THE STRUCTURE OF COMPACT DISJOINTNESS PRESERVING OPERATORS ON CONTINUOUS FUNCTIONS

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Abstract. Let T be a compact disjointness preserving linear operator from $C_0(X)$ into $C_0(Y)$, where X and Y are locally compact Hausdorff spaces. We show that T can be represented as a norm convergent countable sum of disjoint rank one operators. More precisely, $T = \sum_n \delta_{x_n} \otimes h_n$ for a (possibly finite) sequence $\{x_n\}_n$ of distinct points in X and a norm null sequence $\{h_n\}_n$ of mutually disjoint functions in $C_0(Y)$. Moreover, we develop a graph theoretic method to describe the spectrum of such an operator.

1. Introduction

In the setting of Banach lattices, a linear operator is a lattice homomorphism if and only if it is positive and disjointness preserving (see, for example, [2, p. 88]). In [1], Abramovich developed the basic theory of such operators. The concept of disjointness preserving operators has been widely studied in the setting of continuous functions as a good test case (see, e.g., [17, 3, 9, 6, 11]).

Let X and Y be locally compact Hausdorff spaces, and let $C_0(X)$ be the Banach algebra of continuous (real or complex) functions on X vanishing at infinity. In this paper, we discuss *disjointness preserving* (linear) operators T from $C_0(X)$ into $C_0(Y)$; namely, $Tf \cdot Tg = 0$ in $C_0(Y)$ whenever $f \cdot g = 0$ in $C_0(X)$. Hence a disjointness preserving operator preserves disjointness of cozeros of functions. Here the *cozero* of f in $C_0(X)$ is defined to be the open set $\operatorname{coz} f = \{x \in X : f(x) \neq 0\}$.

In case X is a *compact* Hausdorff space, utilizing the Arzela–Ascoli Theorem, Kamowitz [12, 13] showed that every compact algebraic endomorphism, and indeed every compact

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disjointness preserving operator, of $C(X)$ is of finite rank. For disjointness preserving operators on locally compact spaces, however, the theory is much richer.

In Section 2, with a new approach, we show that every compact disjointness preserving operator T from $C_0(X)$ into $C_0(Y)$ is indeed infinite nuclear, that is, carrying a countably infinite or finite sum decomposition $T = \sum_n \delta_{x_n} \otimes h_n$ of disjoint rank one operators (Theorem 2.6). As a consequence, compactness, weak compactness and complete continuity are equivalent for disjointness preserving operators on continuous functions. However, we note that this equivalences do not hold in the case of vector-valued functions as shown in [10].

In Section 3, we develop a spectral theory for compact disjointness preserving operators on $C_0(X)$. By using our characterization of such operators $T = \sum_n \delta_{x_n} \otimes h_n$ on $C_0(X)$, we associate to T a graph G with countably many vertices $\{\infty, x_1, x_2, \ldots\}$. The structure of G gives rise to a complete description of eigenvalues and eigenfunctions of T (Theorem 3.5). Our results extend and generalize those of Kamowitz [13] and Uhlig [16], and also apply to power compact bounded disjointness preserving operators of $C_0(X)$ in [14].

In Section 4, we provide sufficient and necessary conditions for a disjointness preserving operator between spaces of continuous functions to be compact. In line with the Bartle-Dunford-Schwartz Theorem [4], Corollary 4.3 states that T is compact if and only if the image of the dual unit ball of the range under the dual map T^* of T is dominated by a positive atomic measure. Moreover, if the image of T^* is controlled by a positive atomic measure whose support is discrete and has compact closure in $X \cup \{\infty\}$, Corollary 4.4 ensures that T is compact. In [7, 8], Jarchow showed that every weakly compact linear operator T from $C_0(X)$, or more generally a C*-algebra, into a Banach space F can be uniformly approximated by operators which factor through a Hilbert space. Some of his tools rely on controlling and dominating measures. We hope our new results can be used to lead to a better understanding of such operators that also preserve disjointness.

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2. A structure theorem for compact disjointness preserving operators

Let X be a locally compact Hausdorff space with one point compactification $X \cup \{\infty\}$. We find the following identification useful:

$$C_0(X) = \{f \in C(X \cup \{\infty\}) : f(\infty) = 0\}.$$

Let T be a bounded disjointness preserving linear operator from $C_0(X)$ into $C_0(Y)$. Let δ_y be the point mass at y in $Y \cup \{\infty\}$, and thus $\delta_y \circ T(f) = Tf(y)$ for all f in $C_0(X)$. Set

$$Y_\infty = \{y \in Y \cup \{\infty\} : \delta_y \circ T = 0\} \quad \text{and} \quad Y_T = \{y \in Y \cup \{\infty\} : \delta_y \circ T \neq 0\}.$$

For each y in Y_T, the linear functional $\delta_y \circ T$ defined by $\delta_y \circ T(f) = Tf(y)$ is nonzero. By the disjointness preserving property, the support of $\delta_y \circ T$ consists of a single point x in X. Set $\varphi(y) = x$. By Urysohn's Lemma, we have $\ker \delta_{\varphi(y)} \subseteq \ker \delta_y \circ T$. Hence $\delta_y \circ T = h(y)\delta_{\varphi(y)}$ for some nonzero scalar $h(y)$. Consequently, $Tf(y) = h(y)f(\varphi(y))$ for all y in Y_T. As a result, T is a weighted composition operator

$$Tf|_{Y_T} = h \cdot f \circ \varphi \quad \text{and} \quad Tf|_{Y_\infty} = 0. \tag{2.1}$$

It is then routine to see that h is a non-vanishing scalar continuous function on the open set Y_T and $\varphi : Y_T \to X$ is continuous (see, e.g., [9, 6, 11]).

In this section, we assume that $T : C_0(X) \to C_0(Y)$ is a *completely continuous* disjointness preserving linear operator of the form (2.1). In other words, the operator T sends weak null sequences in $C_0(X)$ to norm null sequences in $C_0(Y)$. It is clear that compact operators are weakly compact and completely continuous. For disjointness preserving linear operators between continuous functions, we shall see in the following Theorem 2.6 that the converses do hold. We start with an elementary result.

