DISJOINTNESS PRESERVING LINEAR OPERATORS OF WIENER RING

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Abstract. Using elementary arguments, we shall show that every bounded disjointness preserving linear functional of the Wiener ring $L^1(\mathbb{Z})$ assumes the form

$$\varphi(f) = \lambda \sum_{n=-\infty}^{\infty} f(n)z^n,$$

for some scalar λ in $\mathbb{C}$ and for some z in the unit circle $\mathbb{T}$. Consequently, every bounded disjointness preserving linear operator Φ from $L^1(\mathbb{Z})$ into itself assumes the form $\Phi(f) = \Phi(\mathbf{e_0}) * H(f)$, where H is an algebra homomorphism of $L^1(\mathbb{Z})$.

1. Introduction

A linear functional φ of an algebra A is said to be *disjointness preserving* if $\varphi(f)\varphi(g) = 0$ whenever $fg = 0$. In the case when A is a function algebra on X, a disjointness preserving map is the map that preserves the disjointness of cozeroes, where the cozero of a function f is the set $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$. Recently, many authors studied disjointness preserving maps between function algebras, group algebras, algebras of differentiable functions and general Banach algebras. See e.g., [1, 6, 5, 8, 7, 3, 2].

In this note, using elementary arguments, we shall study bounded disjointness preserving linear functionals φ of the Wiener ring

$$L^1(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \sum_{n=-\infty}^{\infty} |f(n)| < +\infty\},$$

with multiplication defined by convolution. Our results state that such φ will assume a form,

$$\varphi(f) = \lambda \sum_{n=-\infty}^{\infty} f(n)z^n,$$

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for some scalar λ in $\mathbb{C}$ and for some z in the unit circle $\mathbb{T}$. Using this, we show that every bounded disjointness preserving linear operator Φ is the convolution of an algebra homomorphism H of $L^1(\mathbb{Z})$ and the function $\Phi(e_0)$; namely,

$$\Phi(f) = \Phi(e_0) * H(f), \quad \forall f \in L^1(\mathbb{Z}).$$

2. Notations and preliminaries

Recall that $L^1(\mathbb{Z})$ is a commutative Banach algebra with norm

$$\|f\| = \sum_{n=-\infty}^{\infty} |f(n)|$$

and multiplication

$$f * g(n) = \sum_{k=-\infty}^{\infty} f(n-k)g(k).$$

The (multiplicative) unit in $L^1(\mathbb{Z})$ is the function e_0 in $L^1(\mathbb{Z})$ defined by $e_0(0) = 1$ and $e_0(n) = 0$ for $n \in \mathbb{Z} - \{0\}$.

It is also well-known that every complex homomorphism of a group algebra $L^1(G)$ arises exactly from the dual group $\hat{G}$ of G. More precisely, let G be a locally compact abelian group and $\gamma : G \to \mathbb{T}$ be a continuous homomorphism. Define $\hat{f}(\gamma)$ by

$$\hat{f}(\gamma) = \int_G f(x)\gamma(x^{-1})dx$$

for every f in $L^1(G)$. Then $f \longmapsto \hat{f}(\gamma)$ is a nonzero complex homomorphism of $L^1(G)$. Conversely, if $\varphi : L^1(G) \to \mathbb{C}$ is a nonzero homomorphism, there is a continuous homomorphism $\gamma : G \to \mathbb{T}$ such that $\varphi(f) = \hat{f}(\gamma)$. See, e.g., [4, p. 226].

Suppose $\gamma : \mathbb{Z} \to \mathbb{T}$ satisfies that $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $x, y \in \mathbb{Z}$. Then either $\gamma \equiv 0$ or there exists $n \in \mathbb{Z}$ such that $\gamma(n) = a^n$ for all $a \in \mathbb{T}$. In other words, the dual group of $\mathbb{Z}$ is the circle group $\mathbb{T}$, i.e., $\hat{\mathbb{Z}} = \mathbb{T}$. Hence, every complex homomorphism φ of $L^1(\mathbb{Z})$ assumes the form

$$\varphi(f) = \sum_{n=-\infty}^{\infty} f(n)z^n, \quad \forall f \in L^1(\mathbb{Z}),$$

for some z in $\mathbb{T}$. In other words, the maximal ideal space of the Banach algebra $L^1(\mathbb{Z})$ is homeomorphic to the unit circle $\mathbb{T}$.

