

ALGORITHM FOR GENERALIZED CO-COMPLEMENTARITY PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce a new class of generalized co-complementarity problems in Banach spaces. An iterative algorithm for finding approximate solutions of these problems is considered. Some convergence results for this iterative algorithm are derived and several existence results are obtained.

1. INTRODUCTION

Let B be a real Banach space with dual space B^* and pairing (x, f) between $x \in B$ and $f \in B^*$. Let $C(B)$ be the family of nonempty compact subsets of B . Suppose $T, A, G, m : B \rightarrow B, V : B \rightarrow C(B)$, and X is a fixed closed convex cone of B . Define $K : B \rightarrow 2^B$ by

$$K(x) = m(x) + X, \quad \forall x \in B.$$

In this paper we shall study the following generalized co-complementarity problem(GCCP): find $x \in B, y \in V(x)$ such that $Gx \in K(x)$ and

$$Tx + Ay \in (J(K(x) - Gx))^*$$

where $J : B \rightarrow B^*$ is the normalized duality mapping and $(J(K(x) - Gx))^*$ is the dual cone of the set $J(K(x) - Gx)$.

Recall that the normalized duality operator $J : B \rightarrow B^*$ is defined for arbitrary Banach space by the condition

$$\|Jx\|_{B^*} = \|x\| \quad \text{and} \quad (x, Jx) = \|x\|^2, \quad \forall x \in B.$$

Some examples and properties of the mapping J can be found in [1]. When B is a Hilbert space, $Jx = x$ reduces to the identity mapping. Note that every nonzero x in B is weak* continuous and thus attains its norm on the weak* compact unit ball of B^* . In case B^* is strictly convex, the point x attains its norm on the ball of B^* is unique, namely, $Jx/\|x\|$. In this paper, we are mainly interested in uniformly smooth Banach space B . Therefore, the construction of J is concrete to us here.

Before we proceed any further, we make a few observations. There are evidence that our results generalize many known important complementarity problems studied in the literature.

- (i) If B is a Hilbert space, then (GCCP) reduces to finding $x \in B, y \in V(x)$ such that $Gx \in K(x)$,

$$Tx + Ay \in (K(x) - Gx)^*,$$

which is the generalized multi-valued complementarity problem studied by Jou and Yao [5].

- (ii) If B is a Hilbert space and G is the identity mapping, then (GCCP) reduces to finding $x \in K(x), y \in V(x)$ such that

$$Tx + Ay \in (K(x) - x)^*,$$

which is known as the generalized strongly nonlinear quasi-complementarity problem studied by Chang and Huang [2].

- (iii) If B is a Hilbert space and G as well as V are the identity mappings, then (GCCP) is equivalent to finding $x \in K(x)$ such that

$$Tx + Ay \in (K(x) - x)^*,$$

which is known as the strongly nonlinear quasi-complementarity problem studied by Noor [10].

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- (iv) If B is a Hilbert space, G and V are the identity mappings and A is the zero mapping, then (GCCP) is equivalent to finding $x \in K(x)$ such that

$$Tx + Ay \in (K(x) - x)^*,$$

which is known as the generalized quasi-complementarity problem studied by Noor [7].

- (v) If B is a Hilbert space, G and V are the identity mappings and m is the zero mapping, then (GCCP) is equivalent to finding $x \in X$ such that

$$Tx + Ax \in X^* \text{ and } (Tx + Ax, x) = 0,$$

which is known as the mildly nonlinear complementarity problem studied by Noor [8].

- (vi) If B is a Hilbert space, V is the identity mappings and A as well as m are the zero mappings, then (GCCP) is equivalent to finding $x \in B$ such that $Gx \in X$,

$$Tx \in X^* \text{ and } (Tx, Gx) = 0,$$

which is known as the general nonlinear complementarity problem studied by Noor [9].

- (vii) If B is a Hilbert space, G and V are the identity mappings, A and m are the zero mappings, then (GCCP) is equivalent to finding $x \in X$ such that

$$Tx \in X^* \text{ and } (Tx, x) = 0,$$

which is known as the generalized complementarity problem studied by Habetler [4] and Karamardian [6].