Lemma 2.1. *For $n \in \mathbb{N}$, let f_n be in $C_0(X)$ with $\|f_n\| = 1$. If $f_n f_m = 0$ for $n \neq m$, then $f_n \to 0$ weakly.*

Lemma 2.2. *For $n \in \mathbb{N}$, let $x_n = \varphi(y_n)$ be distinct points in $\varphi(Y_T)$. Then $\lim_{n\to\infty} h(y_n) = 0$.*

Proof. We first make an elementary observation.

Claim. For any $\{u_n\}$ of countably infinitely many distinct points in X, there is at most one point u in X such that every neighborhood of u contains all but finitely many u_n's.

Indeed, if z is an other such point in X, then each neighborhood of z contains infinitely many u_n's and thus intersects with every neighborhood of u. Since X is Hausdorff, $z = u$.

We now lead a proof by contradiction. Suppose there was an $\epsilon > 0$ such that $|h(y_n)| \geq \epsilon$ for all n's. Using the Claim and passing to a subsequence if necessary, we can assume that each x_n has a compact neighborhood V_n such that $V_n \cap V_m = \emptyset$ whenever $n \neq m$. Choose f_n in $C_0(X)$ such that $\operatorname{coz} f_n \subseteq V_n$, $f_n(x_n) = 1$ and $0 \leq f_n \leq 1$. Then $f_n f_m = 0$ whenever $n \neq m$. By Lemma 2.1, $f_n \to 0$ weakly. Since T is completely continuous, $Tf_n \to 0$ in norm. But

$$\|Tf_n\| \geq |Tf_n(y_n)| = |h(y_n)f_n(x_n)| = |h(y_n)| \geq \epsilon,$$

a contradiction. □

Lemma 2.3. *For each x in $\varphi(Y_T)$, $\varphi^{-1}(x)$ is an open subset of Y.*

Proof. Suppose $\varphi^{-1}(x)$ was not open in Y, and thus not relatively open in the open set Y_T, either. Let $y_\lambda \in Y_T \backslash \varphi^{-1}(x)$ such that $y_\lambda \to y$ in $\varphi^{-1}(x) \subseteq Y_T$. Then $\lim_{\lambda\to\infty} h(y_\lambda) = h(y) \neq 0$. By Lemma 2.2, the range of the net $\{\varphi(y_\lambda)\}_\lambda$ consists of only finitely many points in X. However, $x_\lambda = \varphi(y_\lambda) \to \varphi(y) = x$. This forces $x_\lambda = x$ for all λ eventually, a contradiction. Hence $\varphi^{-1}(x)$ is open in Y. □

For each x in $\varphi(Y_T)$, let

$$\begin{aligned} Y_x = \varphi^{-1}(x) &= \{y \in Y_T : \varphi(y) = x\} \\ &= \{y \in Y \cup \{\infty\} : \ker \delta_y \circ T = \ker \delta_x\}. \end{aligned}$$

In comparison, we remark that

$$\begin{aligned} Y_\infty &= \{y \in Y \cup \{\infty\} : \delta_y \circ T = 0\} \\ &= \{y \in Y \cup \{\infty\} : \ker \delta_y \circ T = \ker \delta_\infty = C_0(X)\}. \end{aligned}$$

Note that $Y_T = \bigcup_{x \in \varphi(Y_T)} Y_x$ is a disjoint union.

Let $h_x = h\chi_{Y_x}$, where χ_{Y_x} is the characteristic function of the set Y_x. Then $h_x h_{x'} = 0$ whenever $x \neq x'$ in $\varphi(Y_T)$.

Corollary 2.4. *Each Y_x is relatively closed and open in Y_T. For each x in $\varphi(Y_T)$, h_x is in $C_0(Y)$. More precisely, h_x can be extended continuously to $Y \cup \{\infty\} = Y_\infty \cup Y_T$ by setting*

$$h_x|_{Y_\infty} = 0.$$

Proof. It is clear that Y_x is relatively closed and open in Y_T by Lemma 2.3. Thus h_x is continuous on Y_T. Let $\{y_\lambda\}_\lambda$ be a net in Y_x such that $y_\lambda \to y$ for some y in Y_∞. If $h_x(y_\lambda)$ does not converge to 0 then, passing to a subnet if necessary, there is an $\epsilon > 0$ such that

$$|Tf(y_\lambda)| = |h(y_\lambda)f(\varphi(y_\lambda))| = |h_x(y_\lambda)f(x)| \geq \epsilon|f(x)|,$$

for all λ and all f in $C_0(X)$. Hence $|Tf(y)| \geq \epsilon|f(x)|$. Since $y \in Y_\infty$, we have $Tf(y) = 0$, and hence $f(x) = 0$ for all f in $C_0(X)$, a contradiction. Therefore, h_x can be extended continuously to $Y \cup \{\infty\}$ by setting $h_x|_{Y_\infty} = 0$. □

Lemma 2.5. *For $n = 1, 2, \ldots$, the set $\{x \in \varphi(Y_T) : \|h_x\| > \frac{1}{n}\}$ is finite. Thus, $\varphi(Y_T)$ is a countable set. Moreover, if there are infinitely many distinct points x_n in $\varphi(Y_T)$, then $\|h_{x_n}\| \to 0$.*

Proof. Suppose our assertion was not true, then there exist $n \in \mathbb{N}$ and infinitely many distinct $x_1, x_2, \ldots$ in $\varphi(Y_T)$ such that $\|h_{x_k}\| > \frac{1}{n}$ for all k. For each k, let $y_k \in Y_T$ such that $|h_{x_k}(y_k)| > \frac{1}{n}$ and thus $\varphi(y_k) = x_k$. But by Lemma 2.2, $\lim_{k\to\infty} h_{x_k}(y_k) = 0$, a contradiction. Hence the set $\{x \in \varphi(Y_T) : \|h_x\| > \frac{1}{n}\}$ is finite. Consequently,

$$\varphi(Y_T) = \bigcup_{n=1}^{\infty} \left\{x \in \varphi(Y_T) : \|h_x\| > \frac{1}{n}\right\}$$

is countable. □

Now we are ready for a structure theory of compact disjointness preserving operators.

