*–algebra ℓ_∞) but not quasi–complete (because $(\ell_1, \|\cdot\|_{\ell_1})$ is not reflexive).

2. Bornologically surjective hulls of operator ideals on LCS's

Let X be a LCS and $\mathcal{D}(X)$ be the family of all bounded disks in X. To each B in $\mathcal{D}(X)$ we associate a *normed* subspace $L_1(B)$ of $\ell_1(B)$ defined by $L_1(B) = \{(\lambda_b)_{b\in B} : \sum_b \lambda_b \cdot b$ converges in $X\}$. In case X is infracomplete, $L_1(B) = \ell_1(B)$. Define X^1 to be the locally convex direct sum $X^1 = \oplus\{L_1(B) : B \in \mathcal{D}(X)\}$ equipped with the direct sum topology. Define $Q_X^1 : X^1 \to X$ by $Q_X^1(\bigoplus_B \lambda_B) = \sum_B \sum_b \lambda_{B,b} \cdot b$ where $B \in \mathcal{D}(X)$ and $\lambda_B = (\lambda_{B,b})_{b\in B} \in L_1(B)$.

Lemma 2.1. *Q_X^1 is a bornological surjection of X^1 onto X.*

Proof. It is apparent that Q_X^1 is linear and surjective. Since the mapping $L_1(B) \to X$ sending $\lambda_B = (\lambda_{B,b})_{b\in B}$ to $\sum_b \lambda_{B,b} \cdot b$ is continuous for each B in $\mathcal{D}(X)$, Q_X^1 is continuous. Moreover, if B is a bounded disk in X then $\cup_{L_1(B)}$ is a bounded disk in X^1 and $Q_X^1(\cup_{L_1(B)}) \supset B$. That is, Q_X^1 is a bornological surjection. □

Lemma 2.2. *Let X and Y be LCS's and $T \in \mathfrak{L}(X,Y)$. Then we have a T_1 in $\mathfrak{L}(X^1, Y^1)$ such that $TQ^1_X = Q^1_Y T_1$.*

Proof. For each B in $\mathfrak{D}(X)$, $TB \in \mathfrak{D}(Y)$. Define $T_B : L_1(B) \to L_1(TB)$ by $T_B(\lambda) = \beta$ where $\lambda = (\lambda_b)_{b\in B}$ and $\beta = (\beta_c)_{c\in TB}$ with $\beta_c = \sum_{\substack{b\in B\\ Tb=c}} \lambda_b$. Note $|\beta_c| \leq \sum_{Tb=c} |\lambda_b| \leq \|\lambda\|_{L_1(B)} < \infty$ and $\|\beta\|_{L_1(TB)} = \Sigma|\beta_c| \leq \Sigma|\lambda_b| = \|\lambda\|_{L_1(B)}$. So T_B is a well–defined continuous operator. We define T_1 in $\mathfrak{L}(X^1, Y^1)$ by the commutative diagrams

$$\begin{array}{ccc} X^1 & \xrightarrow{T_1} & Y^1 \\ \uparrow & & \uparrow \\ L_1(B) & \xrightarrow{T_B} & L_1(TB) \end{array}$$

where the vertical arrows represent the corresponding canonical embeddings and B runs through all members in $\mathfrak{D}(X)$. Finally if $\lambda = \oplus\lambda_B \in X^1$ with $\lambda_B = (\lambda_{B,b})_{b\in B} \in L_1(B)$,

$$\begin{aligned} TQ^1_X(\lambda) &= T\left(\sum_B \sum_{b\in B} \lambda_{B,b} \cdot b\right) \\ &= \sum_B \sum_{b\in B} \lambda_{B,b} \cdot Tb \\ &= \sum_B \sum_{c\in TB} \beta_{TB,c} \cdot c \end{aligned}$$

where $\beta_{TB,c} = \sum_{Tb=c} \lambda_{B,b}$, and

$$\begin{aligned} Q^1_Y T_1(\lambda) &= Q^1_Y \left(\bigoplus_B (\beta_{TB,c})_{c\in TB}\right) \\ &= \sum_B \sum_{c\in TB} \beta_{TB,c} \cdot c. \end{aligned}$$

