ISOMETRIC SHIFTS ON $C_0(X)$

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ABSTRACT. For a linear isometry $T: C_0(X) \longrightarrow C_0(Y)$ of finite corank, there is a cofinite subset Y_1 of Y such that $Tf_{|Y_1} = h \cdot f \circ \varphi$ is a weighted composition operator and X is homeomorphic to a quotient space of Y_1 modulo a finite subset. When $X = Y$, such a T is called an isometric quasi-n-shift on $C_0(X)$. In this case, the action of T can be implemented as a shift on a tree-like structure, called a T-tree, in $M(X)$ with exactly n joints. The T-tree is total in $M(X)$ when T is a shift. With this tools, we can analyze the structure of T.

1. INTRODUCTION

Let T be a linear isometry from an infinite dimensional separable Hilbert space H into H of finite corank n. The von Neumann–Wold Decomposition Theorem (see e.g. [4, p. 112]) states that T can be written as a direct sum of a unitary and a product of n copies of the unilateral shift. More precisely, $H_u = \bigcap_{m=1}^{\infty} T^m H$ is a reducing subspace of T. Its orthogonal complement $H_s = H \ominus H_u$ is the infinite orthogonal sum $\bigoplus_{m=0}^{\infty} T^m N$, where $N = H \ominus TH$ is of dimension n. Now, $T_{|H_u}$ is a unitary and $T_{|H_s}$ shifts each n-dimensional subspace $T^m N$ onto $T^{m+1}N$ for $m = 0, 1, 2, \ldots$. In this sense, we may call T an isometric quasi-n-shift on H.

We are interested in generalizing the notion of shifts and quasi-shifts to Banach spaces in a basis free setting. Generalizing a notion of Crownoven [5], we call a (necessarily bounded) linear operator S from a Banach space E into E an *n-shift* if

(a) S is injective and has closed range;
(b) S has corank n;
(c) The intersection $\bigcap_{m=1}^{\infty} S^m E$ of the range spaces of all powers S^m of S is zero.

S is called a *quasi-n-shift* if S satisfies conditions (a) and (b). When $n = 1$, we will simply call S a *shift* or a *quasi-shift* accordingly.

In this paper, we study isometric (quasi-)n-shifts on continuous function spaces. Let X be a locally compact Hausdorff spaces. Let $C_0(X)$ be the Banach space of continuous (real- or complex-valued) functions defined on X vanishing at infinity. In [10], Holub proved that the *real* Banach space $C(X, \mathbb{R})$ of continuous real-valued functions defined on X admits no shift at all if X is compact and connected. When the underlying field is the complex,

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however, many concrete examples of such shifts was provided in [9]. The general theory of isometric shifts and quasi-shifts on continuous function spaces was built up in [8], in which Gutek *et. al.* posed also a number of open problems. Farid and Varadarajan [6], Rajagopalan, Rassias and Sundaresan [14, 15, 16] and Haydon [9] answered some of them. More recently, Araujo and Font [1, 7, 2] discussed related questions in this direction. The current paper extends the theory to n-shifts for $n \geq 1$ (and in a locally compact space setting). In particular, we provide new tools in analyzing the range spaces of such shifts.

In Section 2, we study linear isometries T from $C_0(X)$ into $C_0(Y)$ of finite corank. We shall give a full description of such operators and, especially, the structure of their range spaces. In particular, we show that there is a cofinite subset Y_1 of Y such that $Tf_{|Y_1} = h \cdot f \circ \varphi$ is a weighted composition operator and X is homeomorphic to a quotient space of Y_1 modulo a finite subset. These results are applied in Section 3 to isometric n-shifts and quasi-n-shifts on $C_0(X)$. In particular, we show that every isometric quasi-n-shift on $C_0(X)$ is implemented by a shift on a countable set with a tree-like structure, called a T-tree, with exactly n joints in the dual space $M(X)$ of $C_0(X)$. The action of the quasi-n-shift is implemented as a shift on the T-tree. The T-tree is total in $M(X)$ when T is a shift. An open problem stated in [8, p. 119] asks if X is separable when $C_0(X)$ admits an isometric shift. We shall show that if X does not contain infinitely many isolated points or the T-tree satisfies some conditions then the existence of an isometric n-shift T on $C_0(X)$ ensuring the separability of X.

2. Isometries with finite corank

For a locally compact Hausdorff space X, we let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X and let $C_0(X) = \{f \in C(X_\infty) : f(\infty) = 0\}$ be the Banach space of continuous functions on X vanishing at infinity and equipped with the supremum norm. Note that the point ∞ at infinity is isolated in X_∞ if and only if X is compact. Let $M(X)$ be the Banach dual space of $C_0(X)$, which consists of all bounded regular Borel measures on X. Denote by $M_1(X)$ the closed unit ball of $M(X)$ and the set of its extreme points by

$$\operatorname{ext} M_1(X) = \{\lambda\delta_x \in M_1(X) : |\lambda| = 1 \text{ and } x \in X\},$$

which consists of all unimodular scalar multiples of point masses δ_x at x in X.

In this section, X and Y are locally compact Hausdorff and $T : C_0(X) \longrightarrow C_0(Y)$ is a linear isometry with the dual map $T^* : M(Y) \longrightarrow M(X)$. Clearly, $T^*\delta_y \in M_1(X)$ for all y in Y.