Theorem 2.6. *Let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving linear operator. Then the following assertions are equivalent.*

(i) *T is compact;*
(ii) *T is weakly compact;*
(iii) *T is completely continuous;*

(iv) *There are at most countably many distinct points* $\{x_n\}$ *in* X *and mutually disjoint functions* $\{h_n\}$ *in* $C_0(Y)$ *such that*

$$Tf = \sum_n f(x_n)h_n \quad \textit{for all} \quad f \in C_0(X).$$

In case there are infinitely many such x_n *and* h_n*, we have* $\|h_n\| \to 0$ *and thus the sum converges uniformly.*

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) are clear. The implication (ii) ⇒ (iii) follows from the well known fact that $C_0(X)$ has the Dunford-Pettis Property (see, e.g., [5, p. 494]); that is, every weakly compact operator T from $C_0(X)$ into a Banach space F is completely continuous. Indeed, T is weakly compact on $C_0(X)$ if and only if T is completely continuous by [15, Theorem 12]. For (iv) ⇒ (i), we note that as a countable sum of rank one operators, T is compact.

Finally, for the implication (iii) ⇒ (iv), we assume that T is completely continuous. In view of Lemma 2.5, we can write $\varphi(Y_T) = \{x_1, x_2, \ldots\}$ (this set can be finite). Each $Y_n = \varphi^{-1}(x_n)$ is relatively closed and open in the open set Y_T by Lemma 2.3. Let $h_n = h\chi_{Y_n}$ for all $n = 1, 2, \ldots$, we have $h_n h_m = 0$ for all $n \neq m$. Moreover, $h_n \in C_0(Y)$ by Corollary 2.4, and $\|h_n\| \to 0$ by Lemma 2.5. Observe that for each x_n in $\varphi(Y_T)$ we have $(h \cdot f \circ \varphi)\chi_{Y_n} = f(x_n)h_n$. Hence

$$Tf|_{Y_T} = h \cdot f \circ \varphi = \sum_n (h \cdot f \circ \varphi)\ \chi_{Y_n} = \sum_n f(x_n)h_n.$$

By Corollary 2.4 and Lemma 2.5, we can even write

$$Tf = \sum_n f(x_n)h_n,$$

where the sum converges uniformly on Y. □

3. A spectral theory for compact disjointness preserving operators

In this section, denote by X a locally compact Hausdorff space of infinite cardinality and by $T : C_0(X) \to C_0(X)$ a compact disjointness preserving complex linear operator. Here we have $X = Y$ comparing to the previous section, and in particular,

$$X_\infty = \{x \in X \cup \{\infty\} : \delta_x \circ T = 0\} \quad \text{and} \quad X_T = X \setminus X_\infty.$$

By Theorem 2.6, we have

$$Tf = \sum_n f(x_n)h_n \quad \text{for all} \quad f \in C_0(X).$$

Here,

$$X_n = \operatorname{coz} h_n$$

is a relatively open and closed subset of the open set X_T for all $n = 1, 2, \ldots$. Note that we always have $\infty \in X_\infty$ and that $X_T = \bigcup_{n\geq 1} X_n$ is a countable disjoint union.

We now define a graph G in $X \cup \{\infty\}$ associated to T with the vertex set $\{\infty, x_1, x_2, \ldots\}$. We assign a directed edge from x_m to x_n, denoted by $x_m \succ\!\!\!-\, x_n$, whenever $x_n \in X_m$. In this case, we say that x_m is a *parent* of x_n (or x_n is a *child* of x_m). We call x_m an *ancestor* of x_n, if x_m is a parent of x_n, or in case there are finitely many vertices $x_m = x_{m_1}, x_{m_2}, \ldots,$ $x_{m_n} = x_n$ in G such that x_{m_j} is a parent of $x_{m_{j-1}}$ for all $1 \leq j \leq n-1$. Note that every vertex in G has a unique parent but it may have many children or no child at all, and ∞ is always the parent of itself. A *branch* B is a maximal connected family in G. Clearly, two vertices x_n, x_m are in the same branch in G if and only if they have a common ancestor. The *territory* X_B of a branch B of G is defined to be the open set

$$X_B = \bigcup_{x_n \in B} X_n = \bigcup_{x_n \in B} \operatorname{coz} h_n.$$

Then

$$X_T = \bigcup_B X_B$$

is a countable disjoint union of open sets. Write

$$T = \sum_B T_B, \tag{3.1}$$

where

$$T_B f = Tf|_{X_B} = \sum_{x_n \in B} f(x_n)h_n \quad \text{for all} \quad f \in C_0(X).$$

By Theorem 2.6, each T_B is again a compact disjointness preserving linear operator on $C_0(X)$, and the graph of T_B is $B \cup \{\infty\}$. Note that $T_B f = 0$ whenever $\operatorname{coz} f \cap X_B = \emptyset$. Moreover,

$$\operatorname{coz} T_B f \subseteq X_B \quad \text{for all} \quad f \in C_0(X). \tag{3.2}$$

Hence T_{B_1}, T_{B_2} are disjoint whenever $B_1 \neq B_2$, i.e.,

$$T_{B_1} T_{B_2} = T_{B_2} T_{B_1} = 0;$$

moreover,

$$T_{B_1} f \cdot T_{B_2} g = 0 \quad \text{for any} \quad f, g \in C_0(X).$$

By Lemma 2.5, the disjoint sum in (3.1) is norm convergent.

Lemma 3.1. *The spectrum of T is the union of the spectra of T_B for all branches B of the graph of T. That is,*

$$\sigma(T) = \bigcup_B \sigma(T_B).$$

Proof. We first note that 0 belongs to both sides of the equality since a compact operator of the infinite dimensional space $C_0(X)$ cannot be invertible. Let $\lambda \in \sigma(T) \setminus \{0\}$. Then there is a nonzero f in $C_0(X)$ with $\lambda f = Tf = \sum_n f(x_n) h_n$. In particular, we have a decomposition $f = \sum_B f_B$, where

$$f_B = \frac{1}{\lambda} \sum_{x_n \in B} f(x_n) h_n.$$

Clearly, $\operatorname{coz} f_B \subseteq \bigcup_{x_n \in B} \operatorname{coz} h_n = X_B$. Note also that the (possibly finite) disjoint sum converges uniformly on X by Lemma 2.5. Since $f \neq 0$, there is at least one $f_B \neq 0$. It follows from $T_B f_B = T_B f = Tf|_{X_B} = \lambda f_B$ that $\lambda \in \sigma(T_B)$.