Hence $TQ^1_X = Q^1_Y T_1$. □

Lemma 2.3. *Let X and Y be LCS's and T be a bornological surjection from X onto Y. Then there is a T_{-1} in $\mathfrak{L}(Y^1, X^1)$ such that $T_1 T_{-1} = id_{Y^1}$.*

Proof. Let $C \in \mathfrak{D}(Y)$. Since T is a bornological surjection there exists a B_C in $\mathfrak{D}(X)$ such that $TB_C = C$. Let δ_C be a (set–theoretical) bijection from C onto a subset $\delta_C(C)$ of B_C such that $T\delta_C(c) = c$ for every c in C. Define T_C from $L_1(C)$ into $L_1(B_C)$ by $T_C(\beta) = \lambda$ where $\beta = (\beta_c)_{c\in C}$ and $\lambda = (\lambda_b)_{b\in B_C}$ with $\lambda_b = \beta_c$ if $b = \delta_C(c)$ for some c in C and $\lambda_b = 0$, otherwise. Clearly T_C is linear. The equalities

$$\|\lambda\| = \sum_{b\in B_C} |\lambda_b| = \sum_{c\in C} |\beta_c| = \|\beta\|$$

say that T_C is continuous. We define a continuous operator T_{-1} from Y^1 into X^1 such that the diagrams

$$\begin{array}{ccc} X^1 & \xleftarrow{T_{-1}} & Y^1 \\ \uparrow & & \uparrow \\ L_1(B_C) & \xleftarrow{T_C} & L_1(C) \end{array}$$

are all commutative for each C in $\mathfrak{D}(Y)$. It is not difficult to see that $T_1T_{-1} = id_{Y^1}$. □

Theorem 2.4. *Let $\mathfrak{A}$ be an operator ideal on LCS's. The bornologically surjective hull of $\mathfrak{A}$ is given by*

$$\mathfrak{A}^{\mathrm{bsur}}(X,Y) = \{T \in \mathfrak{L}(X,Y) : TQ_X^1 \in \mathfrak{A}(X,Y^1)\}$$

for every pair X and Y of LCS's.

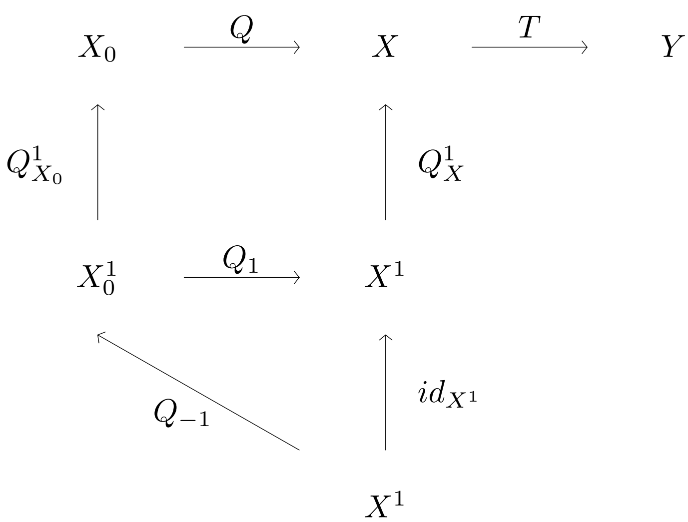
Proof. We first check that $\mathfrak{A}^{\mathrm{bsur}}$ is an operator ideal on LCS's. Let X and Y be $LCS's$. It is obvious that $\mathfrak{A}^{\mathrm{bsur}}(X,Y)$ contains all continuous operators of finite rank from X into Y since $\mathfrak{A}(X,Y^1)$ does. If S and T belong to $\mathfrak{A}^{\mathrm{bsur}}(X,Y)$ then so do $S+T$. Hence $\mathfrak{A}^{\mathrm{bsur}}(X,Y)$ is a nonempty linear subspace of $\mathfrak{L}(X,Y)$. Let $S \in \mathfrak{L}(X_0,X)$, $T \in \mathfrak{A}^{\mathrm{bsur}}(X,Y)$ and $R \in \mathfrak{L}(Y,Y_0)$ for some LCS's X_0, X, Y_0 and

Y. The commutative diagram

$$\begin{array}{ccccc} X^1 & \xrightarrow{Q_X^1} & X & \xrightarrow{T} & Y \\ \uparrow{\scriptstyle S_1} & & \uparrow{\scriptstyle S} & & \downarrow{\scriptstyle R} \\ X_0^1 & \xrightarrow[Q_{X_0}^1]{} & X_0 & \xrightarrow[RTS]{} & Y_0 \end{array}$$

shows that $RTS \in \mathfrak{A}^{\mathrm{bsur}}(X_0, Y_0)$.