Definition 2.1. We define the *vanishing set* of T to be

$$Y_0 = \{y \in Y : T^*\delta_y = 0\},$$

the *Holsztyński set to be*

$$Y_1 = \{y \in Y : T^*\delta_y \in \operatorname{ext} M_1(X)\},$$

and the *exceptional set to be*

$$Y_e = Y \setminus (Y_0 \cup Y_1).$$

The following result is known as Holsztyński's Theorem ([11, 12]). We include a sketch of the proof here for completeness.

Lemma 2.2 (Holsztyński)**.** *There is a continuous surjective map φ from Y_1 onto X and a unimodular scalar continuous function h on Y_1 such that*

$$Tf_{|Y_1} = h \cdot f \circ \varphi, \quad \forall f \in C_0(X).$$

In other words, $Tf(y) = h(y)f(\varphi(y))$ for all y in Y_1.

Sketch of the proof. Let $F = \operatorname{ran} T$ be the (necessarily closed) range space of the isometry T. The dual map $T^* : M(Y) \longrightarrow M(X)$ induces an affine homeomorphism Φ from the closed dual ball F_1^* of F onto $M_1(X)$ in weak* topologies. In particular, Φ maps $\operatorname{ext} F_1^*$ onto $\operatorname{ext} M_1(X)$. Hence, for each x in X there is an extreme point η in F_1^* of norm one such that $\Phi(\eta) = \delta_x$. Since the set of all norm one extensions of η to $C_0(X)$ is a non-empty weak* closed face of $M_1(Y)$, there is a (not necessarily unique) extreme point $\frac{\delta_y}{\lambda}$ in $\operatorname{ext} M_1(Y)$ such that $T^*(\frac{\delta_y}{\lambda}) = \delta_x$, or $T^*\delta_y = \lambda\delta_x$. In particular, $y \in Y_1$. Set $\varphi(y) = x$ and $h(y) = \lambda$ whenever $T^*(\delta_y) = \lambda\delta_x$. It is then routine to verify that φ and h are well-defined and continuous on Y_1 such that $\varphi(Y_1) = X$, $|h(y)| = 1$ on Y_1, and $Tf_{|Y_1} = h \cdot f \circ \varphi$ for all f in $C_0(X)$. □

We remark that $\varphi(y) = x$ if and only if $|Tf(y)| = |f(x)|$ for all f in $C_0(X)$, as the latter ensures $\ker T^*\delta_y = \ker \delta_x$.

Lemma 2.3. *The set $Y_0 \cup Y_1$ is closed in Y. Moreover, if we extend φ by setting*

$$\varphi_{|Y_0 \cup \{\infty\}} \equiv \infty$$

then the surjective map $\varphi : Y_0 \cup Y_1 \cup \{\infty\} \longrightarrow X \cup \{\infty\}$ is again continuous.

Proof. Let $\{y_\lambda\}_\lambda$ be a net in $Y_0 \cup Y_1$ such that $y_\lambda \to y$ for some y in Y. We want to verify that $y \in Y_0 \cup Y_1$. Suppose y does not belong to the closed set $Y_0 = \bigcap\{(Tf)^{-1}\{0\} : f \in C_0(X)\}$, we can then assume that y_λ is in Y_1 for all λ. Let $x_\lambda = \varphi(y_\lambda) \in X$. We have

$$|f(x_\lambda)| = |Tf(y_\lambda)| \to |Tf(y)|, \quad \forall f \in C_0(X). \tag{1}$$

Note that there is some f in $C_0(X)$ with $Tf(y) \neq 0$ since $y \notin Y_0$. Let x be any cluster point of the net $\{x_\lambda\}_\lambda$ in $X \cup \{\infty\}$. By (1), we have $x \neq \infty$ and

$$|Tf(y)| = |f(x)|, \quad \forall f \in C_0(X).$$

Hence, $y \in Y_1$ and $\varphi(y) = x$. Thus, $Y_0 \cup Y_1$ is closed in Y.

To verify the continuity of φ, we show that $\varphi(y_\lambda) \to \varphi(y)$ in $X \cup \{\infty\}$ whenever $y_\lambda \to y$ in $Y_0 \cup Y_1 \cup \{\infty\}$. By first half of the proof, it suffices to check that $\varphi(y_\lambda) \to \infty$ whenever $y \in Y_0 \cup \{\infty\}$. Indeed, this follows from the observation

$$0 = |Tf(y)| = \lim_\lambda |Tf(y_\lambda)| = \lim_\lambda |f(\varphi(y_\lambda))|, \quad \forall f \in C_0(X).$$

□

Note that as an open subset of a closed subset of a locally compact space, Y_1 is locally compact of its own.

Lemma 2.4. *Let $T_1 : C_0(X) \longrightarrow C_0(Y_1)$ be the linear isometry defined by $T_1 f = Tf_{|Y_1}$ for all f in $C_0(X)$. Then*

$$\operatorname{ran} T_1 = \{g \in C_0(Y_1) : \frac{g(a)}{h(a)} = \frac{g(b)}{h(b)} \text{ whenever } a, b \in Y_1 \text{ such that } \varphi(a) = \varphi(b)\}.$$

Proof. One inclusion is plain since $T_1 f = h \cdot f \circ \varphi$. Suppose g in $C_0(Y_1)$ satisfies that $\frac{g(a)}{h(a)} = \frac{g(b)}{h(b)}$ whenever $\varphi(a) = \varphi(b)$. Define a function f on X by

$$f(x) = \frac{g(y)}{h(y)} \quad \text{if } y \in Y_1 \text{ and } \varphi(y) = x.$$