On the other hand, suppose $T_B f = \lambda f \neq 0$. For any other branch $B' \neq B$, we have $T_{B'} f = \frac{1}{\lambda} T_{B'} T_B f = 0$. Consequently, $Tf = \lambda f$ and $\lambda \in \sigma(T)$. □

Definition 3.2. A vertex x in G is called *noble* if it has infinitely many ancestors or it has ∞ as an ancestor. It is easy to see that, if a branch has a noble vertex, all its vertices are noble. We call such a branch a *noble branch.* A branch is called *active* if it is not noble.

Lemma 3.3. *For each noble branch B of G, we have $\sigma(T_B) = \{0\}$.*

Proof. Suppose on the contrary that there was a nonzero eigenvalue λ of T_B such that

$$\lambda f = T_B f = \sum_{x \in B} f(x) h_x \text{ for some } f \neq 0 \text{ in } C_0(X).$$

Then there is $z_1 \neq \infty$ in B such that $f(z_1) \neq 0$ and $f(z_2)h_{z_2}(z_1) = \lambda f(z_1) \neq 0$, where $\infty \neq z_2 \succ\!\!-\!\!- z_1$, and we have $f(z_2) \neq 0$. Similarly, $\lambda f(z_2) = f(z_3)h_{z_3}(z_2) \neq 0$, where $\infty \neq z_3 \succ\!\!-\!\!- z_2 \succ\!\!-\!\!- z_1$ and $f(z_3) \neq 0$. Continuing this process, we have

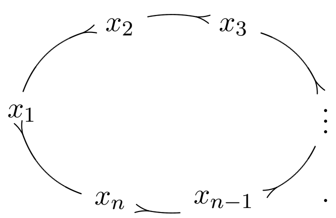
$$\lambda f(z_n) = f(z_{n+1})h_{z_{n+1}}(z_n) \neq 0,$$

where $z_{n+1} \succ\!\!-\!\!- z_n \succ\!\!-\!\!- \cdots \succ\!\!-\!\!- z_2 \succ\!\!-\!\!- z_1$ is a chain in the noble branch B consisting of distinct vertices none of which is ∞. On the other hand,

$$\begin{aligned}
\lambda^n f(z_1) &= \lambda^{n-1} f(z_2)h_{z_2}(z_1) \\
&= \lambda^{n-2} f(z_3)h_{z_3}(z_2)h_{z_2}(z_1) \\
&= \cdots \\
&= f(z_{n+1})h_{z_{n+1}}(z_n)\cdots h_{z_2}(z_1).
\end{aligned}$$

Consequently, $|f(z_1)| = |f(z_{n+1})\frac{h_{z_{n+1}}(z_n)}{\lambda}\cdots\frac{h_{z_2}(z_1)}{\lambda}| \leq \|f\|\frac{\|h_{z_{n+1}}\|}{|\lambda|}\cdots\frac{\|h_{z_2}\|}{|\lambda|}$. As all z_n are distinct, $\|h_{z_n}\| \to 0$ as $n \to \infty$ by Lemma 2.5. It forces $f(z_1) = 0$, a contradiction. Hence $\sigma(T_B) = \{0\}$. $\square$

In an active branch B of the graph G of T, every vertex has finitely many ancestors and none of which is ∞. It is not difficult to see that there are finitely many vertices $x_1, x_2, \ldots, x_n$, say, in B such that x_{k+1} is the parent of x_k for $k = 1, 2, \ldots, n-1$ and x_1 is the parent of x_n. We say that they form *the (unique) primitive cycle* of B and denote it by $[x_1; x_2; \ldots; x_n]$, which can be depicted as



Definition 3.4. Suppose B is an active branch of the graph G of T. Let $[x_1; x_2; \ldots; x_n]$ be its primitive cycle. All vertices in the cycle are said to be of *the* 0*th generation.* A vertex in B which is not in the primitive cycle is said to be of *the first generation* if it is a child of a 0th generation vertex. For $m \geq 1$, a vertex in B is said to be of *the* $(m+1)$*th generation* if it is a child of an mth generation vertex.

We now introduce a grading to B. Write

$$B = B_0 \cup B_1 \cup B_2 \cup \cdots , \tag{3.3}$$

where

$$B_0 = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad B_m = \{x_{m1}, x_{m2}, \ldots\} \tag{3.4}$$

are sets of all 0th generation and mth generation vertices for $m \geq 1$, respectively. Similarly, we decompose the territory X_B of B into a possibly finite disjoint union

$$X_B = X_{B_0} \cup X_{B_1} \cup X_{B_2} \cup \cdots ,$$

where the subterritory is defined by

$$X_{B_k} = \bigcup \{\operatorname{coz} h_x : x \text{ is a } k\text{th generation vertex in } B\}$$

for $k = 0, 1, 2, \ldots$.