Next we check that $\mathfrak{A}^{\mathrm{bsur}}$ is bornologically surjective. Let X_0, X and Y be LCS's, $T \in \mathfrak{L}(X, Y)$ and $Q \in \mathfrak{L}(X_0, X)$ be a bornological surjection such that $TQ \in \mathfrak{A}^{\mathrm{bsur}}(X_0, Y)$. Then $TQ_X^1 = TQ_X^1 id_{X^1} = TQ_X^1 Q_1 Q_{-1} = \left((TQ)Q_{X_0}^1\right) Q_{-1} \in \mathfrak{A}(X^1, Y)$ by the commutative diagram



Hence $T \in \mathfrak{A}^{\mathrm{bsur}}(X, Y)$ and thus $\mathfrak{A}^{\mathrm{bsur}}$ is bornologically surjective.

Finally, if $\mathfrak{A}_0$ is another bornologically surjective operator ideal containing $\mathfrak{A}$ and $T \in \mathfrak{A}^{\mathrm{bsur}}(X, Y)$ for some LCS's X and Y then $TQ_X^1 \in \mathfrak{A}(X^1, Y) \subset \mathfrak{A}_0(X^1, Y)$. The bornological surjectivity of Q_X^1 implies $T \in \mathfrak{A}_0(X, Y)$. Therefore, $\mathfrak{A}^{\mathrm{bsur}} \subset \mathfrak{A}_0$ and thus $\mathfrak{A}^{\mathrm{bsur}}$ is the bornologically surjective hull of $\mathfrak{A}$. $\square$

The following result ensures that we can safely substitute the surjectivity for the bornological surjectivity in many cases. Let N be a normed space. Similar to the case of Banach spaces, we define N^{sur} to be the *normed* space $L_1(\cup_N)$ and $Q_N : N^{\text{sur}} \to N$ to be the surjection defined by $Q_N((\lambda_x)_{x\in\cup_N}) = \sum_x \lambda_x \cdot x$.

Proposition 2.5. *Let $\mathfrak{A}$ be an operator ideal on LCS's, N a normed space, Y a LCS and $T \in \mathfrak{L}(N,Y)$. Then $TQ_N^1 \in \mathfrak{A}(N^1,Y)$ if and only if $TQ_N \in \mathfrak{A}(N^{\text{sur}},Y)$.*

Proof. For each B in $sD(N)$, let $\lambda_B > 0$ such that $B \subset \lambda_B \cup_N$. Associate to each f_B in $L_1(B)$ a g_B in $L_1(\cup_N)$ such that $g_B(b) = \lambda_B f_B(\lambda_B b)$ for all b in $\cup_N$ such that $\lambda_B b \in B$ and $g_B(b) = 0$, otherwise. Define P in $\mathfrak{L}(N^1, N^{\text{sur}})$ by $P(\oplus f_B) = \sum_B g_B$ and $J : N^{\text{sur}} \to N^1$ be the canonical embedding. It is easy to see that $Q_N P = Q_N^1$ and $Q_N = Q_N^1 J$. The desired assertion follows from this. □

Corollary 2.6. *Let $\mathfrak{A}$ be an operator ideal on LCS's and N be a normed space. Then*

$$\mathfrak{A}^{\text{bsur}}(N,Y) \subset \mathfrak{A}^{\text{sur}}(N,Y), \qquad \forall LCS\ Y.$$

They are equal if $\mathfrak{A}$ is surjective.

Proof. Let $T \in \mathfrak{L}(N,Y)$. Observe that

$T \in \mathfrak{A}^{\text{bsur}}(N,Y)$

$\Leftrightarrow TQ_N^1 \in \mathfrak{A}(N^1,Y)$ by Theorem 2.4

$\Leftrightarrow TQ_N \in \mathfrak{A}(N^{\text{sur}},Y)$ by Proposition 2.5

$\Rightarrow T \in \mathfrak{A}^{\text{sur}}(N,Y)$ since Q_N is a surjection.