- (viii) If $B = \mathbb{R}^n$, G and A are the identity mappings, T and m are the zero mappings, then (GCCP) is equivalent to finding $x \in X$ and $y \in V(x)$ such that

$$y \in X^* \text{ and } (x, y) = 0,$$

which has been studied by Saigal [11].

The aim of this paper is to construct the projection iterative methods of finding approximate solutions of (GCCP) in (especially uniformly smooth) Banach spaces. Our results are new, interesting, and should be applicable to all those classical complementarity problems mentioned above, with hopefully giving more insights to its algorithmic aspect.

In Section 2, we shall give some preliminaries. In Section 3, we shall derive some characterization of solutions of (GCCP) by employing the sunny nonexpansive retraction method. In the final section, we shall construct an iterative algorithm for finding the approximate solutions of (GCCP) and derive some corresponding convergence and existence results.

2. PRELIMINARIES

We first recall the following definitions.

Definition 2.1. Let B be a Banach space and $A : B \rightarrow B$.

- (i) A is said to be *strongly accretive* if there exists a constant $\gamma > 0$ such that

$$(Ax - Ay, J(x - y)) \geq \gamma \|x - y\|^2, \text{ for all } x, y \in B.$$

- (ii) A is said to be *Lipschitz continuous* if there exists a positive constant β such that

$$\|Ax - Ay\| \leq \beta \|x - y\|, \text{ for all } x, y \in B.$$

- (iii) A is said to be *strongly accretive* with respect to the point-to-set mapping $V : B \rightarrow C(B)$ if there exists a positive constant α such that

$$(Au - Av, J(x - y)) \geq 0, \text{ for all } x, y \in B \text{ and for all } u \in V(x), v \in V(y).$$

Definition 2.2. The mapping $V : B \rightarrow C(B)$ is said to be *H-Lipschitz continuous* if there exists a constant $\eta > 0$ such that

$$H(V(x), V(y)) \leq \eta \|x - y\|, \text{ for all } x, y \in B,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $C(B)$.

We remark that the *uniform convexity* of the space B means that for any given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B$, $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| = \epsilon$ ensure the following inequality

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon\right\}$$

is called *the modulus of the convexity* of the space B .

The *uniform smoothness* of the space B means that for any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon\|y\|$$

holds. The function

$$\rho_B(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}$$

is called *the modulus of the smoothness* of the space B .

We also remark that the space B is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$ and it is uniformly smooth if and only if $\lim_{t \rightarrow 0} t^{-1}\rho_B(t) = 0$. Moreover, B^* is uniformly convex if and only if B is uniformly smooth. In this case, B is reflexive by the Milman Theorem. A Hilbert space is uniformly convex and uniformly smooth. The proof of the following inequalities can be found, e.g., in [1] and hence will be omitted.

Proposition 2.3. *Let B be a uniformly smooth Banach space and J the normalized duality mapping from B into B^* . Then for all $x, y \in B$, we have*

- (i) $\|x + y\|^2 \leq \|x\|^2 + 2(y, J(x + y))$,
- (ii) $(x - y, Jx - Jy) \leq 2C^2\rho_B(4\|x - y\|/d)$ where $d = ((\|x\|^2 + \|y\|^2)/2)^{1/2}$.

Let B be a real Banach space and Ω a nonempty closed convex subset of B . A mapping $Q_\Omega : B \rightarrow \Omega$ is said to be a *retraction* on Ω if $Q_\Omega^2 = Q_\Omega$. The mapping Q_Ω is said to be a *nonexpansive retraction* if, in addition,

$$\|Q_\Omega x - Q_\Omega y\| \leq \|x - y\|, \text{ for all } x, y \in B;$$

and Q_Ω is a *sunny retraction* if for all $x \in B$,

$$Q_\Omega(Q_\Omega x + t(x - Q_\Omega x)) = Q_\Omega x, \quad \forall t \in \mathbb{R}.$$

The following characterization of a sunny nonexpansive retraction mapping can be found, e.g., in [3].

Proposition 2.4. *Q_Ω is a sunny nonexpansive retraction if and only if for all $x, y \in B$,*

$$(x - Q_\Omega x, J(Q_\Omega x - y)) \geq 0.$$

From Proposition 2.4, we have the following retraction shift equality.