Suppose f is an eigenfunction of T_B associated with a nonzero eigenvalue λ arising from an active branch B of the graph of T. We have

$$\lambda f = T_B f = \sum_{x_i \in B} f(x_i) h_i.$$

Using (3.3) and (3.4), we write

$$f = \sum_{x_i \in B} \frac{1}{\lambda} f(x_i) h_i = \sum_{j=1}^{n} \frac{1}{\lambda} f(x_j) h_j + \sum_{i \geq 1} \sum_{x_{ij} \in B_i} \frac{1}{\lambda} f(x_{ij}) h_{ij}.$$

We can thus decompose f as a disjoint sum

$$f = f_0 + f_1 + f_2 + \cdots , \tag{3.5}$$

where

$$f_0 = \sum_{j=1}^{n} \frac{1}{\lambda} f(x_j) h_j \quad \text{and} \quad f_i = \sum_{x_{ij} \in B_i} \frac{1}{\lambda} f(x_{ij}) h_{ij}$$

have cozeros contained in X_{B_i} for each $i = 0, 1, 2, \ldots$. Note that the sum in (3.5) converges in norm by Lemma 2.5. We call (3.5) the *generation decomposition* of the eigenfunction f. Moreover, we have

$$\operatorname{coz} T_B f_0 \subseteq X_{B_0} \cup X_{B_1} \quad \text{and} \quad \operatorname{coz} T_B f_n \subseteq X_{B_{n+1}} \quad \text{for} \quad n \geq 1. \tag{3.6}$$

Remark that $h_m(x_n) \neq 0$ exactly when $x_m \succ\!\!\!- x_n$. Hence

$$Th_m = \sum_{x_m \succ\!\!\!- x_n} h_m(x_n)h_n. \tag{3.7}$$

Theorem 3.5. *Let T be a compact disjointness preserving complex linear operator on $C_0(X)$. A nonzero complex number λ is an eigenvalue of T if and only if there is an active branch B of the graph of T with the primitive cycle $[x_1; \ldots; x_n]$ such that*

$$\lambda^n = h_1(x_n)h_2(x_1)\cdots h_n(x_{n-1}).$$

Here, we assume $T_B = \sum_i \delta_{x_i} \otimes h_i$ as in Theorem 2.6. In this case, a nonzero eigenfunction f of T associated to λ with a generation decomposition $f = f_0 + f_1 + f_2 + \cdots$ can be constructed as follows.

$$f_0 = \frac{h_n(x_{n-1})\cdots h_3(x_2)h_2(x_1)}{\lambda^{n-1}}h_1 + \frac{h_n(x_{n-1})\cdots h_3(x_2)}{\lambda^{n-2}}h_2 + \cdots + h_n,$$
$$f_k = \frac{T^{k-1}(T-\lambda)}{\lambda^k}\, f_0 \quad \textit{for all} \quad k \geq 1.$$

In fact, every eigenfunction of T associated to λ must have this form up to scalar multiples.

Proof. If $\lambda \neq 0$ is an eigenvalue of T, then λ is an eigenvalue of T_B for some active branch B of the graph G of T by Lemmas 3.1 and 3.3. Let $B_0 = [x_1; x_2; \ldots; x_n]$ be the primitive cycle of B. Write $T_B = T_0 + T_\infty$, where $T_0 = \sum_{i=1}^n \delta_{x_i} \otimes h_i$ is the part arising from all 0th generation vertices $x_1, x_2, \ldots, x_n$, and $T_\infty = \sum_{x \in B \setminus B_0} \delta_x \otimes h_x$ is the part arising from all other vertices in B. Note that T_0 and T_∞ are disjoint by (3.2), and are both compact disjointness preserving linear operators on $C_0(X)$ by Theorem 2.6. As a result, $\sigma(T_B) = \sigma(T_0) \cup \sigma(T_\infty)$. By Lemma 3.3, $\sigma(T_\infty) = \{0\}$ since the graph of T_∞ consists of a single noble branch. Hence,

$$\sigma(T_B) = \sigma(T_0) \cup \{0\}.$$

Consequently, the nonzero eigenvalue λ of T is also an eigenvalue of T_0.

Let $f_0 \neq 0$ be an eigenfunction of T_0 associated with λ. Write $f_0 = \sum_{i=1}^n a_i h_i$. By setting $x_0 = x_n$ and $h_0 = h_n$, we have

$$T_0 f_0 = \sum_{i=1}^n a_i h_i(x_{i-1})h_{i-1} = \lambda f_0 = \sum_{i=1}^n \lambda a_i h_i.$$

It follows from the disjointness of h_i's that

$$\begin{bmatrix} 0 & h_2(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n(x_{n-1}) \\ h_1(x_n) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Therefore,

$$\lambda^n = h_1(x_n)\ h_2(x_1) \cdots\ h_n(x_{n-1}),$$

and up to scalar multiples we must have

$$f_0 = \frac{h_n(x_{n-1}) \cdots h_3(x_2) h_2(x_1)}{\lambda^{n-1}} h_1 + \frac{h_n(x_{n-1}) \cdots h_3(x_2)}{\lambda^{n-2}} h_2 + \cdots + h_n. \tag{3.8}$$

Let f be nonzero in $C_0(X)$ such that $T_B f = \lambda f$. Assume that f has the generation decomposition form, then by (3.6) and the disjointness of h_n, up to scalar multiples,

$$f = f_0 + f_1 + f_2 + \cdots$$

as asserted in the statement of this theorem. Clearly, we have $Tf = \lambda f$.

Conversely, suppose B is an active branch of the graph of T with the primitive cycle $[x_1; x_2; \ldots, x_n]$ and $\lambda^n = h_1(x_n)\ h_2(x_1) \cdots\ h_n(x_{n-1})$. Then the nonzero function f_0 given by (3.8) is an eigenfunction of $T_0 = \sum_{i=1}^n \delta_{x_i} \otimes h_i$. Define

$$f_k = \frac{T^{k-1}(T-\lambda)}{\lambda^k} f_0 \quad \text{for all} \quad k = 1, 2, \ldots.$$

It follows from (3.7) that $f_1 = \frac{(T-\lambda)f_0}{\lambda}$ is a linear sum of finitely many disjoint functions h_{1j} of the first generation, and $f_2 = \frac{Tf_1}{\lambda} = \sum_{j\geq 1} \frac{1}{\lambda} f_1(x_{2j}) h_{2j}$ is a linear sum of disjoint functions of the second generation. In general, f_k is a linear sum of disjoint functions of the kth generation. Let

$$C_k = \max\{\|h_{kj}\| : h_{kj} \text{ is of the } k\text{th generation}\} \quad \text{for every} \quad k = 0, 1, 2, \ldots.$$