Since $\varphi(Y_1) = X$ (Lemma 2.2), such an f is well-defined on X. To see f is continuous, we assume on the contrary that $x_\lambda \to x$ in $X \cup \{\infty\}$ but $|f(x) - f(x_\lambda)| > \epsilon$ for some $\epsilon > 0$. Let $x_\lambda = \varphi(y_\lambda)$ for some y_λ in Y_1. Let y be a cluster point of $\{y_\lambda\}_\lambda$ in $Y_0 \cup Y_1 \cup \{\infty\}$. By Lemma 2.3, $\varphi(y)$ is a cluster point of $\{x_\lambda\}_\lambda$. Thus $\varphi(y) = x$. If $x \in X$ then $y \in Y_1$ and $\frac{g(y)}{h(y)} = f(x)$ is a cluster point of $f(x_\lambda) = \frac{g(y_\lambda)}{h(y_\lambda)}$, a contradiction. In case $x_\lambda \to \infty$, we see that $y \in Y_0 \cup \{\infty\}$ by Lemma 2.3. Since h is unimodular, we have $|f(x_\lambda)| = |g(y_\lambda)| \to 0$, a contradiction again. Therefore, $f \in C_0(X)$ and $g = Tf \in \operatorname{ran} T_1$. □

From now on, we assume that $T : C_0(X) \longrightarrow C_0(Y)$ is a linear isometry with finite corank n. Let $\#A$ denote the cardinality of a set A.

Definition 2.5. Let

$$M = \{y \in Y_1 : \varphi^{-1}\{\varphi(y)\} \text{ contains at least two points}\}$$

be the set of *merging points*, $\varphi(M)$ the set of *merged points* and the number $\#M - \#\varphi(M)$ the *merging index* of T. Call also the number $\#Y_0$ the *vanishing index* and $\#Y_e$ the *exception index* of T.

Lemma 2.6.

$$\#M - \#\varphi(M) \leq n,$$

and

$$\#Y_0 + \#Y_e \leq n.$$

Proof. It follows from Lemma 2.4 that $\#M - \#\varphi(M) = \operatorname{corank} T_1 \leq \operatorname{corank} T = n$. For the second inequality, we note that $Y_e = Y \setminus Y_0 \cup Y_1$ is open in Y by Lemma 2.3. Suppose there are distinct $p_1, \ldots, p_k$ in Y_0 and $y_1, \ldots, y_l$ in Y_e. Then we can choose $f_1, \ldots, f_k, g_1, \ldots, g_l$ in $C_0(Y)$ with mutually disjoint supports such that

$$f_1(p_1) = \cdots = f_k(p_k) = g_1(y_1) = \cdots = g_l(y_l) = 1,$$

and

$$g_{j|Y_0 \cup Y_1} \equiv 0, \quad j = 1, \ldots, l.$$

We claim that these $k+l$ functions are linear independent modulo the range space of T. To this end, let

$$Tf = \lambda_1 f_1 + \cdots + \lambda_k f_k + \alpha_1 g_1 + \cdots + \alpha_l g_l$$

for some f in $C_0(X)$ and scalars $\lambda_1, \ldots, \lambda_k, \alpha_1, \ldots, \alpha_l$. By evaluating at each p_i, we get $\lambda_i = 0$ for $i = 1, \ldots, k$. It then follows $Tf_{|Y_0 \cup Y_1} \equiv 0$ and, in particular, $|f(\varphi(y))| = |Tf(y)| = |\sum_{j=1}^{l} \alpha_j g_j(y)| = 0$ for all y in Y_1. Since $\varphi(Y_1) = X$, we have $f = 0$. This makes $\alpha_1 = \cdots = \alpha_l = 0$ since $g_1, \ldots, g_l$ have disjoint supports. As a result, $l + k \leq \operatorname{corank} T = n$. □

Corollary 2.7. *1. Both the vanishing set Y_0 and the exceptional set Y_e are finite.*

2. Y_e consists of isolated points in Y.

3. Suppose X is compact. Then both Y and Y_1 are compact and Y_0 consists of isolated points in Y.

Proof. We mention that Y_e is an open set by Lemma 2.3. In case X is compact, ∞ is isolated in $X \cup \{\infty\}$ and Lemma 2.3 ensures $Y_0 \cup \{\infty\} = \varphi^{-1}\{\infty\}$ is also open. The assertions follow since finite open sets consists of isolated points. □

The following example borrowed from [13] says that Y_0 can contain non-isolated point if X is not compact.

Example 2.8 ([13])**.** Let X be the disjoint union in $\mathbb{R}^2$ of $I_n^+ = \{(n,t)\colon 0 < t \leq 1\}$ and $I_n^- = \{(n,t) : -1 < t < 0\}$ for $n = 1, 2, \ldots$. Let p be the point $(1,1)$ and let $X_1 = X \setminus \{p\}$. Let φ be the homeomorphism from X_1 onto X by sending the intervals $I_1^+ \setminus \{p\}$ onto I_1^-, I_{n+1}^+ onto I_n^+, and I_n^- onto I_{n+1}^- in a canonical way for $n = 1, 2, \ldots$. Then the corank one linear isometry $Tf = f \circ \varphi$ from $C_0(X)$ into $C_0(X)$ has exactly one vanishing point, i.e., p. We note that p is not an isolated point in X. In a similar manner, one can even construct an example in which X is connected (by adjoining each $I_n^{\pm}$ a common base point, for example).