Proposition 2.5. *Let B be a Banach space, Ω a nonempty closed convex subset of B and $m : B \rightarrow B$. Then for all $x, y \in B$, we have*

$$Q_{\Omega+m(x)}x = m(x) + Q_\Omega(x - m(x)).$$

3. CHARACTERIZATION OF SOLUTIONS — ALGORITHM AND CONVERGENCE

In this section we first derive some characterizations of solutions of the generalized co-complementarity problem.

Theorem 3.1. *Let B be a Banach space and X a closed convex cone in B . Let $T, G, A, m : B \rightarrow B$, $V : B \rightarrow C(B)$, $K : B \rightarrow 2^B$ and $K(x) = m(x) + X$ for all $x \in B$. Then the following statements are equivalent.*

- (i) $x \in B, y \in V(x)$ are solutions of (GCCP), i.e., $Gx \in K(x)$ and

$$Tx + Ay \in (J(K(x) - Gx))^*.$$

- (ii) $x \in B, y \in V(x)$ and for some $\tau > 0$,

$$Gx = Q_{K(x)}(Gx - \tau(Tx + Ay)).$$

Proof. We note that $Tx + Ay \in (J(K(x) - Gx))^*$ if and only if

$$(Tx + Ay, J(z - Gx)) \geq 0, \text{ for all } z \in K(x).$$

The result then can be proved by the same arguments as those in [1, Theorem 8.1], [2, Theorem 3.1] and Proposition 2.4. \square

By combining Proposition 2.5 and Theorem 3.1, we have the following result.

Theorem 3.2. *Let B be a Banach space and X a closed convex cone in B . Let $T, G, A, m : B \rightarrow B$, and $V : B \rightarrow C(B)$. Then the following statements are equivalent.*

- (i) $x \in B, y \in V(x)$ are solutions of (GCCP).
- (ii) $x = x - Gx + m(x) + Q_X(Gx - \tau(Tx + Ay) - m(x))$ for some $\tau > 0$.

Next we shall construct an iterative algorithm for finding approximate solutions of (GCCP). Let $\tau > 0$ be fixed. Given $x_0 \in B$, take any $y_0 \in V(x_0)$ and let

$$x_1 = x_0 - Gx_0 + m(x_0) + Q_X(Gx_0 - \tau(Tx_0 + Ay_0) - m(x_0)).$$

Since $V(x_0)$ is a nonempty and compact set, there exists $y_1 \in V(x_1)$ such that

$$\|y_0 - y_1\| \leq H(V(x_0), V(x_1)).$$

Let

$$x_2 = x_1 - Gx_1 + m(x_1) + Q_X(Gx_1 - \tau(Tx_1 + Ay_1) - m(x_1)).$$

By continuing the above process inductively, we can get sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in V(x_n)$,

$$(3.1) \quad \|y_n - y_{n+1}\| \leq H(V(x_n), V(x_{n+1}))$$

and

$$(3.2) \quad x_{n+1} = x_n - Gx_n + m(x_n) + Q_X(Gx_n - \tau(Tx_n + Ay_n) - m(x_n)).$$

Now we have the following convergence and existence result.

Theorem 3.3. *Let B be a uniformly smooth Banach space with $\rho_B(t) \leq Ct^2$ for some $C > 0$. Let X be a closed convex cone of B . Suppose $T, A, G, m : B \rightarrow B, V : B \rightarrow C(B), K : B \rightarrow 2^B$ such that $K(x) = m(x) + X$ for all $x \in B$ and the following conditions.*

- (i) T, G, A and m are Lipschitz continuous with constants β, δ, λ and θ , respectively.
- (ii) G is strongly accretive with constant γ and V is H -Lipschitz continuous with constant η .
- (iii) $(1 - 2\gamma + 64C\delta^2)^{1/2} + 2\theta + \delta + \tau(\beta + \lambda\eta) < 1$ for some $\tau > 0$.