The asserted equality is trivial. □

Remark. The construction of $\mathfrak{A}^{\text{bsur}}$ is deeply influenced by [1]. A direct construction of $\mathfrak{A}^{\text{sur}}$ of the similar sort for an operator ideal $\mathfrak{A}$ on LCS's seems to be impossible.

For example, locally convex direct sums and quotients of Mackey spaces are still Mackey, see e.g. [3]. It forces us to remain in the category of Mackey spaces.

For comparison and later uses we describe the construction of $\mathfrak{A}^{\text{inj}}$. Let X be a LCS and $\mathcal{E}(X')$ be the collection of all $\sigma(X',X)$–closed and equicontinuous disks in X' and let $X^\infty = \Pi\{\ell_\infty(D) : D \in \mathcal{E}(X')\}$ be the product space equipped with the product topology. Define $J_X^\infty : X \to X^\infty$ by setting $J_X^\infty(x) = (J_{X,D}(x))_{D\in\mathcal{E}(X')}$, where $J_{X,D}(x) \in \ell_\infty(D)$ is a bounded scalar function on D with values $J_{X,D}(x)(d') = \langle x, d'\rangle, \forall d' \in D$.

Theorem 2.7 (Franco and Piñeiro [1])**.** *The map $J_X^\infty \in \mathcal{L}(X, X^\infty)$ is an injection for every LCS X. Let X and Y be LCS's and $T \in \mathcal{L}(X,Y)$. There is a T_∞ in $\mathcal{L}(X^\infty, Y^\infty)$ such that $J_Y^\infty T = T_\infty J_X^\infty$. If, in addition, T is an injection then there is a $T_{-\infty}$ in $\mathcal{L}(Y^\infty, X^\infty)$ such that $T_{-\infty}T_\infty = id_{X^\infty}$. Moreover, the injective hull $\mathfrak{A}^{\text{inj}}$ of an operator ideal $\mathfrak{A}$ on LCS's is given by*

$$\mathfrak{A}^{\text{inj}}(X,Y) = \{T \in \mathcal{L}(X,Y) : J_Y^\infty T \in \mathfrak{A}(X, Y^\infty)\}$$

for every pair X and Y of LCS's.

Associate to each normed space N the Banach space $N^{\text{inj}} = \ell_\infty(\cup_{N'})$ and the injection J_N in $\mathcal{L}(N, N^{\text{inj}})$ defined by $J_N(x) = (< x, a >)_{a\in\cup_{N'}}$. Analogous to Proposition 2.5, we have

Proposition 2.8. *Let $\mathfrak{A}$ be an operator ideal on LCS's, X be a LCS, N be a normed space and $T \in \mathcal{L}(X,N)$. $J_N T \in \mathfrak{A}(X, N^{\text{inj}})$ if and only if $J_N^\infty T \in \mathfrak{A}(X, N^\infty)$.*

Proof. Define π to be the canonical projection from N^∞ onto N^{inj}. Let D be a closed and bounded disk in N' and $\lambda_D > 0$ such that $D \subset \lambda_D \cup_{N'}$. Associate to each f in $\ell_\infty(\cup_{N'})$ an f_D in $\ell_\infty(D)$ such that $f_D(d) = \lambda_D f\left(\frac{d}{\lambda_D}\right)$, $\forall d \in D$. Define a j in $\mathfrak{L}(N^{\text{inj}}, N^\infty)$ by $j(f) = (f_D)_{D \in \mathcal{E}(N')}$. It is easy to see that $jJ_N = J_N^\infty$ and $J_N = \pi J_N^\infty$. The assertion is now clear. □

Proposition 2.9. *Let $\mathfrak{A}$ be an operator ideal on LCS's. We have*

$$(\mathfrak{A}^{\text{bsur}})^{\text{inj}} = (\mathfrak{A}^{\text{inj}})^{\text{bsur}}$$

Proof. Follows easily from Theorems 2.4 and 2.7. □

Proposition 2.10. *Let $\mathfrak{A}$ be an operator ideal on LCS's. We have*

$$(\mathfrak{A}^{\text{inj}})^{\text{sur}} \subset (\mathfrak{A}^{\text{sur}})^{\text{inj}}$$

Proof. By Theorem 2.7, it is easy to see that the injective hull of a surjective operator ideal is still surjective. The asserted inclusion is a direct consequence of this. □

3. Injectivity and surjectivity under extensions

Let $\mathfrak{A}$ be an operator ideal on LCS's. We denote by $\mathfrak{A}_{\mathbb{B}}$ the restriction of $\mathfrak{A}$ to Banach spaces.