Theorem 2.9. *The map $\varphi : (Y_1, M) \longrightarrow (X, \varphi(M))$ is a relative homeomorphism. More precisely, $\varphi\colon Y_1 \setminus M \to X \setminus \varphi(M)$ is a homeomorphism, and the induced map $\widetilde{\varphi}\colon Y_1/_{\sim} \to X$ is also a homeomorphism, where "$\sim$" is the equivalence relation such that $y_1 \sim y_2$ if and only if $\varphi(y_1) = \varphi(y_2)$.*

Proof. It suffices to show that $y_\lambda \to y$ in Y_1 whenever $\varphi(y_\lambda) \to \varphi(y)$ in X. Suppose y' is any cluster point of $\{y_\lambda\}$ in $Y_0 \cup Y_1 \cup \{\infty\}$ which is compact by Corollary 2.7. It follows from Lemma 2.3 that $\varphi(y')$ is a cluster point of $\{\varphi(y_\lambda)\}$. Thus $\varphi(y) = \varphi(y')$ and, in particular, $y' \in Y_1$. If y is not a merging point, i.e. $y \notin M$, then $y = y'$. In case y is a merging point, the above argument tells us that the equivalence class $[y] = \varphi^{-1}\{\varphi(y)\}$ contains all cluster points y' of $\{y_\lambda\}$. This shows that the induced map $\widetilde{\varphi}$ is also a homeomorphism. □

Lemma 2.10. *Fix each y' in Y_e, there is a μ' in $M(Y)$ supported by Y_1 such that*

$$g(y') = \int_{Y_1} \frac{g(y)}{h(y)} d\mu'(y), \quad \forall g \in \operatorname{ran} T.$$

Proof. Let $\nu = \delta_{y'} \circ T \in M(X)$. If $\nu(x) = 0$ for all x in $\varphi(M)$, then we can set $\mu'(\{y\}) = 0$ for all y in M and $\mu'(B) = \nu(\varphi(B \cap Y_1))$ for all Borel subsets B of Y disjoint from M. It follows from Theorem 2.9 that $\mu' \in M(Y)$. Clearly, μ' satisfies the stated condition. In case $\nu(\{x\}) \neq 0$ for some merged point x in $\varphi(M)$ and $\varphi^{-1}\{x\} = \{y_1, \dots, y_k\}$, we may set $\mu'(\{y_1\}) = \cdots = \mu'(\{y_k\}) = \nu(\{x\})/k$. Since $\frac{g(y_1)}{h(y_1)} = \cdots = \frac{g(y_k)}{h(y_k)} = f(x)$ if $Tf = g$, we again have the stated condition. □

Theorem 2.11. *The sum of the vanishing, exception and merging indices of a corank n linear isometry* $T : C_0(X) \longrightarrow C_0(Y)$ *is n. In other words,*

$$\#Y_0 + \#Y_e + \#M - \#\varphi(M) = n.$$

In fact,

$$\operatorname{ran} T = \Big\{ g \in C_0(X) : g_{|Y_0} \equiv 0, \quad g(y') = \int_{Y_1} \frac{g(y)}{h(y)} d\mu'(y) \text{ for all } y' \text{ in } Y_e, \\ \text{and } \frac{g(a)}{h(a)} = \frac{g(b)}{h(b)} \text{ whenever } a, b \in M \text{ such that } \varphi(a) = \varphi(b) \Big\},$$

where $Tf_{|Y_1} = h \cdot f \circ \varphi$ *and* μ' *is the Borel measure in* $M(Y)$ *associated to each* y' *in* Y_e *as in Lemmas 2.2 and 2.10.*

Proof. From Lemmas 2.2, 2.4 and 2.10, we have already had one side inclusion. For the other inclusion, we suppose a g in $C_0(Y)$ satisfies all $\#Y_0 + \#Y_e + \#M - \#\varphi(M)$ linear independent conditions stated on the right hand side. Set

$$f(x) = \frac{g(y)}{h(y)} \quad \text{whenever } y \in Y_1 \text{ and } \varphi(y) = x.$$

By the proof of Lemma 2.4, we have $f \in C_0(X)$ and Tf agrees with g on Y_1. It is plain that Tf also agrees with g on Y_0 and

$$Tf(y') = \int_{Y_1} \frac{Tf(y)}{h(y)} d\mu'(y) = \int_{Y_1} f(\varphi(y)) d\mu'(y) = \int_{Y_1} \frac{g(y)}{h(y)} d\mu'(y) = g(y'), \quad \forall y' \in Y_e.$$

Hence $g = Tf$, and consequently, $\#Y_0 + \#Y_e + \#M - \#\varphi(M) = n$. □

Remark 2.12. (a) In the recent literature, corank 1 linear isometries are of particular interests. In [1], corank 1 linear isometries of function algebras are classified into three types. Recall that a subset of $C_0(Y)$ is said to *separate points in* Y (resp. Y_∞) *strongly* if for any distinct y and y' in Y (resp. Y_∞) there is a g in this subset such that $|g(y)| \neq |g(y')|$. In [1], a corank 1 linear isometry $T : A \longrightarrow B$ between function algebras is said to be of

Type I: if the range of T separates points in Y strongly, except for two of them.

Type II: if the range of T separates points in Y, but not in Y_∞, strongly.

Type III: if the range of T separates points in Y_∞ strongly.

In case $A = C_0(X)$ and $B = C_0(Y)$, our structure theory (Theorem 2.11) simply says that T is of Type I, II, or III if and only if either the merging, the vanishing, or the exception index of T is 1. Our approach seems to be more convenient in the higher dimensional case (cf. [7]).