From the disjointness of h_{k_j}, we have $\|T^{k-1} f_1\| \leq aC_1C_2 \cdots C_k$ for some constant $a > 0$ and all $k \geq 1$. By Lemma 2.5, $C_k \to 0$ as $k \to \infty$. For every $\epsilon > 0$ there is an i_0 such that $C_i < \frac{|\lambda|}{2}\epsilon$ for all $i > i_0$. We obtain

$$\frac{\|T^{k-1} f_1\|}{|\lambda|^{k-1}} \leq \frac{aC_1C_2 \cdots C_k}{|\lambda|^{k-1}} \leq A \left(\frac{\epsilon}{2}\right)^{k-i_0}$$

for some positive constant A and all $k > i_0 + 1$. Therefore, $\sum_{k\geq 1} \frac{T^{k-1}(T-\lambda)}{\lambda^k} f_0$ converges in norm in $C_0(X)$. And it is clear that

$$f = f_0 + \sum_{k\geq 1} f_k = \left[I + \sum_{k\geq 1} \frac{T^{k-1}(T-\lambda)}{\lambda^k} \right] f_0$$

is an eigenfunction of T with $Tf = \lambda f$. The proof is now complete. □

Remark 3.6. We note that there might be more than one (but finitely many) active branches B of G giving rise to the same λ. All such functions constructed as above for all these branches form a basis of the (finite dimensional) eigenspace of T associated with this nonzero eigenvalue λ.

Example 3.7. Let $T : C_0(\mathbb{N}) \to C_0(\mathbb{N})$ be defined by

$$\begin{aligned} & T(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \ldots) \\ = \quad & (x_2, \tfrac{x_3}{2}, \tfrac{x_1}{3}, \tfrac{x_1}{4}, \tfrac{x_2}{5}, \tfrac{x_3}{6}, \tfrac{x_4}{7}, \tfrac{x_5}{8}, \tfrac{x_6}{9}, \ldots). \end{aligned}$$

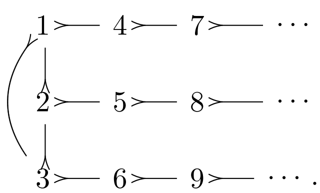
Then $Tf = \sum_{n=1}^{\infty} f(n)h_n$ is a compact disjointness preserving linear operator on $C_0(\mathbb{N})$, where

$$h_1 = \frac{1}{3}\chi_3 + \frac{1}{4}\chi_4, \quad h_2 = \chi_1 + \frac{1}{5}\chi_5, \quad h_3 = \frac{1}{2}\chi_2 + \frac{1}{6}\chi_6,$$

and

$$h_n = \frac{1}{n+3}\chi_{n+3} \quad \text{for all} \quad n \geq 4.$$

As usual, χ_n denotes the (continuous) characteristic function of the closed and open subset $\{n\}$ of $\mathbb{N}$. Here the graph of T consists of two branches — the noble branch $\{\infty\}$ and the active branch B



We note that the primitive cycle of the unique active branch B depicted above is $[1; 2; 3]$, Theorem 3.5 ensures that all nonzero eigenvalues λ of T arise from the equation

$$\lambda^3 = h_1(3)h_2(1)h_3(2) = 1/6.$$

For each one of those three λ, we may set

$$\begin{aligned} f_0 &= \tfrac{h_3(2)h_2(1)}{\lambda^2}h_1 + \tfrac{h_3(2)}{\lambda}h_2 + h_3 = \tfrac{1}{2\lambda^2}h_1 + \tfrac{1}{2\lambda}h_2 + h_3 \\ &= \tfrac{1}{2\lambda}\,\chi_1 + \tfrac{1}{2}\,\chi_2 + \tfrac{1}{6\lambda^2}\,\chi_3 + \tfrac{1}{8\lambda^2}\,\chi_4 + \tfrac{1}{10\lambda}\,\chi_5 + \tfrac{1}{6}\,\chi_6, \end{aligned}$$

and, for $n \geq 1$,

$$\begin{aligned} f_n &= \tfrac{3}{4\cdot 7\cdots(3n+1)\lambda^{n-1}}h_{3n+1} + \tfrac{1}{2\cdot 5\cdot 8\cdots(3n+2)\lambda^{n+1}}h_{3n+2} + \tfrac{1}{6\cdot 9\cdots(3n+3)\lambda^n}h_{3n+3} \\ &= \tfrac{3}{4\cdot 7\cdots(3n+4)\lambda^{n-1}}\,\chi_{3n+4} + \tfrac{1}{2\cdot 5\cdot 8\cdots(3n+5)\lambda^{n+1}}\,\chi_{3n+5} + \tfrac{1}{6\cdot 9\cdots(3n+6)\lambda^n}\,\chi_{3n+6}. \end{aligned}$$

Consequently, the eigenspace of T associated with this eigenvalue λ is spanned by

$$\begin{aligned} f &= \textstyle\sum_{n=0}^{\infty} f_n \\ &= \tfrac{1}{2\lambda}\,\chi_1 + \tfrac{1}{2}\,\chi_2 + \tfrac{1}{6\lambda^2}\,\chi_3 + \tfrac{1}{8\lambda^2}\,\chi_4 + \tfrac{1}{10\lambda}\,\chi_5 + \tfrac{1}{6}\,\chi_6 \\ &\qquad + \textstyle\sum_{n=1}^{\infty}\left[\tfrac{3}{4\cdot 7\cdots(3n+4)\lambda^{n-1}}\,\chi_{3n+4} + \tfrac{1}{2\cdot 5\cdot 8\cdots(3n+5)\lambda^{n+1}}\,\chi_{3n+5} + \tfrac{1}{6\cdot 9\cdots(3n+6)\lambda^n}\,\chi_{3n+6}\right] \\ &= \left(\tfrac{1}{2\lambda}, \tfrac{1}{2}, \tfrac{1}{6\lambda^2}, \tfrac{1}{8\lambda^2}, \tfrac{1}{10\lambda}, \tfrac{1}{6}, \tfrac{3}{28}, \tfrac{1}{80\lambda^2}, \tfrac{1}{54\lambda}, \ldots\right). \end{aligned}$$

4. More properties of compact disjointness preserving operators

Let $T : E \to F$ be a bounded linear operator between Banach spaces. It is well known that T is compact if and only if T has a compact factorization through a closed subspace of c_0 (see, e.g., [18, Theorem 19.4]). A result of Terzioglu states that T has a compact factorization through the whole of c_0 if and only if there exists a sequence (η_n) in c_0, an equicontinuous sequence $\{f_n\}_n$ in the dual space E^* of E and a summable sequence $\{y_n\}_n$ in F such that

$$Tx = \sum_n \eta_n \langle x, f_n\rangle\, y_n \quad \text{for all} \quad x \in E,$$

where $\langle\cdot,\cdot\rangle$ is the dual pair of E and E^* (see, e.g., [18, Theorem 19.6]).