The Gelfand transform $\Gamma : L^1(\mathbb{Z}) \to C(\mathbb{T})$, sending f to $\hat{f}$, takes the following form

$$\hat{f}(z) = \sum_{n=-\infty}^{\infty} f(n)z^n, \quad f \in L^1(\mathbb{Z}), z \in \mathbb{T}.$$

It is clearly linear, bounded, injective and multiplicative. We note that the range of the Gelfand transform contains $C^2(\mathbb{T})$, via, e.g., the Fourier series. Moreover, this map expends to an isometry from $L^2(\mathbb{Z})$ onto $L^2(\mathbb{T})$ (Plancherel Theorem).

As usual, we regard $C(\mathbb{T})$ the space of all continuous periodic functions defined on the real line with period 2π. Then the following well-known results are applicable. We sketch the proofs here for completeness.

Lemma 1. *Let $[a, b] \subset (c, d)$. There exists a function $\mathcal{K}$ in $C^\infty(\mathbb{R})$ such that $\mathcal{K} = 1$ on $[a, b]$, $\mathcal{K} = 0$ outside (c, d), and $0 \leq \mathcal{K} \leq 1$ on $\mathbb{R}$.*

Proof. Define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$g(x) = f(x)f(1 - x).$$

Then g is of class C^∞; furthermore, g is positive for $0 < x < 1$ and vanishes elsewhere. Define

$$h(x) = \frac{\int_0^x g(t)dt}{\int_0^1 g(t)dt},$$

and

$$\mathcal{K}(x) = \begin{cases} 0 & \text{for } x \leq c \\ h(\frac{x-c}{a-c}) & \text{for } c \leq x \leq a \\ 1 & \text{for } a \leq x \leq b \\ h(\frac{d-x}{d-b}) & \text{for } b \leq x \leq d \\ 0 & \text{for } x \geq d\ . \end{cases}$$

It is straightforward to see that $\mathcal{K}$ is smooth on $\mathbb{R}$ with the stated properties. ∎

Corollary 2.

(1) Let U, V be two nonempty disjoint closed subsets of $\mathbb{T}$. Then there is an $f \in L^1(\mathbb{Z})$ such that $\hat{f} = 1$ on U and $\hat{f} = 0$ on V.

(2) Let $\{U_1, \ldots, U_n\}$ be an open covering of a compact subset K of $\mathbb{T}$. Then there are $f_1, \ldots, f_n$ in $L^1(\mathbb{Z})$ such that $0 \leq \hat{f}_i \leq 1$, $\operatorname{coz} \hat{f}_i \subset U_i$, for $i = 1, \ldots, n$, and $\hat{f}_1 + \cdots + \hat{f}_n = 1$ on K.

Lemma 3. *Let* $f \in L^1(\mathbb{Z})$ *such that* $\hat{f}(z_0) = 0$ *for some* $z_0 \in \mathbb{T}$. *Then for any* $\epsilon > 0$, *there is an* $f_\epsilon \in L^1(\mathbb{Z})$ *such that* $\hat{f}_\epsilon$ *vanishes in a neighborhood of* z_0 *and* $\|f - f_\epsilon\| < \epsilon$.