(b) By Corollary 2.7, Y_e consists of isolated points. Consequently, if Y is connected then every corank 1 linear isometry T from $C_0(X)$ into $C_0(Y)$ must be of Type I or Type II. In general, T is of Type I or Type II if and only if T is disjointness preserving, i.e. $fg = 0$ implies $TfTg = 0$. Hence, we may also divide linear isometries of finite corank into two classes: ones preserve disjointness and the others do not.

(c) If X is compact then Y_0 consists of isolated points by Corollary 2.7. However, if X is *not* compact then Y_0 can contain non-isolated points as shown in Example 2.8. This example provides us more insights into a result in [1, Theorem 6.1], which deals with the preservation of Shilov boundaries of function algebras by a corank 1 linear isometry.

3. Isometric (quasi-)n-shifts on $C_0(X)$

Recall that an isometric quasi-n-shift T on $C_0(X)$ is a corank n linear isometry from $C_0(X)$ into itself. All results in Section 2 thus apply. In particular, we have the following generalization of [8, Theorem 2.6].

Proposition 3.1. *Let X be a compact Hausdorff space with at most finitely many isolated points. If $C(X)$ admits an isometric quasi-n-shift T, then there is a finite subset M of X and a relative homeomorphism $\varphi : (X, M) \longrightarrow (X, \varphi(M))$ such that $n = \#(M) - \#(\varphi(M))$. Moreover, the induced quotient map $\widetilde{\varphi}$: ${X}/{\sim} \to X$ is a homeomorphism, where $\sim$ is the equivalence relation on X such that $x \sim x'$ if and only if $\varphi(x) = \varphi(x')$.*

Proof. By Lemma 2.2, $Tf = h \cdot f \circ \varphi$ for a continuous unimodular scalar function h on X and a surjective continuous map φ from X_1 onto X. By Corollary 2.7, both X_0 and X_e are empty since X is compact and contains at most finitely many isolated points; for else the set $\{\varphi^{-n}\{x\} : n = 1, 2, \dots\}$ would contain infinitely many isolated points in X for any x in $X_0 \cup X_e$. Hence, $X = X_1$. The assertions now follow from Theorem 2.9. □

Corollary 3.2. *Let X be a path-connected compact Hausdorff space in which points are strong deformation retract of compact neighborhoods. If $C(X)$ admits an isometric quasi-n-shift then the first homological group $H_1(X)$ of X has infinitely many free generators.*

Proof. Suppose $x \in \varphi(M)$ and $\varphi^{-1}\{x\} = \{y_1, \dots, y_l\}$. Consider the long exact sequence:

$$\cdots \to H_1(\{y_1, \dots, y_l\}) \to H_1(X) \to H_1(X, \{y_1, \dots, y_l\}) \\ \to H_0(\{y_1, \dots, y_l\}) \to H_0(X) \to H_0(X, \{y_1, \dots, y_l\}).$$

Since X is path-connected and points are strong deformation retract of compact neighborhoods in X, the above long exact sequence gives a short exact sequence

$$0 \to H_1(X) \to H_1({X}/{\sim_x}) \to \mathbb{Z}^{l-1} \to 0,$$

where $\sim_x$ is the equivalence relation defined on X by identifying $y_1, \dots, y_l$. Hence,

$$H_1({X}/{\sim_x}) \cong H_1(X) \oplus \mathbb{Z}^{l-1}.$$

Let x' be another point in $\varphi(M)$ and $\varphi^{-1}\{x'\} = \{y'_1, \ldots, y'_k\}$. Applying the same argument to $X/\sim_x$, we get

$$H_1(X/\sim_{x,x'}) \cong H_1(X/\sim_x) \oplus \mathbb{Z}^{k-1} \cong H_1(X) \oplus \mathbb{Z}^{l+k-2},$$

where $\sim_{x,x'}$ is the equivalence relation defined on X by identifying $y_1, \ldots, y_l$ and identifying $y'_1, \ldots, y'_k$. In this manner, we would get

$$H_1(X/\sim) \cong H_1(X) \oplus \mathbb{Z}^n$$

since $n = \#M - \#\varphi(M)$, where $\sim$ is the equivalent relation defined as in Proposition 3.1. Because $X/\sim$ and X are homeomorphic, the assertion follows. □

We note that the first homological group of any finite-dimensional compact topological manifold is finitely generated (see e.g. [17, p. 163]). Suggested by [8, Corollary 2.4], we extend [6, Theorem 6.1] in the following

Corollary 3.3. *There is no finite-dimensional compact topological manifold X such that $C(X)$ admits any isometric quasi-n-shift.*

Remark 3.4. In a similar manner, results in [3] can be applied so that Corollaries 3.2 and 3.3 are also valid for disjointness preserving quasi-n-shifts.

Let T be an isometric quasi-n-shift on $C_0(X)$ such that $Tf = h \cdot f \circ \varphi$ on X_1 (Lemma 2.2). In the following, we discuss the structure of the range spaces of the powers T^k of T. For convenience, we extend h to X_∞ be setting $h \equiv 1$ on $X_\infty \setminus X_1$. Note that h is not necessarily continuous unless X is compact (Corollary 2.7).