Lemma 3.1. *Let $\mathfrak{A}$ be an operator ideal on LCS's. We have*

(a) $(\mathfrak{A}^{\text{inj}})_{\mathbb{B}} = (\mathfrak{A}_{\mathbb{B}})^{\text{inj}}$, *and*

(b) $(\mathfrak{A}^{\text{bsur}})_{\mathbb{B}} = (\mathfrak{A}_{\mathbb{B}})^{\text{sur}} \subset (\mathfrak{A}^{\text{sur}})_{\mathbb{B}}$,

where the injective hull and the surjective hull of $\mathfrak{A}_{\mathbb{B}}$ are, of course, taken within the category of Banach spaces.

Proof. Follows easily from Propositions 2.5 and 2.8 and Corollary 2.7. □

Corollary 3.2. *Let $\mathfrak{A}$ be an operator ideal on Banach spaces.*

(a) *If $\mathfrak{A}$ is injective then $\mathfrak{A}^{\text{sup}}$ is injective, too.*

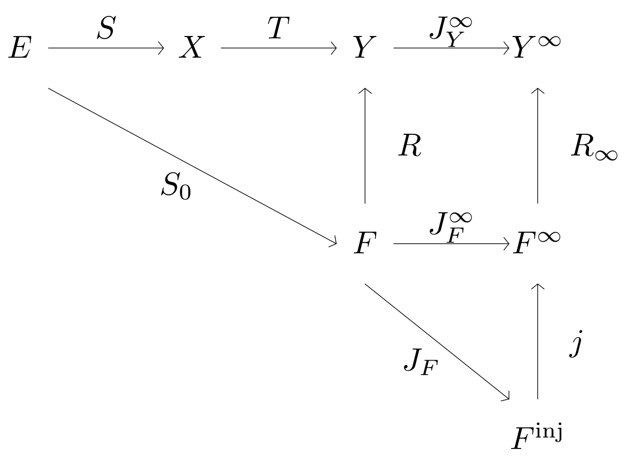
(b) *If $\mathfrak{A}$ is surjective then $\mathfrak{A}^{\text{sup}}$ is bornologically surjective, too.*

Proposition 3.3. *Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We have*

(a) $(\mathfrak{A}^{\text{inj}})^{\text{lup}} \subset (\mathfrak{A}^{\text{lup}})^{\text{inj}}$*; and* $(\mathfrak{A}^{\text{inj}})^{\text{lup}}(X,Y) = (\mathfrak{A}^{\text{lup}})^{\text{inj}}(X,Y)$ *for every LCS X and every sequentially complete LCS Y.*

(b) $(\mathfrak{A}^{\text{sur}})^{\text{rup}} \subset (\mathfrak{A}^{\text{rup}})^{\text{bsur}}$*;* $(\mathfrak{A}^{\text{sur}})^{\text{rup}}(X,Y) = (\mathfrak{A}^{\text{rup}})^{\text{bsur}}(X,Y)$ *for every bornological LCS X and every LCS Y.*

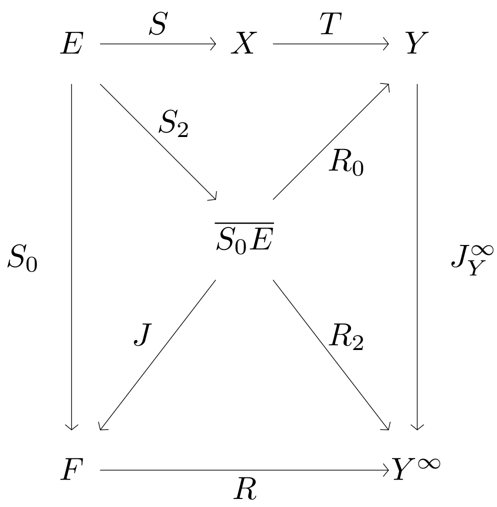
Proof. Let X and Y be LCS's and $T \in \mathfrak{L}(X,Y)$. For (a), assume that $T \in (\mathfrak{A}^{\text{inj}})^{\text{lup}}$ and verify $T \in (\mathfrak{A}^{\text{lup}})^{\text{inj}}$, or equivalently, $J_Y^\infty T \in \mathfrak{A}^{\text{lup}}(X, Y^\infty)$. Let $S \in \mathfrak{L}(E,X)$ where E is a Banach space. Since $T \in (\mathfrak{A}^{\text{inj}})^{\text{lup}}$, we have a Banach space F, an S_0 in $\mathfrak{A}^{\text{inj}}(E,F)$ and an R in $\mathfrak{L}(F,Y)$ such that $TS = RS_0$. Consider the following commutative diagram:



where the map j is defined in Proposition 2.8 and R_∞ is the one in Theorem 2.7 (cf. [1]). Now we have $(J_Y^\infty T)S = (R_\infty j)(J_F S_0)$ and $J_F S_0 \in \mathfrak{A}(E, F^{\text{inj}})$ by Proposition

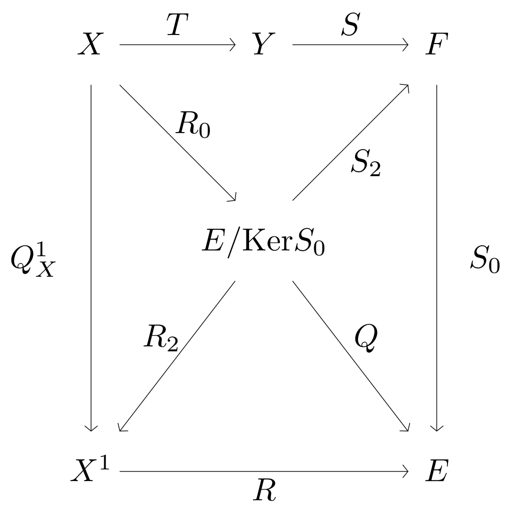
2.8. Thereby, we can infer that $J_Y^\infty T \in \mathfrak{A}^{\mathrm{lup}}(X, Y^\infty)$.

Conversely, assume $T \in (\mathfrak{A}^{\mathrm{lup}})^{\mathrm{inj}}(X, Y)$ and Y is sequentially complete. Let E be a Banach space and $S \in \mathfrak{L}(E, X)$. We have a factorization of $J_Y^\infty TS = RS_0$ for some R in $\mathfrak{L}(F, Y^\infty)$ and S_0 in $\mathfrak{A}(E, F)$ where F is a Banach space. The goal is to establish a similar factorization of TS. Consider the following commutative diagram:



Here J is the natural embedding of the norm closure of the range space S_0E of S_0 into F and S_2 in $\mathfrak{L}(E, \overline{S_0E})$ and R_2 in $\mathfrak{L}(\overline{S_0E}, Y^\infty)$ are the maps induced by S_0 and R, respectively. Since $J_Y^\infty TS = RS_0$, Y is sequentially complete and J_Y^∞ is an injection, we can define an R_0 in $\mathfrak{L}(\overline{S_0E}, Y)$ such that $J_Y^\infty R_0 = R_2$. Now $TS = R_0S_2$ and $S_2 \in \mathfrak{A}^{\mathrm{inj}}(E, \overline{S_0E})$ since $JS_2 = S_0 \in \mathfrak{A}(E, F)$ and J is an injection. It implies that $T \in (\mathfrak{A}^{\mathrm{inj}})^{\mathrm{lup}}(X, Y)$.

The proof of (b) is similar to the above except we shall use the map P defined in the proof of Proposition 2.6 instead of j. For the second part, we refer the readers to the following commutative diagram and ask them to fill in the detail.



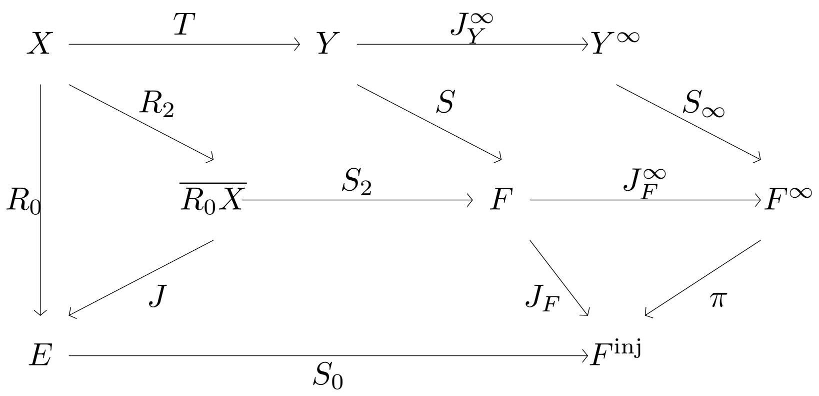
$\square$

Proposition 3.4. *Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We have*

(a) $(\mathfrak{A}^{\mathrm{rup}})^{\mathrm{inj}} \subset (\mathfrak{A}^{\mathrm{inj}})^{\mathrm{rup}}$, *and*