Proof. We can assume $z_0 = 1$. As $\hat{f}(1) = 0$, we have

$$\sum_{m=-\infty}^{+\infty} f(m) = 0. \tag{1}$$

On the other hand, $\sum_{m=-\infty}^{+\infty} |f(m)| < +\infty$ ensures that for any $\delta > 0$, there is a positive integer N such that

$$\sum_{m=-N}^{N} |f(m)| < \delta.$$

Let

$$U = \bigcap_{m=-N}^{N} \{\theta \in (-\pi, \pi) : |1 - e^{im\theta}| < \delta\}.$$

Then U is an open neighborhood of 0 in $\mathbb{R}$. Assume $(-3\alpha, 3\alpha) \subseteq U$. Set

$$C_t = \{e^{i\theta} : -t\alpha < \theta < t\alpha\}, \quad \text{for } t = 1, 2, 3.$$

Let $g, h \in L^2(\mathbb{Z})$ such that $\hat{g}$ and $\hat{h}$ are the characteristic functions of C_1 and C_2, respectively. Define

$$k = \frac{\pi g h}{\alpha} \in L^1(\mathbb{Z}).$$

Then $\hat{k} = \frac{\pi}{\alpha}\hat{g} * \hat{h}$, and

$$\hat{k}(z) = \frac{\pi}{\alpha}\int_{\mathbb{T}} \hat{g}(z')\hat{h}(\frac{z}{z'})\, dz' = \frac{\pi}{\alpha}\int_{C_1} \hat{h}(\frac{z}{z'})\, dz'.$$

Hence, $\hat{k} = 1$ on C_1, $\hat{k} = 0$ outside C_3, and $0 \leq \hat{k} \leq 1$ on $\mathbb{T}$. Moreover,

$$\|k\| \leq \frac{\pi}{\alpha}\|g\|_2\|h\|_2 = \frac{\pi}{\alpha}\sqrt{\alpha/\pi}\sqrt{2\alpha/\pi} = \sqrt{2}.$$

By (1), we have

$$f * k(n) = \sum_{m=-\infty}^{+\infty} f(m)k(n-m) = \sum_{m=-\infty}^{+\infty} f(m)[k(n-m) - k(n)].$$

Denoting by $k_m(n) = k(m-n)$, we have

$$\begin{aligned}\|f * k\| &= \sum_{n=-\infty}^{+\infty} \left| \sum_{m=-\infty}^{+\infty} f(m)[k_m(n) - k(n)] \right| \\ &\leq \sum_{m=-\infty}^{+\infty} |f(m)| \|k_m - k\| \\ &= \sum_{m=-N}^{N} |f(m)| \|k_m - k\| + \sum_{|m|>N} |f(m)| \|k_m - k\| \\ &\leq \sum_{m=-N}^{N} |f(m)| \|k_m - k\| + 2\sqrt{2}\delta.\end{aligned}$$

Observe that

$$\frac{\alpha}{\pi}(k_m - k) = g_m h_m - gh = g_m(h_m - h) + (g_m - g)h.$$

Here,

$$\|g_m(h_m - h)\| \leq \|g_m\|_2 \|h_m - h\|_2 = \|\hat{g}_m\|_2 \|\hat{h}_m - \hat{h}\|_2.$$

Let $\hat{\gamma}_m(z) = z_m$ be the character of $\mathbb{T}$ associated to m in $\mathbb{Z}$. Then

$$\begin{aligned}\|\hat{h}_m - \hat{h}\|_2^2 &= \|(\hat{\gamma}_m - 1)\hat{h}\|_2^2 \\ &\leq \frac{1}{2\pi} \int_{-2\alpha}^{2\alpha} \left| e^{im\theta} - 1 \right|^2 d\theta \\ &< \frac{2\alpha\delta^2}{\pi}, \quad \forall\, |m| \leq N.\end{aligned}$$

As $\|\hat{g}_m\|_2 = \|\hat{g}\|_2 = \sqrt{\frac{\alpha}{\pi}}$, we have

$$\|g_m(h_m - h)\| < \frac{\sqrt{2}\alpha\delta}{\pi}.$$

Similarly, we have

$$\|(g_m - g)h\| < \frac{\sqrt{2}\alpha\delta}{\pi}.$$

Hence,

$$\|k_m - k\| < 2\sqrt{2}\delta, \quad \forall\, |m| \leq N.$$

Consequently,

$$\|f * k\| < 2\sqrt{2}\|f\|\delta + 2\sqrt{2}\delta.$$

Setting $\delta < \frac{\epsilon}{2\sqrt{2}(1+\|f\|)}$ and $f_\epsilon = f - f * k$, we will have the desired conclusion. □