Let $X_e = \{q_1, \ldots, q_m\}$ be the exceptional set of T. For each q in X_e, let μ be the bounded regular Borel measure in $M(X)$ supported by X_1 defined as in Lemma 2.10 such that

$$Tf(q) = \int_{X_1} \frac{Tf(y)}{h(y)} d\mu(y), \quad \forall f \in C_0(X).$$

In a similar manner, we can construct a sequence $\{\mu_k\}$ of bounded regular Borel measures in $M(X)$ supported by X_1 such that $T^*(\frac{\mu_{k+1}}{h}) = \mu_k$ for $k = 0, 1, \ldots$. Here we set $\mu_0 = T^*\delta_q$ and $\mu_1 = \mu$. In general, let $\mu_{k+1}(B) = \mu_k(\varphi(B \cap X_1))$ for all Borel subsets B of X disjoint from the merging set M, and for each merged point x in $\varphi(M)$ we let $\mu_{k+1}(\{y_1\}) = \cdots = \mu_{k+1}(\{y_k\}) = \mu_k(\{x\})/k$ if $\varphi^{-1}\{x\} = \{y_1, \ldots, y_k\}$. Moreover, we identify points x in X with point evaluations δ_x in $M(X)$, and ∞ with the zero measure.

Definition 3.5. A *T-branch originated at a point* x in X_∞ is defined to be the set

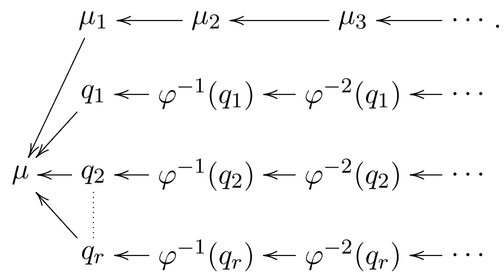
$$B_x = \bigcup \left\{ \varphi^{-n}(x) \colon n = 0, 1, 2, \ldots \right\},$$

where $\varphi^0(x) = \{x\}$ and $\varphi^{-n}(x) = \{y \in X \colon \varphi^n(y) = x\}$ for $n = 1, 2, \ldots$. We note that $x = \varphi(y)$ if and only if $T^*(\frac{\delta_y}{h}) = \delta_x$. Suppose μ is the bounded Borel measure in $M(X)$ associated simultaneously to q_1, q_2, $\ldots$, q_r in X_e, i.e. $T^*\delta_{q_i} = \mu$ for $i = 1, 2, \ldots, r$. We define the *T-branch B_μ originated at* μ to be the union of the sequence $\{\mu_k\}$ and B_{q_i} for $i = 1, 2, \ldots, r$. The *T-tree* is a directed graph, whose vertex set is the union of all T-branches B_x originated at some point x in $\varphi(M)$ (and also at $x = \infty$ if $Y_0 \neq \emptyset$) and all

T-branches B_μ originated at some μ associated to a point q in X_e. There is a directed edge from μ to ν if and only if $T^*(\mu/h) = \nu$. In case μ and ν are point masses at y and x in X_∞ respectively, we will write $x \leftarrow y$ instead. Note that this is equivalent to $\varphi(y) = x$.

The branch of the T-tree originated at μ may look like:

(2)



This T-branch has at least r "joints" (and maybe more "joints" at some subsequent vertex $\varphi^{-1}(q_j)$). In general, the whole T-tree has exactly n "joints" if T is a quasi-n-shift.

Example 3.6. Let ℓ_∞ be the Banach space of bounded scalar sequences. We can identify ℓ_∞ as $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of the natural numbers $\mathbb{N}$. Define an isometric shift T on ℓ_∞ be $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. The T-tree is

$$\circlearrowright \infty \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots.$$

Note that the T-tree is dense in $X = \beta\mathbb{N}$.

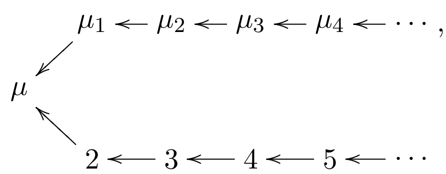
Example 3.7. Let $c_0 = C_0(\mathbb{N})$ be the Banach space of null sequences. Let $T\colon c_0 \to c_0$ be defined by

$$T\big((x_1, x_2, x_3, \dots)\big) = \left(x_1, -\frac{x_1 + x_2}{2}, x_2, x_3, \dots\right).$$

Then T is an isometric quasi-shift on c_0. In this case, $X = \mathbb{N}$, $X_0 = \emptyset$, $X_e = \{2\}$, $X_1 = \mathbb{N} \setminus \{2\}$, $h \equiv 1$ on $\mathbb{N} \setminus \{2\}$, and $\varphi\colon \mathbb{N} \setminus \{2\} \to \mathbb{N}$ is a homeomorphism defined by $\varphi(1) = 1$ and $\varphi(n+1) = n$ for $n = 2, 3, \dots$. Moreover, we have $M = \varphi(M) = \emptyset$ and

$$\mu = T^*\delta_2 = -\frac{\delta_1 + \delta_2}{2},$$

where δ_m is the point evaluation at m in $\mathbb{N}$. The T-tree is



where

$$\mu_1 = \mu \circ \varphi = -\frac{\delta_{\varphi^{-1}(1)} + \delta_{\varphi^{-1}(2)}}{2} = -\frac{\delta_1 + \delta_3}{2},$$
$$\mu_2 = \mu \circ \varphi^2 = -\frac{\delta_{\varphi^{-2}(1)} + \delta_{\varphi^{-2}(2)}}{2} = -\frac{\delta_1 + \delta_4}{2},$$
$$\vdots$$

and, in general, for $m = 1, 2, \dots$,

$$\mu_m = \mu \circ \varphi^m = -\frac{\delta_{\varphi^{-m}(1)} + \delta_{\varphi^{-m}(2)}}{2} = -\frac{\delta_1 + \delta_{m+2}}{2}.$$