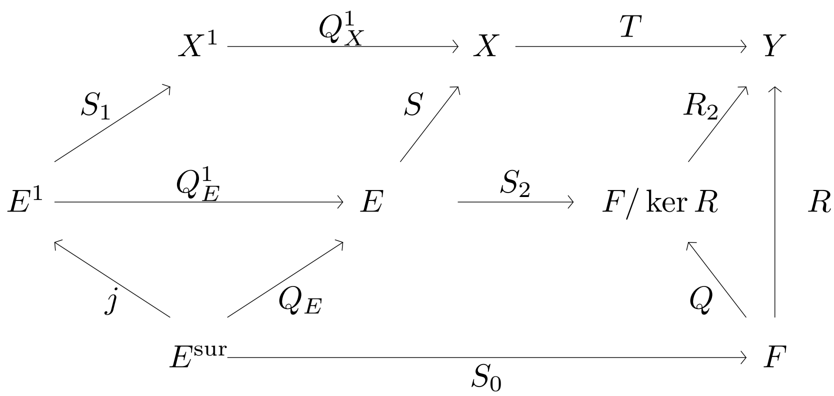
(b) $(\mathfrak{A}^{\mathrm{lup}})^{\mathrm{bsur}} \subset (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{lup}}$.

Proof. (a) follows easily from [4, p. 398] but we would like to provide another proof. Let $T \in (\mathfrak{A}^{\mathrm{rup}})^{\mathrm{inj}}(X, Y)$ and $S \in \mathfrak{L}(Y, F)$ for some LCS's X and Y and Banach space F. We use the following commutative diagram to obtain a factorization of $ST = S_2R_2$ with S_2 in $\mathfrak{A}^{\mathrm{inj}}$ and hence $T \in (\mathfrak{A}^{\mathrm{inj}})^{\mathrm{rup}}$. Note that $J_Y^{\infty}T \in \mathfrak{A}^{\mathrm{rup}}(X, Y^{\infty})$ ensures a factorization of $\pi S_{\infty} J_Y^{\infty} T = S_0R_0$ for some R_0 in $\mathfrak{L}(X, E)$, S_0 in $\mathfrak{A}(E, F^{\mathrm{inj}})$ and Banach space E.



Here J is the canonical embedding from the norm closure $\overline{R_0X}$ of the range space of R_0 in E into E, S_∞ and J_F^∞ are defined in Theorem 2.7 (cf. [1]), π is defined in Proposition 2.8, R_2 and S_2 are induced by R_0 and S_0, respectively. Now $J_F S_2 = S_0 J \in \mathfrak{A}(\overline{RE}, F^{\text{inj}})$, and thus $S_2 \in \mathfrak{A}^{\text{inj}}(\overline{RE}, F)$ by Proposition 2.8, as asserted.

(b) is essentially identical except that we shall use the following commutative diagram instead.



The detail is left to the readers. □

Proposition 3.5. *Let $\mathfrak{A}$ be an operator ideal on Banach spaces and E and F be Banach spaces. Then*

(a) $(\mathfrak{A}^{\text{rup}})^{\text{inj}}(X, F) = (\mathfrak{A}^{\text{inj}})^{\text{rup}}(X, F)$, *and*

(b) $(\mathfrak{A}^{\text{lup}})^{\text{bsur}}(E,Y) = (\mathfrak{A}^{\text{sur}})^{\text{lup}}(E,Y)$ *hold for all LCS's X and Y.*