3. Bounded disjointness preserving operators of $L^1(\mathbb{Z})$

Theorem 4. *If φ is a nonzero bounded disjointness linear functional of $L^1(\mathbb{Z})$, then $\varphi = \lambda h$, where h is a complex homomorphism of $L^1(\mathbb{Z})$, and λ is a scalar.*

Proof. For each point z in the dual group $\hat{\mathbb{Z}} = \mathbb{T}$, let I_z (resp. M_z) consist of all functions f in $L^1(\mathbb{Z})$ such that $\hat{f}$ vanishes in a neighborhood of z (resp. vanishing at z). By Lemma 3, I_z is norm dense in M_z.

We claim that $\varphi(I_z) = \{0\}$ for exactly one z in $\mathbb{T}$.

Suppose, on contrary, that for each z in $\mathbb{T}$ there is an $\hat{f}_z$ vanishing in an open neighborhood V_z of z with $\varphi(f_z) \neq 0$. Let $f \in L^1(\mathbb{Z})$. The compact support of $\hat{f}$ is covered by a finite union of the open sets V_{x_i}. By Corollary 2, we can write $f = f_1 + \cdots + f_n$ for some $f_i \in L^1(\mathbb{Z})$ with $\operatorname{coz}(\hat{f}) \subseteq V_{x_i}$, $i = 1, \ldots, n$. Now $\hat{f}_i \hat{f}_{x_i} = 0$ implies $f_i * f_{x_i} = 0$. Since φ preserves disjointness, $\varphi(f_i) = 0$ since $\varphi(f_{x_i}) \neq 0$, a contradiction.

For the uniqueness of z, assume that $\varphi(I_z) = \varphi(I_y) = \{0\}$ for some $y \neq z$. By Corollary 2, for every f in $L^1(\mathbb{Z})$ we can write $f = f_1 + f_2$ with $f_1 \in I_z$ and $f_2 \in I_y$. It follows $\varphi(f) = 0$ for all f in $L^1(\mathbb{Z})$. And thus $\varphi = 0$, again a contraction.

Finally, it follows from the boundedness of φ that $\varphi(M_z) = \{0\}$. In other words, the kernel of φ contains that of the complex homomorphism h in $\mathbb{T}$ defined by $h(f) = \hat{f}(z)$. Hence the assertion follows. □

Corollary 5. *Every bounded disjointness preserving linear functional φ of $L^1(\mathbb{Z})$ is in the form of*

$$\varphi(f) = \lambda \sum_{n=-\infty}^{\infty} f(n) z^n, \quad \forall n \in \mathbb{Z}$$

for some z in $\mathbb{T}$ and scalar λ.

Theorem 6. *Let $\Phi : L^1(\mathbb{Z}) \to L^1(\mathbb{Z})$ be a bounded disjointness preserving linear operator. Then there is an algebra endomorphism H of $L^1(\mathbb{Z})$ such that*

$$\Phi(f) = \Phi(e_0) * H(f), \quad \forall f \in L^1(\mathbb{Z}).$$

Proof. For each z in $\mathbb{T}$, $\widehat{\Phi(\cdot)}(z)$ is a bounded disjointness preserving linear functional of $L^1(\mathbb{Z})$. By Theorem 4, we have

$$\widehat{\Phi(f)}(z) = \lambda_z h_z(f), \quad \forall f \in L^1(\mathbb{Z}).$$

Here, λ_z is a scalar and h_z is a complex homomorphism of $L^1(\mathbb{Z})$. In case $\widehat{\Phi(\cdot)}(z)$ is zero, we have $\lambda_z = 0$. It is clear that $\widehat{\Phi(e_0)}(z) = \lambda_z$ for all $z \in \mathbb{T}$. Define, by duality, a map H from $L^1(\mathbb{Z})$ into itself by asking that

$$\widehat{H(f)}(z) = h_z(f), \quad \forall f \in L^1(\mathbb{Z}).$$

We see that H is an algebra homomorphism. The assertion follows. □

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