We verify that T is a shift on c_0, i.e., $\bigcap_{m=1}^{\infty} \operatorname{ran} T^m = \{0\}$. It follows from Theorem 2.11 that the range space of T is

$$\operatorname{ran} T = \left\{ g = (g_m)_{m=1}^{\infty} \in c_0 \colon g_2 = -\frac{g_1 + g_3}{2} \right\}.$$

It is also easy to see that

$$\begin{aligned} T^2 f(3) &= Tf(\varphi(3)) = Tf(2) = \int f(x) d\mu(x) \\ &= \int f(\varphi(y)) d\mu(\varphi(y)) = \int Tf(y) d\mu_1(y) = \int Tf(\varphi(z)) d\mu_1(\varphi(z)) = \int T^2 f(z) d\mu_2(z) \end{aligned}$$

for all f in c_0. Hence,

$$\operatorname{ran} T^2 = \left\{ g = (g_m)_{m=1}^{\infty} \in c_0 \colon g_2 = -\frac{g_1 + g_3}{2} \text{ and } g_3 = -\frac{g_1 + g_4}{2} \right\}.$$

In this manner, for any $g = (g_m)$ in c_0, we have

$$\begin{aligned} g \in \bigcap_{m=1}^{\infty} \operatorname{ran} T^m &\Leftrightarrow g(m+1) = \int g \, d\mu_m, \quad \forall m = 1, 2, \dots. \\ &\Leftrightarrow g_{m+1} = -\frac{g_1 + g_{m+2}}{2}, \quad \forall m = 1, 2, \dots. \\ &\Leftrightarrow g_1 = -2g_{m+1} - g_{m+2}, \quad \forall m = 1, 2, \dots. \end{aligned}$$

As a result, $g_1 = 0$ and thus

$$2g_{m+1} + g_{m+2} = 0, \quad \forall m = 1, 2, \dots.$$

Consequently,

$$|g_{m+1}| = \frac{|g_{m+2}|}{2} = \frac{|g_{m+3}|}{2^2} = \cdots = \frac{|g_{m+k}|}{2^k} \to 0, \qquad \text{as } k \to \infty.$$

Hence, $g = 0$, and thus $\bigcap_{m=1}^{\infty} \operatorname{ran} T^m = \{0\}$. □

Suppose T is an isometric quasi-n-shift on $C_0(X)$ and $Tf = h \cdot f \circ \varphi$ on X_1. Denote by

$$h \circ \varphi_{k!}(x) = h(x) h\big(\varphi(x)\big) \cdots h\left(\varphi^{k-1}(x)\right), \quad \forall x \in X_\infty, \forall k = 1, 2, \dots.$$

We set $h_{|X_\infty \setminus X_1} = 1$ for convenience.

Definition 3.8. A g in $C_0(X)$ is said to be *h-equipotential on the T-tree at level k* if we have

$$\int \frac{g}{h \circ \varphi_{k!}} d\mu_k = \int \frac{g}{h \circ \varphi_{k!}} d\nu_k$$

whenever the two vertices μ_k and ν_k in the T-tree are connected forward by k directed edges to the same vertex. Note that points x in X are identified with point masses δ_x in $M(X)$.

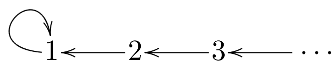
The following result is obtained by the same argument given in Example 3.7.

Proposition 3.9. *Let T be an isometric quasi-n-shift on $C_0(X)$. The range space of the power T^m is given by*

$$\operatorname{ran} T^m = \Big\{ g \in C_0(X) \colon g \text{ is } h\text{-equipotential on the } T\text{-tree at levels } 1, 2, \ldots, m \Big\}.$$

Corollary 3.10. *The T-tree is weak* total in $M(X)$ whenever T is an isometric n-shift on $C_0(X)$.*

We remark that the converse of Corollary 3.10 is not true. For example, consider the isometric quasi-shift $T(x_1, x_2, x_3, \ldots) = (x_1, x_1, x_2, x_3, \ldots)$ on $c = C(\mathbb{N} \cup \{\infty\})$. The T-tree



is dense in $X = \mathbb{N} \cup \{\infty\}$ although T is not a shift.

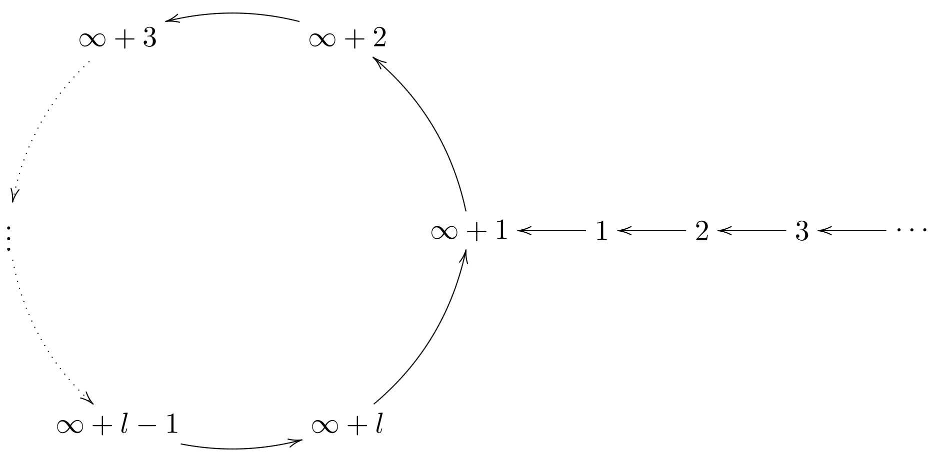
Example 3.11. Let

$$X = \Big\{1, 2, 3, \ldots, \infty, \infty + 1, \infty + 2, \ldots, \infty + l\Big\}$$

be the disjoint union of $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and a discrete set of l points. Let $T \colon C(X) \to C(X)$ be the isometric shift defined by

$$\begin{aligned} &T\big((x_1, x_2, x_3, \ldots, x_\infty, x_{\infty+1}, \ldots, x_{\infty+l-1}, x_{\infty+l})\big) \\ &\quad = (x_{\infty+1}, x_1, x_2, \ldots, x_\infty, x_{\infty+2}, \ldots, x_{\infty+l}, -x_{\infty+1}). \end{aligned}$$