Then for any given $x_0 \in B$ and $y_0 \in V(x_0)$, the sequences x_n and y_n generated by (3.2) and (3.1), respectively, converge strongly to some $x \in B$ and $y \in V(x)$ which solve the (GCCP).

Proof. By the iterative schemes (3.1) and (3.2), we have

$$(3.3) \quad \begin{aligned} & \|x_{n+1} - x_n\| \\ &= \|x_n - Gx_n + m(x_n) + Q_X(Gx_n - \tau(Tx_n + Ay_n) - m(x_n)) \\ &\quad - (x_{n-1} - Gx_{n-1} + m(x_{n-1})) - Q_X(Gx_{n-1} - \tau(Tx_{n-1} + Ay_{n-1}) - m(x_{n-1}))\| \\ &\leq \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \|(Gx_n - Gx_{n-1})\| + \tau\|Tx_n - Tx_{n-1}\| + \tau\|Ay_n - Ay_{n-1}\|. \end{aligned}$$

By Proposition 2.3, we have

$$(3.4) \quad \begin{aligned} & \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 + 2(-(Gx_n - Gx_{n-1}), J(x_n - x_{n-1} - (Gx_n - Gx_{n-1}))) \\ &= \|x_n - x_{n-1}\|^2 + 2(-(Gx_n - Gx_{n-1}), J(x_n - x_{n-1})) \\ &\quad + 2(-(Gx_n - Gx_{n-1}), J(x_n - x_{n-1} - (Gx_n - Gx_{n-1})) - J(x_n - x_{n-1})) \\ &\leq \|x_n - x_{n-1}\|^2 - 2\gamma\|x_n - x_{n-1}\|^2 + 4d^2\rho_B(4\|Gx_n - Gx_{n-1}\|/d) \\ &\leq \|x_n - x_{n-1}\|^2 - 2\gamma\|x_n - x_{n-1}\|^2 + 64C\|Gx_n - Gx_{n-1}\|^2 \\ &\leq (1 - 2\gamma + 64C\delta^2)\|x_n - x_{n-1}\|^2. \end{aligned}$$

It follows from the Lipschitz property of the corresponding functions that

$$(3.5) \quad \|m(x_n) - m(x_{n-1})\| \leq \theta \|x_n - x_{n-1}\|,$$

$$(3.6) \quad \|Tx_n - Tx_{n-1}\| \leq \beta \|x_n - x_{n-1}\|,$$

$$(3.7) \quad \|Ay_n - Ay_{n-1}\| \leq \lambda \eta \|x_n - x_{n-1}\|,$$

and

$$(3.8) \quad \|Gx_n - Gx_{n-1}\| \leq \delta \|x_n - x_{n-1}\|.$$

From (3.3)-(3.8), we have the following inequality

$$\|x_{n+1} - x_n\| \leq \kappa \|x_n - x_{n-1}\|,$$

where

$$\kappa = (1 - 2\gamma + 64C\delta^2)^{1/2} + 2\theta + \delta + \tau(\beta + \lambda\eta)$$

and $0 < \kappa < 1$ by (iii).

Consequently, $\{x_n\}$ is a Cauchy sequence and thus converges to some $x \in B$. By (3.1), we have

$$\|y_n - y_{n-1}\| \leq H(V(x_n), V(x_{n-1})) \leq \eta \|x_n - x_{n-1}\|,$$

and hence $\{y_n\}$ is also a Cauchy sequence in B . Let $\{y_n\}$ converge to some $y \in B$. Since Q_X, G, T, A, V and m are all continuous, we have

$$x = x - Gx + m(x) + Q_X(Gx - \tau(Tx + Ay) - m(x)).$$

It remains to show that $y \in V(x)$. In fact,

$$\begin{aligned} d(y, V(x)) &\leq \|y - y_n\| + d(y_n, V(x)) \\ &\leq \|y - y_n\| + H(V(x_n), V(x)) \\ &\leq \|y - y_n\| + \eta \|x - x_n\|, \end{aligned}$$

where $d(y, V(x)) = \inf\{\|y - z\| : z \in V(x)\}$. Letting n go to infinity, we have $d(y, V(x)) = 0$ and therefore $y \in V(x)$. The result then follows from Theorem 3.2. \square

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