Corollary 4.1. *Let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving linear operator. Then T is compact if and only if T has a compact factorization through c_0.*

Proof. Let T be a compact disjointness preserving linear operator of $C_0(X)$. The case that T is of finite rank is trivial. Therefore, by Theorem 2.6, we suppose $Tf = \sum_{n=1}^{\infty} f(x_n)h_n$, where $x_n \neq x_m$ in X, $h_nh_m = 0$ in $C_0(Y)$ whenever $n \neq m$, and h_n is nonzero but convergent to zero in norm. Let

$$S(f) = (f(x_n)\|h_n\|^{1/2}) \quad \text{for all} \quad f \in C_0(X),$$

$$R(z) = \sum_{n=1}^{\infty} z_n \frac{h_n}{\|h_n\|^{1/2}} \quad \text{for all} \quad z = (z_n) \in c_0.$$

Then $T = RS$ factors through c_0. The other direction is trivial. □

Corollary 4.2. *Let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving operator of the form* (2.1)*. Then T is compact if and only if the following conditions hold.*

(i) *The image of φ is a countable set $\varphi(Y_T) = \{x_1, x_2, \ldots\}$ such that each $Y_n = \varphi^{-1}(x_n)$ is open; and*

(ii) *$h \in C_0(Y_T)$, that is, h can be continuously extended to the whole of $Y \cup \{\infty\}$ by setting $h = 0$ outside Y_T.*

Proof. The necessity follows from Theorem 2.6. We verify the sufficiency. For each $n = 1, 2, \ldots$, let $h_n(y) = h(y)$ on Y_n and 0 on $Y \cup \{\infty\} \setminus Y_n$. Then $h_n = h\chi_{Y_n}$ is continuous on $Y \cup \{\infty\}$ and $Tf(y) = \sum_n f(x_n)h_n(y)$ pointwise on Y. We claim that the sum $T = \sum_n \delta_{x_n} \otimes h_n$ converges uniformly. To this end, let $\epsilon > 0$ and observe that the set $\{y \in Y_T : |h(y)| \geq \epsilon\}$ is a compact subset of $Y_T = \bigcup_n Y_n$. Hence there is a positive integer N such that $|h(y)| < \epsilon$ whenever $y \notin \bigcup_{n=1}^{N} Y_n$. In other words, $\|h_n\| < \epsilon$ for all $n > N$. Thus the sum converges uniformly, and so T is compact. □

Let $M(X)$ denote the Banach dual space of $C_0(X)$ consisting of bounded Radon measures on X. A subset S of $M(X)$ is said to be *controlled* by a finite positive measure μ on X if every ν in S is absolutely continuous with respect to μ. The set S is said to be *dominated* by μ if for all $\epsilon > 0$ there is a $\delta > 0$ such that for any Borel subset B of X with $\mu(B) < \delta$, we have $|\nu(B)| < \epsilon$ for all ν in S. It is clear that a dominating measure of S is also a controlling measure of S. By the Bartle-Dunford-Schwartz Theorem [4], a bounded linear operator T from $C_0(X)$ into a Banach space F is weakly compact if and only if the image $T^*U_{F^*}$ of the dual unit ball of F under the dual map T^* has a dominating measure.

Corollary 4.3. *Let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving linear operator. The following two assertions are equivalent.*

(i) *T is compact;*

(ii) $T^*U_{C_0(Y)^*}$ *is dominated by a positive bounded atomic measure on* X.

Proof. Assume T is compact. By Theorem 2.6, we write $Tf = \sum_{n=1}^{\infty} f(x_n)h_n$. Without loss of generality, we can assume $\|T\| < 1$ and thus $0 < \|h_n\| < 1$ for all n. Let

$$P_m = \{x_n : 2^{-m} < \|h_n\| \leq 2^{-m+1}\},$$

and $p_m = \#P_m$ be the finite cardinality of P_m (Lemma 2.5). Define a bounded atomic measure on X by

$$\mu = \sum_{p_m \neq 0} \frac{1}{2^m p_m} \sum_{x_n \in P_m} \delta_{x_n}.$$

For all $\epsilon > 0$, there is a positive integer N such that $\sum_{m=N+1}^{\infty} \frac{1}{2^{m-1}} < \epsilon$. Let $0 < \delta \leq \frac{1}{2^n p_n}$ for all $1 \leq n \leq N$ with $p_n \neq 0$. Then for all Borel subsets A of X with $\mu(A) < \delta$, we have

$$A \cap P_m = \emptyset \quad \text{for all} \quad 1 \leq m \leq N. \tag{4.1}$$

On the other hand, let ν be any Borel measure on Y with $\|\nu\| \leq 1$ and $f \in C_0(X)$. We have

$$\begin{aligned}(T^*\nu)(f) &= \int_Y Tf\, d\nu = \int_Y \sum_n f(x_n)h_n\, d\nu \\ &= \sum_n f(x_n) \int_Y h_n\, d\nu = \sum_{p_m \neq 0} \sum_{x_n \in P_m} f(x_n) \int_Y h_n\, d\nu.\end{aligned}$$