Then $Tf = h \cdot f \circ \varphi$ for all f in $C(X)$. Here, $h(\infty + l) = -1$ and $h \equiv 1$ elsewhere. The relative homeomorphism $\varphi : (X, \{1, \infty{+}l\}) \longrightarrow (X, \{\infty{+}1\})$ is represented by the following T-tree, which is the branch originated at the merged point $\infty + 1$. Here, $a \leftarrow b$ indicates $\varphi(b) = a$. Moreover, $\varphi(\infty) = \infty$.



We verify that $\cap_{m=1}^{\infty} \operatorname{ran} T^m = \emptyset$. By Proposition 3.9,

$$\operatorname{ran} T = \{g \in C(X) : \ g(1) = -g(\infty + l)\},$$

$$\operatorname{ran} T^2 = \{g \in C(X) : \ g(1) = -g(\infty + l), \ g(2) = -g(\infty + l - 1)\},$$

$$\vdots$$

In fact, a g in $C(X)$ is h-equipotential on the T-tree at level k if and only if

$$\frac{g(k)}{h \circ \varphi_{k!}(k)} = \frac{g(\infty + l - k_1)}{h \circ \varphi_{k!}(\infty + l - k_1)},$$

or

$$g(k) = (-1)^r g(\infty + l - k_1),$$

where $k = rl - k_1$ and $0 \leq k_1 < l$. This makes $g(k) = 0$ for $k = 1, 2, \ldots, \infty$. It then in turn forces $g(\infty + k) = 0$ for $k = 1, 2, \ldots, l$. Hence, $g = 0$ as asserted.

Note that the T-tree has exactly one joint at $\infty + 1$, and it is dense in X. In fact, only the limit point ∞ is missing from the T-tree above. □

Remark 3.12. In [8] and [6], the authors considered the notion of types. Example 3.11 was used in [6] to show that there is a type I isometric 1-shift T such that T is a weighted composition operator on $X \setminus \{q\}$ and the set

$$D = \{q, \varphi^{-1}(q), \varphi^{-2}(q), \ldots\}$$

is not dense in X. In this case, $q = 1$ and $D = \mathbb{N}$. But we have seen above that the T-tree is dense in X, indeed. It seems to us that the notions of types of shifts and the set D (and F in their notations) can be misleading in some situations.

Note that the unilateral shift defined only on separable Hilbert spaces. The action of the unilateral shift can be thought of a shift on a countable orthonormal basis. Although it is now a basis free theory for isometric shifts T on $C_0(X)$, the T-tree can be considered as a "basis" for the shift T. Corollary 3.10 says this countable "basis" is total in $M(X)$. Thus $M(X)$ is weak* separable. We are interested in knowing when X is separable. Recall that a measure μ in $M(X)$ is *separately supported* if the support $\mathrm{supp}(\mu)$ of μ is a separable subset of X.

Theorem 3.13. *Suppose $C_0(X)$ admits an isometric n-shift T. If all measures $\mu' = \delta_{y'} \circ T$ arising from points y' in X_e are separately supported then X is separable.*

Proof. We first note that the assumption implies all measures appearing in the T-tree are separately supported. In fact, every such measure is either a point mass or the one obtained by successively composing those μ' with φ in a finite steps. For the latter, the supports is separable since $\varphi : (X_1, M) \longrightarrow (X, \varphi(M))$ is a relative homeomorphism and M is a finite set. Let S be the countable union of the supports of all the measures appearing in the T-tree. Then S has a countable dense subset. Finally, we claim S is dense in X. It is plain that every g in $C_0(X)$ vanishing on S is zero at each vertex in the T-tree. By Corollary 3.9, all such g are in the range of T^m for $m = 1, 2, \ldots$. This forces g being constantly zero since T is an n-shift. Hence S is dense in X, as asserted. □

Corollary 3.14. *Suppose $C_0(X)$ admits an isometric n-shift T. Then X is separable if any one of the following holds.*

1. *X does not contain infinitely many isolated points.*
2. *The range space of T cannot strongly separate points in X_∞ unless at least n points are removed.*

3. T is disjointness preserving.
4. X_e is empty.
5. The T-tree is contained in X_∞.

Proof. It follows from the structure of the range space of T (Theorem 2.11) that T is disjointness preserving, if and only if, X_e is empty, if and only if, the T-tree is contained in X_∞. On the other hand, if q is a point in X_e then q is isolated by Corollary 2.7. Consequently, the T-branch originated at q consists of infinitely many isolated points in X. Hence the first condition also implies X_e is empty. Finally, the second condition ensures that the merging index $\#M - \#\varphi(M)$ of T is exactly n. Thus $X_e = \emptyset$ again. In all cases, Theorem 3.13 applies. □

To end this paper, we remark that Araujo and Font [2] recently showed that if X is a (not necessarily compact) metrizable space such that the Banach space $C_b(X)$ of bounded continuous functions on X admits an isometric shift then X is separable.

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