Proof. We prove (b) only and (a) is similar. In view of Proposition 3.4, it suffices to verify that every T in $(\mathfrak{A}^{\text{sur}})^{\text{lup}}(E,Y)$ belongs to $(\mathfrak{A}^{\text{lup}})^{\text{bsur}}(E,Y)$ whenever E is a Banach space and Y is a LCS. By Proposition 2.5, it is equivalent to that $TQ_E \in \mathfrak{A}^{\text{lup}}(E^{\text{sur}},Y)$. Since E is a Banach space, we have a factorization of $Tid_E = RK$ for some K in $\mathfrak{A}^{\text{sur}}(E,F)$ and R in $\mathfrak{L}(F,Y)$ where F is a Banach space. Now $TQ_E = Tid_EQ_E = RKQ_E$ and $KQ_E \in \mathfrak{A}(E^{\text{sur}},F)$ ensure the assertion. □

Proposition 3.6. *Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We have*

(a) $(\mathfrak{A}^{\text{inj}})^{\text{inf}} \subset (\mathfrak{A}^{\text{inf}})^{\text{inj}}$; $(\mathfrak{A}^{\text{inj}})^{\text{inf}}(X,Y) = (\mathfrak{A}^{\text{inf}})^{\text{inj}}(X,Y)$ *for every LCS X and every infracomplete LCS Y.*

(b) $(\mathfrak{A}^{\text{sur}})^{\text{inf}} \subset (\mathfrak{A}^{\text{inf}})^{\text{bsur}}$; $(\mathfrak{A}^{\text{sur}})^{\text{inf}}(X,Y) = (\mathfrak{A}^{\text{inf}})^{\text{bsur}}(X,Y)$ *for every bornological LCS X and every LCS Y.*

Proof. The inclusions in (a) and (b) follow easily from Lemma 3.1. For (a), let $T \in (\mathfrak{A}^{\text{inf}})^{\text{inj}}(X,Y)$. We need to show that there is a continuous seminorm q on X and a bounded σ–disk B in Y such that the induced map T_{Bq} by T belongs to $\mathfrak{A}^{\text{inj}}(\widetilde{X}_q, Y(B))$. By assumption there is a continuous seminorm q on X and a bounded σ–disk C in Y^∞ such that $J_Y^\infty TV_q \subset C$ and the induced map R from $\widetilde{X}_q$ into $Y^\infty(C)$ by $J_Y^\infty T$ belongs to $\mathfrak{A}(\widetilde{X}_q, Y^\infty(C))$ where $V_q = \{x \in X : q(x) \leq 1\}$. Since J_Y^∞ is an injection, and Y is assumed to be infracomplete, the bounded disk $B = (J_Y^\infty)^{-1}C$ is σ–disked in Y. Moreover, it is clear that $TV_q \subset B$. Let T_{Bq} in $\mathfrak{L}(\widetilde{X}_q, Y(B))$ and S in $\mathfrak{L}(Y(B), Y^\infty(C))$ be the maps induced by T and J_Y^∞, respectively. Since $ST_{Bq} = R$ belongs to $\mathfrak{A}(\widetilde{X}_q, Y^\infty(C))$ and S is an injection, $T_{Bq} \in \mathfrak{A}^{\text{inj}}(\widetilde{X}_q, Y(B))$, as asserted. For (b), let $T \in (\mathfrak{A}^{\text{inf}})^{\text{bsur}}(X,Y)$. We want to verify that $T \in (\mathfrak{A}^{\mathrm{bsur}})^{\mathrm{inf}}$. By assumption, TQ_X^1 has a factorization $TQ_X^1 = ST_0R$ for some R in $\mathfrak{L}(X^1, E)$, S in $\mathfrak{L}(F, Y)$ and T_0 in $\mathfrak{A}(E, F)$, where E and F are Banach spaces. Let $E_0 = E/\mathrm{Ker}ST_0$ and Q be the quotient map from E onto E_0. Define a linear operator R_0 from X into E_0 by the relation $R_0x = QRy$ where $Q_X^1 y = x$. Since Q_X^1 is bornologically surjective, R_0 is locally bounded. R_0 is thus continuous as X is assumed to be bornological. Let T_2 in $\mathfrak{L}(E_0, F)$ be the map induced by T_0. $T_2Q = T_0 \in \mathfrak{A}$ implies $T_2 \in \mathfrak{A}^{\mathrm{sur}}$. Now, $T = ST_2R_0 \in (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inf}}$, as asserted. □

We leave the cases of left and right inferior extensions of operator ideals on Banach spaces to interested readers.

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