Hence,

$$\begin{aligned}|(T^*\nu)(A)| &= \left| \int_X \chi_A\, d(T^*\nu) \right| = \left| \sum_{p_m \neq 0} \sum_{x_n \in P_m} \chi_A(x_n) \int_Y h_n\, d\nu \right| \\ &\leq \sum_{p_m \neq 0} \sum_{x_n \in P_m} \chi_A(x_n) \frac{|\nu|(\operatorname{coz} h_n)}{2^{m-1}} \\ &= \sum_{p_m \neq 0} \frac{1}{2^{m-1}} \sum_{x_n \in A \cap P_m} |\nu|(\operatorname{coz} h_n) \\ &\leq \sum_{A \cap P_m \neq \emptyset} \frac{1}{2^{m-1}}\end{aligned}$$

since h_n has disjoint cozeros and $\|\nu\| \leq 1$. If $\mu(A) < \delta$, then

$$|(T^*\nu)(A)| \leq \sum_{m=N+1}^{\infty} \frac{1}{2^{m-1}} < \epsilon$$

by (4.1). Therefore, the sufficiency is verified. The converse follows from Theorem 2.6 and the theorem of Bartle, Dunford and Schwartz [4]. □

Corollary 4.4. *Let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving linear operator. Suppose the range of the dual map T^* is controlled by a finite positive atomic measure $\mu = \sum_n \lambda_n \delta_{x_n}$. If $\{x_1, x_2, \ldots\}$ is a discrete subset of X with compact closure in X, then T is compact.*

Proof. By assumption, for every y in Y, we have $T^*\delta_y \ll \mu$. The Radon-Nikodym Theorem ensures $T^*\delta_y = \sum_{n=1}^\infty h_n(y)\delta_{x_n}$ for some scalars $h_n(y)$. So we have

$$Tf = \sum_{n=1}^\infty f(x_n)h_n.$$

Since T is disjointness preserving and $\{x_1, x_2, \ldots\}$ is discrete, we obtain $h_n \in C_0(Y)$ and $h_n h_m = 0$ for $n \neq m$ by the complete regularity of X. Let $f \in C_0(X)$ such that $f \equiv 1$ on $\operatorname{supp}\mu = \overline{\{x_n : n = 1, 2, \ldots\}}$. Then $Tf = \sum_{n=1}^\infty h_n \in C_0(Y)$. For $\epsilon > 0$, the compact set $\{y \in Y : |Tf(y)| \geq \epsilon\} \subseteq \{y \in Y : Tf(y) \neq 0\} = \bigcup_{n=1}^\infty \operatorname{coz} h_n$. Thus there is an integer N such that

$$\{y \in Y : \sup_n |h_n(y)| \geq \epsilon\} \subseteq \bigcup_{n=1}^N \operatorname{coz} h_n.$$

Since $\operatorname{coz} h_n \cap \operatorname{coz} h_m = \emptyset$ for $n \neq m$, we have $\bigcup_{n>N} \operatorname{coz} h_n \subseteq \{y \in Y : \sup_n |h_n(y)| < \epsilon\}$. Hence, $\|h_n\| < \epsilon$ for all $n > N$. Consequently, $\{h_n\}_n$ converges to zero in norm. Therefore, $T = \sum_{n=1}^\infty \delta_{x_n} \otimes h_n$ is compact from $C_0(X)$ into $C_0(Y)$. □

We close the paper with some examples. The first two examples show that both the discreteness and the compactness in Corollary 4.4 are essential. The second one also shows that Corollary 4.2 is sharp. The third one shows that the converse of Corollary 4.4 does not hold.

Example 4.5. (a) Let $X = [0, 1]$ and $Y = \bigcup_{n=1}^\infty [\frac{1}{2n}, \frac{1}{2n-1}] \cup \{0\}$. Let $\varphi : Y \to X$ be the continuous map defined by

$$\varphi(y) = \begin{cases} \frac{1}{2n}, & y \in [\frac{1}{2n}, \frac{1}{2n-1}]; \\ 0, & y = 0. \end{cases}$$

Define $T : C(X) \to C(Y)$ by

$$Tf(y) = f(\varphi(y)) = \begin{cases} f(\frac{1}{2n}), & \frac{1}{2n} \leq y \leq \frac{1}{2n-1}; \\ f(0), & y = 0. \end{cases}$$

Then T is bounded and disjointness preserving with $T^*\nu \ll \delta_0 + \sum_n \frac{\delta_{1/2n}}{2^n}$ for all bounded Borel measures ν on Y. It is easy to see that $Y_T = Y$, $h \equiv 1 \in C(Y_T)$, and $\varphi(Y_T) = \{0, 1/2, 1/4, 1/6, \ldots\}$ with 0 as its unique cluster point. Set $h_0 = \chi_{\{0\}}$ and $h_n = \chi_{[\frac{1}{2n}, \frac{1}{2n-1}]} \in C(Y)$ for $n = 1, 2, \ldots$. Note that $h_0 \notin C(Y)$, $\varphi^{-1}(0) = \{0\}$ is not open in Y, and $Tf = f(0)h_0 + \sum_{n=1}^{\infty} f(1/2n)h_n$ pointwise. But T is not compact.

(b) Let $T : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be defined by

$$Tf(x) = \sum_{n=1}^{\infty} f(n)(\sin \pi x)\chi_{(n,n+1)}(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Then T is disjointness preserving such that $T^*\nu \ll \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}$ for all bounded Borel measures ν on $\mathbb{R}$. But T is not compact on $C_0(\mathbb{R})$. Note that $\{1, 2, \ldots\}$ does not have compact closure in $\mathbb{R}$ and $h = \sin \pi x \notin C_0(\mathbb{R})$.

(c) Enumerate the rational numbers $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Let $\{h_n\}_n$ be a sequence of disjoint nonzero functions in $C_0(\mathbb{R})$ with $\lim_{n\to\infty} \|h_n\| = 0$. As a uniformly convergent sum of rank one operators,

$$T = \sum_{n=1}^{\infty} \delta_{r_n} \otimes h_n$$

is a compact disjointness preserving operator on $C_0(\mathbb{R})$. Note that

$$T^*\nu = \sum_{n=1}^{\infty} \left(\int h_n d\nu \right) \delta_{r_n} \quad \text{for all} \quad \nu \in C_0(\mathbb{R})^*.$$

It is easy to see that neither the condition $\{x_1, x_2, \ldots\}$ being discrete nor the closure of $\{x_1, x_2, \ldots\}$ in $\mathbb{R}$ being compact is satisfied by any atomic controlling measure $\sum_n \lambda_n \delta_{x_n}$ of the range of T^*. $\square$

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